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Some Results on Gaussian Leonardo P Numbers

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Abstract In this paper, we define the Gaussian Leonardo p numbers and we examine sum formula of the Gaussian Leonardo p numbers and then we define Q matrix of the Gaussian Leonardo p numbers. Also we give Binet formula of these numbers. We give the Gaussian Leonardo p numbers relations with the Leonardo p numbers and the Fibonacci p numbers.

Keywords: *Leonardo numbers, Gaussian Leonardo numbers, Leonardo p numbers*

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1. Introduction

The Fibonacci sequence, a well-known example of an integer sequence, finds applications in mathematics computer science, and art. The Fibonacci and Lucas sequences are widely recognized as being among the most prominent integer sequences. In the realm complex numbers, Gaussian integers are characterized by having integer values for both their real and imaginary components. Gaussian integers, with their captivating properties, have drawn the attention of numerous researchers throughout history. Gauss's groundbreaking contributions mathematics included the introduction of Gaussian integers in [1]. Complex Fibonacci numbers, introduced by Horadam, have found applications in various fields including number theory, algebra, and physics in [2]. The concepts of Gaussian Fibonacci and Gaussian Lucas numbers were brought forth by J.H. Jordan's innovative research in [3] and he expanded the applicability of established relationships between Fibonacci sequences to Gaussian Fibonacci numbers. Pethe and Horadam introduced generalized form of Gaussian Fibonacci numbers, expanding the mathematical landscape in [4]. Berzsenyi's work marked a significant milestone in the extension of Fibonacci numbers to the complex plane [5]. Furthermore, Stakhov and Rozin derived explicit Binet formulas for Fibonacci and Lucas p-numbers, extending the classical Binet formulas in [6]. Tasci studied in [7,8,9,10] complex Fibonacci p numbers, Gaussian Padovan and Gaussian Pell-Padovan sequences, Gaussian Mersenne numbers, Gauss balancing numbers, and Gauss Lucas-balancing numbers.

Fibonacci and Lucas numbers are defined recursively for n≥1:

$F_{n+1} = F_n + F_{n-1}$, with the initial conditions $F_0 = 0, F_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$, with the initial conditions $L_0 = 2, L_1 = 1$, respectively see [11]. Gaussian Fibonacci numbers GF_n are also [3] defined recursively: $GF_n = GF_{n-1} + GF_{n-2}$, $n \geq 2$ with the initial values

$GF_0 = i, GF_1 = 1$ .It is clear that these numbers are closely related to Fibonacci numbers : $GF_n = F_n + iF_{n-1}$, where $i = \sqrt{-1}$. Similarly Gaussian Lucas numbers GL_n are defined [3] recursively for $n \geq 2$: $GL_n = GL_{n-1} + GL_{n-2}$, with the initial conditions $GL_0 = 2 + i, GL_1 = 1 + 2i$. Leonardo numbers are studied by Catarino and Burgers in [12,13,14]. They defined these numbers by the second order inhomogeneous recurrence relation : $Le_n = Le_{n-1} + Le_{n-2} + 1$, $n \geq 2$, with the initial conditions $Le_0 = Le_1 = 1$. Also, these numbers can be defined as : $Le_{n+1} = 2Le_n - Le_{n-2}$,$n \geq 2$. Gaussian Leonardo numbers are studied by D.Tasci in [15]. Gaussian Leonardo numbers GLe_n are defined as $GLe_n = GLe_{n-1} + GLe_{n-2} + (1+i)$, $n \geq 2$, with the initial conditions $GLe_0 = 1 - i, GLe_1 = 1 + i$. Some Gaussian Leonardo numbers : $1-i, 1+i, 3+i, 5+3i, 9+5i, \ldots$ It is clear that Gaussian Leonardo numbers are closely related to Leonardo numbers: $GLe_n = Le_n + iLe_{n-1}$, where Le_n denotes the nth Leonardo number. Also for $n \geq 1$ it holds that $GLe_{n+2} = 2GLe_{n+1} - GLe_{n-1}$ .There are many important generalizations of Fibonacci numbers. The generalized Fibonacci p- numbers [16] examples of them and are defined by:

$F_p(n)=F_p(n-1)+F_p(n-p-1)$ with the initial conditions $F_p(0)=0$ and $F_p(1)=F_p(2)=...=F_p(p)=1$. The generalized Lucas p-numbers are defined by : $L_p(n)=L_p(n-1)+L_p(n-p-1)$ with the initial conditions $L_p(0)=p+1$ and $L_p(n)=1, n=1,2,...p$ where p=0,1,2 and $n=0,\pm1,\pm2,\pm3,...$. Let p be an integer the Gaussian Fibonacci p-numbers $\{GF_p(n)\}_{n=0}^{\infty}$ are defined [17] by the following recurrence relation

$$GF_p(n)=GF_p(n-1)+GF_p(n-p-1), n\rangle p$$

with the initial conditions $GF_p(0)=i$ and $GF_p(1)=GF_p(2)=...=GF_p(p)=1$. It can be easily seen that $GF_p(n)=F_p(n)+iF_p(n-p)$ where $F_p(n)$ is the nth Fibonacci p-number. For any given integer p>0, the Leonardo p-sequence $\{Le_p(n)\}_{n=0}^{\infty}$ is defined [18] by the following non-homogenous relation :

