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# Some New Ostrowski Type İnequalities for Co-Ordinated Convex Functions

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**Abstract** In this paper, we obtain new identity for function of two variables and apply them to give new Ostrowski type integral inequality for double integrals involving functions whose derivatives are co-ordinates convex function on $\Delta := [a,b]\times[c,d]$ in $\mathrm{R}^2$ with $a<b,\ c<d$ .

***Keywords:*** *Ostrowski type inequalities, coordinated convex functions, Hölder's inequality*

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## 1. Introduction

Let $f:[a,b]\to\mathrm{R}$ be a differentiable mapping on $(a,b)$ whose derivative $f':(a,b)\to\mathrm{R}$ is baunded on $(a,b)$, i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \quad (1.1)$$

for all $x\in[a,b]$ (see, [13]). The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.* For some results which generalize, improve and extend the inequality (1.1) see ([5,6,7,14,15,16,17]) and the references therein.

Let us consider now a bidimensional interval $\Delta := [a,b]\times[c,d]$ in $\mathrm{R}^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathrm{R}$ is said to be convex on $\Delta$ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le\lambda f(x,y)+(1-\lambda)f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$. The mapping $f$ is said to be concave on the co-ordinates on if the above inequality holds in reverse direction, for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

A formal definition for coordinated convex function may be stated as follows:

**Definition 1.** A function $f:\Delta\to\mathrm{R}$ will be coordinated convex on $\Delta$, for all $t,s\in[0,1]$ and $(x,y),(u,v)\in\Delta$, if the following inequality holds:

$$\begin{aligned}&f(tx+(1-t)y,su+(1-s)v)\\&\le tsf(x,u)+s(1-t)f(y,u)\\&\quad+t(1-s)f(x,v)+(1-t)(1-s)f(y,v).\end{aligned}$$

Clearly, every convex function is coordinated convex. Furthermore, there exist coordinated convex function which is not convex, (see, [3]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathrm{R}^2$, we refer the reader to ([1,2,3,4,8-12,18,19]).

Also, in [3], Dragomir establish the following Hermite-Hadamard's type inequality for coordinated convex mapping on a rectangle from the plane $\mathrm{R}^2$.

**Theorem 1.** Suppose that $f:\Delta\to\mathrm{R}$ is coordinated convex on $\Delta$. Then one has the inequalities:

$$\begin{aligned}&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\&\le\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\&\quad\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\&\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}\quad (1.2)$$

The above inequalities are sharp.

In a recent paper [5], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals:

**Theorem 2.** Let $f : \Delta \to \mathrm{R}$ be continuous on $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x \partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,

$$\left\|f''_{x,y}\right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right| < \infty.$$

Then, we have the inequality:

$$\begin{aligned}&\left|\int_a^b\int_c^d f(s,t)dtds-(d-c)(b-a)f(x,y)\right.\\&\left.-\left[(b-a)\int_c^d f(x,t)dt+(d-c)\int_a^b f(s,y)ds\right]\right|\\&\le\left[\frac{1}{4}(b-a)^2+(x-\frac{a+b}{2})^2\right]\\&\times\left[\frac{1}{4}(d-c)^2+(y-\frac{d+c}{2})^2\right]\left\|f''_{x,y}\right\|_\infty\end{aligned}\tag{1.3}$$

for all $(x,y)\in[a,b]\times[c,d]$.

The main aim of this paper is to establish some new Ostrowski type inequalities for double integrals involving functions whose derivatives are co-ordinates convex function on $\Delta := [a,b]\times[c,d]$ in $\mathrm{R}^2$ with $a<b,\ c<d$.

## 2. Main Results

To establish our main results, we need the following identity:

**Lemma 1.** Let $f : \Delta\subset \mathrm{R}^2 \to \mathrm{R}$ be a partial differentiable mapping on $\Delta$. If $f_{\lambda\alpha}=\frac{\partial^2 f}{\partial\lambda\partial\alpha}\in L_1(\Delta)$, then for any $(x,y)\in\Delta$, we have the equality:

$$\begin{aligned}f(x,y)&=\frac{1}{d-c}\int_c^d f(x,s)ds+\frac{1}{b-a}\int_a^b f(t,y)dt\\&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\\&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x-t)(y-s)\\&\times\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)d\alpha d\lambda dsdt.\end{aligned}\tag{2.1}$$

**Proof** For any $t,x\in[a,b]$ and $y,s\in[c,d], t\neq x$, $y\neq s$, we have

$$\begin{aligned}\int_t^x\int_s^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma&=\int_t^x\left[f_\sigma(\sigma,y)-f_\sigma(\sigma,s)\right]d\sigma\\&=\left[f(\sigma,y)-f(\sigma,s)\right]_t^x\\&=f(x,y)-f(x,s)-f(t,y)+f(t,s)\end{aligned}$$

and

$$\begin{aligned}f(x,y)&=f(x,s)+f(t,y)-f(t,s)\\&+\int_t^x\int_s^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma.\end{aligned}$$

For $\sigma=\lambda x+(1-\lambda)t$ and $\tau=\alpha y+(1-\alpha)s$, we obtain

$$\begin{aligned}f(x,y)&=f(x,s)+f(t,y)-f(t,s)+(x-t)(y-s)\\&\times\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)d\alpha d\lambda.\end{aligned}\tag{2.2}$$

By integrating (2.2) with respect to $t,s$ on $\Delta$ and divide by $(b-a)(d-c)$, we get the desired equality (2.1).

**Theorem 3.** Let $f : \Delta\to\mathrm{R}$ be a partial differentiable mapping on $\Delta$ and $\left|\frac{\partial^2 f}{\partial\lambda\partial\alpha}\right|=|f_{\lambda\alpha}|$ is co-ordinates convex function on $\Delta$.

**(i)** If $f_{\lambda\alpha}\in L_\infty(\Delta)$ then for any $(x,y)\in\Delta$,

$$\begin{aligned}&\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\&\le\frac{(b-a)(d-c)}{4}\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|+\|f_{\lambda\alpha}(x,.)\|_\infty\\+\|f_{\lambda\alpha}(.,y)\|_\infty+\|f_{\lambda\alpha}\|_\infty\end{array}\right]\\&\times\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right].\end{aligned}$$

**(ii)** If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ then for any $(x,y)\in\Delta$,

$$\begin{aligned}&\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\&\le\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\\&\times\left[\left(\frac{d-y}{d-c}\right)^{q+1}+\left(\frac{y-c}{d-c}\right)^{q+1}\right]^{\frac{1}{q}}\\&\times\left\|\begin{array}{l}|f_{\lambda\alpha}(x,y)|^p+|f_{\lambda\alpha}(x,.)|^p\\+|f_{\lambda\alpha}(.,y)|^p+|f_{\lambda\alpha}|^p\end{array}\right\|_p.\end{aligned}$$

**(iii)** If $f_{\lambda\alpha}\in L_1(\Delta)$ then for any $(x,y)\in\Delta$,

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]$$
$$\times\Big[(b-a)(d-c)\left|f_{\lambda\alpha}(x,y)\right|+(b-a)\left\|f_{\lambda\alpha}(x,.)\right\|_1+(d-c)\left\|f_{\lambda\alpha}(.,y)\right\|_1+\left\|f_{\lambda\alpha}\right\|_1\Big]$$

**Proof (i).** Using (2.1), convexity of $|f_{\lambda\alpha}|$ and taking the modulus, it follows that

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s|\times\int_0^1\int_0^1\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|d\alpha d\lambda dsdt$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\int_0^1\int_0^1|x-t||y-s|\times\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|d\alpha d\lambda dsdt$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\int_0^1\int_0^1|x-t||y-s|\times\left[\begin{array}{l}\lambda\alpha\left|f_{\lambda\alpha}(x,y)\right|+\lambda(1-\alpha)\left|f_{\lambda\alpha}(x,s)\right|\\+(1-\lambda)\alpha\left|f_{\lambda\alpha}(t,y)\right|\\+(1-\lambda)(1-\alpha)\left|f_{\lambda\alpha}(t,s)\right|\end{array}\right]d\alpha d\lambda dsdt$$
$$=\frac{1}{4(b-a)(d-c)}\times\int_a^b\int_c^d|x-t||y-s|\times\left[\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,s)\right|+\left|f_{\lambda\alpha}(t,y)\right|+\left|f_{\lambda\alpha}(t,s)\right|\right]dsdt$$