$$Le_p(n)=Le_p(n-1)+Le_p(n-p-1)+p, n>p$$

with the initial values $Le_p(0)=Le_p(1)=...=Le_p(p)=1$. The non-homogenous recurrence relation of Leonardo p numbers can be converted to the following homogenous recurrence relation :

$$Le_p(n)=Le_p(n-1)+Le_p(n-p)-Le_p(n-2p-1), n>2p$$

Now we will define the Gaussian Leonardo p-numbers, we will give some results of these numbers.

2. Main Results

We start by giving the definition of the Gaussian Leonardo p-sequence.

Definition 2.1

For any given integer p >0, the Gaussian Leoanardo p –sequence $\{GLe_p(n)\}_{n=0}^{\infty}$ is defined by the following non-homogenous relation :

$$GLe_p(n)=GLe_p(n-1)+GLe_p(n-p-1)+p+pi, n>p$$

with the initial values $GLe_p(0)=GLe_p(1)=...=GLe_p(p-1)=1-pi$ and $GLe_p(p)=1+i$.

It is clear that Gaussian Leonardo p-numbers are closely related to Leoanrdo p-numbers:

$$GLe_p(n)=Le_p(n)+iLe_p(n-p), n>p$$

The non-homogenous recurrence relation of Gaussian Leonardo p-numbers can be converted to the following homogenous recurrence relation :

$$GLe_p(n)=GLe_p(n-1)+GLe_p(n-p)$$
$$-GLe_p(n-2p-1), n>2p$$

For p=1, we get Lemma 2.1 in [17]

$$GLe_1(n)=GLe_1(n-1)+GLe_1(n-1)-GLe_1(n-2-1)$$
$$GLe_1(n)=2GLe_1(n-1)-GLe_1(n-3)$$
$$GLe_1(n+2)=2GLe_1(n+1)-GLe_1(n-1)$$

For p=1, p=2 and p=3 we give some of the numbers as follows

$GLe_1(0)=1-i$ $GLe_2(0)=1-2i$ $GLe_3(0)=1-3i$
$GLe_1(1)=1+i$ $GLe_2(1)=1-2i$ $GLe_3(1)=1-3i$
$GLe_1(2)=3+i$ $GLe_2(2)=1+i$ $GLe_3(2)=1-3i$
$GLe_1(3)=5+3i$ $GLe_2(3)=4+i$ $GLe_3(3)=1+i$
$GLe_1(4)=9+5i$ $GLe_2(4)=7+i$ $GLe_3(4)=5+i$
$GLe_1(5)=15+9i$ $GLe_2(5)=10+4i$ $GLe_3(5)=9+i$
$GLe_1(6)=25+15i$ $GLe_2(6)=16+7i$ $GLe_3(6)=13+i$
$GLe_1(7)=41+25i$ $GLe_2(7)=25+10i$ $GLe_3(7)=17+5i$
$GLe_1(8)=67+41i$ $GLe_2(8)=37+16i$ $GLe_3(8)=25+9i$

Theorem 2.1

For $n\geq 0$,we have

$$GLe_p(n)=\left[(p+1)F_p(n+1)-p\right]+i\left[(p+1)F_p(n-p+1)-p\right]$$

where $F_p(n)$ denotes the n-th Fibonacci p-numbers.

Proof 2.1

For $n\geq 0$, we have $Le_p(n)=(p+1)F_p(n+1)-p$ and then $GLe_p(n)=Le_p(n)+iLe_p(n-p)$ we can get

$$GLe_p(n)=\left[(p+1)F_p(n+1)-p\right]+i\left[(p+1)F_p(n-p+1)-p\right]$$

Corollary 2.1

For p =1 , we get Theorem 2.1 in [17]

$$GLe_1(n)=\left[2F_1(n+1)-1\right]+i\left[2F_1(n-1+1)-1\right]$$
$$GLe_1(n)=\left[2F_1(n+1)-1\right]+i\left[2F_1(n)-1\right]$$

Theorem 2.2

For $n\geq 0$, we have

$$GLe_p(n)=\left[L_p(n+p+1)-F_p(n+p+1)-p\right]$$
$$+i\left[L_p(n+1)-F_p(n+1)-p\right]$$

where $GLe_p(n), L_p(n)$ and $F_p(n)$ denotes respectively the n-th Gaussian Leonardo p –numbers,Lucas p-numbers and Fibonacci p-numbers.