Since $f_{\lambda\alpha}\in L_\infty(\Delta)$, we get

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s|dsdt\times ess\sup_{(t,s)\in\Delta}\left\{\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,s)\right|\\+\left|f_{\lambda\alpha}(t,y)\right|+\left|f_{\lambda\alpha}(t,s)\right|\end{array}\right\}$$
$$=\frac{1}{4(b-a)(d-c)}\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left\|f_{\lambda\alpha}(x,.)\right\|_\infty\\+\left\|f_{\lambda\alpha}(.,y)\right\|_\infty+\left\|f_{\lambda\alpha}\right\|_\infty\end{array}\right]\times\left(\int_a^b|x-t|dt\right)\left(\int_c^d|y-s|ds\right)$$
$$=\frac{1}{4(b-a)(d-c)}\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left\|f_{\lambda\alpha}(x,.)\right\|_\infty\\+\left\|f_{\lambda\alpha}(.,y)\right\|_\infty+\left\|f_{\lambda\alpha}\right\|_\infty\end{array}\right]\times\left(\frac{(x-a)^2+(b-x)^2}{2}\right)\left(\frac{(y-c)^2+(d-y)^2}{2}\right)$$
$$=\frac{(b-a)(d-c)}{4}\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left\|f_{\lambda\alpha}(x,.)\right\|_\infty\\+\left\|f_{\lambda\alpha}(.,y)\right\|_\infty+\left\|f_{\lambda\alpha}\right\|_\infty\end{array}\right]\times\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right].$$

**(ii).** As above, we can write

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4(b-a)(d-c)}\times\int_a^b\int_c^d|x-t||y-s|\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,s)\right|\\+\left|f_{\lambda\alpha}(t,y)\right|+\left|f_{\lambda\alpha}(t,s)\right|\end{array}\right]dsdt.$$

Using Hölder's ineguality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4(b-a)(d-c)}\left(\int_a^b\int_c^d|x-t|^q|y-s|^q dsdt\right)^{\frac{1}{q}}\times\left(\int_a^b\int_c^d\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,s)\right|\\+\left|f_{\lambda\alpha}(t,y)\right|+\left|f_{\lambda\alpha}(t,s)\right|\end{array}\right]^p dsdt\right)^{\frac{1}{p}}$$
$$=\frac{1}{4(b-a)(d-c)}\left(\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right)^{\frac{1}{q}}\times\left(\frac{(y-c)^{q+1}+(d-y)^{q+1}}{q+1}\right)^{\frac{1}{q}}$$
$$\times\left\|\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,.)\right|+\left|f_{\lambda\alpha}(.,y)\right|+\left|f_{\lambda\alpha}\right|\right\|_p.$$

**(iii).** As above, we obtain the following inequality

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq \frac{1}{4(b-a)(d-c)}\times\int_a^b\int_c^d |x-t||y-s| \times\begin{bmatrix}|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|\\+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\end{bmatrix}dsdt$$

Using convexity of $|f_{\lambda\alpha}|$, we obtain

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq \frac{1}{4(b-a)(d-c)}\sup_{t\in[a,b]}|x-t|\sup_{s\in[c,d]}|y-s| \times\int_a^b\int_c^d\begin{bmatrix}|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|\\+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\end{bmatrix}dsdt$$

$$=\frac{1}{4(b-a)(d-c)}\max\{x-a,b-x\}\max\{y-c,d-y\}\times\Big[(b-a)(d-c)|f_{\lambda\alpha}(x,y)|+(b-a)\|f_{\lambda\alpha}(x,.)\|_1+(d-c)\|f_{\lambda\alpha}(.,y)\|_1+\|f_{\lambda\alpha}\|_1\Big]$$

$$=\frac{1}{4}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]\times\begin{bmatrix}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|\\+(b-a)\|f_{\lambda\alpha}(x,.)\|_1\\+(d-c)\|f_{\lambda\alpha}(.,y)\|_1+\|f_{\lambda\alpha}\|_1\end{bmatrix}$$

This completes the proof.

**Corollary 1.** With the assumptions of Theorem 3 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have the inequality

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq\frac{(b-a)(d-c)}{64}\left[\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_\infty+\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_\infty+\|f_{\lambda\alpha}\|_\infty\right],$$

provided $f_{\lambda\alpha}\in L_\infty(\Delta)$.

If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{16(q+1)^{\frac{2}{q}}}\left\|\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^p+\left|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right|^p+\left|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right|^p+|f_{\lambda\alpha}|^p\right\|_p.$$

If $f_{\lambda\alpha}\in L_1(\Delta)$, then

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq\frac{1}{16}\begin{bmatrix}(b-a)(d-c)\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\+(b-a)\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_1\\+(d-c)\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_1+\|f_{\lambda\alpha}\|_1\end{bmatrix}.$$

**Theorem 4.** Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta$ and $\left|\frac{\partial^2 f}{\partial\lambda\partial\alpha}\right|^p=|f_{\lambda\alpha}|^p$, $p>1$ is co-ordinates convex function on $\Delta$.

**(i)** If $f_{\lambda\alpha}\in L_\infty(\Delta)$ then for any $(x,y)\in\Delta$,

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq\frac{(b-a)(d-c)}{4^{\frac{1}{p}}}\times\begin{bmatrix}|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_\infty^p\\+\|f_{\lambda\alpha}(.,y)\|_\infty^p+\|f_{\lambda\alpha}\|_\infty^p\end{bmatrix}^{\frac{1}{p}}\times\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right].$$

**(ii)** If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ then for any $(x,y)\in\Delta$,

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}(q+1)^{\frac{2}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}$$
$$x\left[\left(\frac{d-y}{d-c}\right)^{q+1}+\left(\frac{y-c}{d-c}\right)^{q+1}\right]^{\frac{1}{q}}$$
$$\times\begin{bmatrix}(b-a)(d-c)\left|f_{\lambda\alpha}(x,y)\right|^p+\left\|f_{\lambda\alpha}(x,.)\right\|_p^p\\+\left\|f_{\lambda\alpha}(.,y)\right\|_p^p+\left\|f_{\lambda\alpha}\right\|_p^p\end{bmatrix}^{\frac{1}{p}}.$$

**(iii)** If $f_{\lambda\alpha}\in L_p(\Delta)$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ then for any $(x,y)\in\Delta$,

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]$$
$$\times\begin{bmatrix}(b-a)(d-c)\left|f_{\lambda\alpha}(x,y)\right|^p\\+\left\|f_{\lambda\alpha}(x,.)\right\|_p^p+\left\|f_{\lambda\alpha}(.,y)\right\|_p^p+\left\|f_{\lambda\alpha}\right\|_p^p\end{bmatrix}^{\frac{1}{p}}.$$

**Proof** As in the proof of Theorem 3, we can write

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dtt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsd\right|$$
$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s|$$
$$\int_0^1\int_0^1\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|d\alpha d\lambda dsdt$$

for any $(x,y)\in\Delta$. From Hölder's inequality, we get

$$\int_0^1\int_0^1\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|d\alpha d\lambda$$
$$\leq\left(\int_0^1\int_0^1 1^q\,d\alpha d\lambda\right)^{\frac{1}{q}}\left(\int_0^1\int_0^1\left|\begin{matrix}f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y\\+(1-\alpha)s)\end{matrix}\right|^p d\alpha d\lambda\right)^{\frac{1}{p}}$$

for any $(x,y)\in\Delta$.

Since $|f_{\lambda\alpha}|^p$ is a co-ordinates convex function on $\Delta$, we get

$$\left(\int_0^1\int_0^1\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|^p d\alpha d\lambda\right)^{\frac{1}{p}}$$
$$\leq\left(\frac{1}{4}\begin{bmatrix}\left|f_{\lambda\alpha}(x,y)\right|^p+\left|f_{\lambda\alpha}(x,s)\right|^p\\+\left|f_{\lambda\alpha}(t,y)\right|^p+\left|f_{\lambda\alpha}(t,s)\right|^p\end{bmatrix}\right)^{\frac{1}{p}}$$

for any $(x,y)\in\Delta$. Therefore

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s| \tag{2.3}$$
$$\times\left(\begin{bmatrix}\left|f_{\lambda\alpha}(x,y)\right|^p+\left|f_{\lambda\alpha}(x,s)\right|^p\\+\left|f_{\lambda\alpha}(t,y)\right|^p+\left|f_{\lambda\alpha}(t,s)\right|^p\end{bmatrix}\right)^{\frac{1}{p}}dsdt.$$

**(i).** Now, if $f_{\lambda\alpha}\in L_\infty(\Delta)$ then

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}$$
$$\times\sup_{(t,s)\in\Delta}\begin{bmatrix}\left|f_{\lambda\alpha}(x,y)\right|^p+\left|f_{\lambda\alpha}(x,s)\right|^p\\+\left|f_{\lambda\alpha}(t,y)\right|^p+\left|f_{\lambda\alpha}(t,s)\right|^p\end{bmatrix}^{\frac{1}{p}}$$
$$\times\int_a^b\int_c^d|x-t||y-s|dsdt$$
$$=\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}$$
$$\times\begin{bmatrix}\left|f_{\lambda\alpha}(x,y)\right|^p+\left\|f_{\lambda\alpha}(x,.)\right\|_\infty^p\\+\left\|f_{\lambda\alpha}(.,y)\right\|_\infty^p+\left\|f_{\lambda\alpha}\right\|_\infty^p\end{bmatrix}^{\frac{1}{p}}$$
$$\times\left(\frac{(x-a)^2+(b-x)^2}{2}\right)\left(\frac{(y-c)^2+(d-y)^2}{2}\right)$$

for any $(x,y)\in\Delta$.

**(ii).** If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, by using Hölder's inequality in (2.3), we have

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\left(\int_a^b\int_c^d|x-t|^q|y-s|^q dsdt\right)^{\frac{1}{q}}$$

$$\times\left(\int_a^b\int_c^d\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|^p+|f_{\lambda\alpha}(x,s)|^p\\+|f_{\lambda\alpha}(t,y)|^p+|f_{\lambda\alpha}(t,s)|^p\end{array}\right]dsdt\right)^{\frac{1}{p}}$$

$$\leq\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\left(\frac{(x-a)^2+(b-x)^2}{2}\right)\left(\frac{(y-c)^2+(d-y)^2}{2}\right)$$

$$\times\left[\begin{array}{l}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_p^p\\+\|f_{\lambda\alpha}(.,y)\|_p^p+\|f_{\lambda\alpha}\|_p^p\end{array}\right]^{\frac{1}{p}}$$

$$=\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}(q+1)^{\frac{2}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}$$

$$\times\left[\left(\frac{d-y}{d-c}\right)^{q+1}+\left(\frac{y-c}{d-c}\right)^{q+1}\right]^{\frac{1}{q}}$$

$$\times\left[\begin{array}{l}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_p^p\\+\|f_{\lambda\alpha}(.,y)\|_p^p+\|f_{\lambda\alpha}\|_p^p\end{array}\right]^{\frac{1}{p}}$$

for any $(x,y)\in\Delta$.

**(iii)** If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then, by Hölder's inequality, we have

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s|\times\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|^p+|f_{\lambda\alpha}(x,s)|^p\\+|f_{\lambda\alpha}(t,y)|^p+|f_{\lambda\alpha}(t,s)|^p\end{array}\right]^{\frac{1}{p}}dsdt$$

$$=\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\sup_{t\in[a,b]}|x-t|\sup_{s\in[c,d]}|y-s|\left(\int_a^b\int_c^d 1^q dsdt\right)^{\frac{1}{q}}$$

$$\times\left(\int_a^b\int_c^d\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|^p+|f_{\lambda\alpha}(x,s)|^p\\+|f_{\lambda\alpha}(t,y)|^p+|f_{\lambda\alpha}(t,s)|^p\end{array}\right]dsdt\right)^{\frac{1}{p}}$$

$$\leq\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}(b-a)(d-c)}\max\{x-a,b-x\}\max\{y-c,d-y\}$$

$$\times\left[\begin{array}{l}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_p^p\\+\|f_{\lambda\alpha}(.,y)\|_p^p+\|f_{\lambda\alpha}\|_p^p\end{array}\right]^{\frac{1}{p}}$$

$$=\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]$$

$$\times\left[\begin{array}{l}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_p^p\\+\|f_{\lambda\alpha}(.,y)\|_p^p+\|f_{\lambda\alpha}\|_p^p\end{array}\right]^{\frac{1}{p}}.$$

**Corollary 2.** With the assumptions of Theorem 4 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have the inequality

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq\frac{(b-a)(d-c)}{4^{2+\frac{1}{p}}}\left[\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^p+\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_\infty^p+\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_\infty^p+\|f_{\lambda\alpha}\|_\infty^p\right]^{\frac{1}{p}},$$

where $f_{\lambda\alpha}\in L_\infty(\Delta)$.

If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ then we have

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{1+\frac{1}{p}}(q+1)^{\frac{2}{q}}}\left[(b-a)(d-c)\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^p+\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_p^p+\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_p^p+\|f_{\lambda\alpha}\|_p^p\right]^{\frac{1}{p}}.$$

If $f_{\lambda\alpha}\in L_p(\Delta)$, $p>1,\frac{1}{p}+\frac{1}{q}=1,$ then we have,

$$\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds \right.$$

$$\left. -\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right|$$

$$\leq \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{1+\frac{1}{p}}}\left[(b-a)(d-c)\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^p\right.$$

$$\left.+\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_p^p+\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_p^p+\left\|f_{\lambda\alpha}\right\|_p^p\right]^{\frac{1}{p}}.$$

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