Proof 2.2

For $n\geq 0$,we have

$$Le_p(n)=L_p(n+p+1)-F_p(n+p+1)-p$$

and then

$$GLe_p(n)=Le_p(n)+iLe_p(n-p)$$

we can get

$$GLe_p(n)=\left[L_p(n+p+1)-F_p(n+p+1)-p\right]$$
$$+i\left[L_p(n+1)-F_p(n+1)-p\right]$$

Corollary 2.2

For p=1, we get theorem 2.2.b in [17]

$$GLe_1(n)=\left[L_1(n+1+1)-F_1(n+1+1)-1\right]$$
$$+i\left[L_1(n+1)-F_1(n+1)-1\right]$$
$$GLe_1(n)=\left[L_1(n+2)-F_1(n+2)-1\right]$$
$$+i\left[L_1(n+1)-F_1(n+1)-1\right]$$
$$GLe_1(n)=\left[L_1(n+2)+iL_1(n+1)\right]$$
$$-\left[F_1(n+2)+iF_1(n+1)\right]-(1+i)$$
$$GLe_1(n)=GL_1(n+2)-GF_1(n+2)-(1+i)$$

Theorem 2.3

The Binet formula of the Gaussian Leonardo p–sequence is

$$GLe_p(n)=(p+1)\left[\sum_{k=1}^{p+1}\frac{\alpha_k^{n+1}}{(p+1)\alpha_k-p}\right]$$
$$-p+i\left((p+1)\left[\sum_{k=1}^{p+1}\frac{\alpha_k^{n-p+1}}{(p+1)\alpha_k-p}\right]-p\right)$$

Corollary 2.3

For p=1, we get theorem 2.4 in [15]

$$GLe_1(n)=(1+1)\left[\sum_{k=1}^{1+1}\frac{\alpha_k^{n+1}}{(1+1)\alpha_k-1}\right]-p+i\left((1+1)\left[\sum_{k=1}^{1+1}\frac{\alpha_k^{n-1+1}}{(1+1)\alpha_k-1}\right]-1\right)$$

$$GLe_1(n)=2\left[\sum_{k=1}^{2}\frac{\alpha_k^{n+1}}{2\alpha_k-1}\right]-1+i\left(2\left[\sum_{k=1}^{2}\frac{\alpha_k^{n-1+1}}{2\alpha_k-1}\right]-1\right)$$

$$GLe_1(n)=2\left[\sum_{k=1}^{2}\frac{\alpha_k^{n+1}}{2\alpha_k-1}\right]-1+i\left(2\left[\sum_{k=1}^{2}\frac{\alpha_k^{n}}{2\alpha_k-1}\right]-1\right)$$

$$GLe_1(n)=2\left(\left[\frac{\alpha_1^{n+1}-\alpha_2^{n+1}}{\alpha_1-\alpha_2}\right]+i\left[\frac{\alpha_1^{n}-\alpha_2^{n}}{\alpha_1-\alpha_2}\right]\right)-(1+i)$$

Theorem 2.4

For $n\geq 0$, we have

$$\sum_{k=0}^{n}GLe_p(k)=\left[Le_p(n+p+1)-(n+1)p-1\right]$$

Proof 2.4

$$+i\left[Le_p(n+1)-(n+1)p-1\right]$$

We have $GLe_p(n)=Le_p(n)+iLe_p(n-p)$,

$\sum_{j=1}^{n+1}F_p(j)=F_p(n+p+2)-F_p(p)-F_p(0)$ and

$$Le_p(n)=(p+1)F_p(n+1)-p$$

$$\sum_{j=0}^{n}GLe_p(j)=\sum_{j=0}^{n}Le_p(j)+i\sum_{j=0}^{n}Le_p(j-p)$$

$$\sum_{j=0}^{n}GLe_p(j)=\sum_{j=0}^{n}\left[(p+1)F_p(j+1)-p\right]$$
$$+i\sum_{j=0}^{n}\left[(p+1)F_p(j-p+1)-p\right]$$
$$\sum_{j=0}^{n}GLe_p(j)=(p+1)\sum_{j=0}^{n}F_p(j+1)$$
$$-\sum_{j=0}^{n}p+i\left[(p+1)\sum_{j=0}^{n}F_p(j-p+1)-\sum_{j=0}^{n}p\right]$$
$$\sum_{j=0}^{n}GLe_p(j)=(p+1)\sum_{j=1}^{n+1}F_p(j)-(n+1)p$$
$$+i\left[(p+1)\sum_{j=1}^{n+1}F_p(j-p)-(n+1)p\right]$$
$$\sum_{j=0}^{n}GLe_p(j)=(p+1)\left[F_p(n+p+2)-1\right]-(n+1)p$$
$$+i\left[(p+1)\left[F_p(n+2)-1\right]-(n+1)p\right]$$
$$\sum_{j=0}^{n}GLe_p(j)=\left[Le_p(n+p+1)-(n+1)p-1\right]$$
$$+i\left[Le_p(n+1)-(n+1)p-1\right]$$

Corallary 2.4

For p=1, we get theorem 2.6 in [15]

$$\sum_{k=0}^{n}GLe_p(k)=\left[Le_p(n+p+1)-(n+1)p-1\right]$$
$$+i\left[Le_p(n+1)-(n+1)p-1\right]$$
$$\sum_{k=0}^{n}GLe_p(k)=\left[Le_1(n+2)-(n+1)-1\right]$$
$$+i\left[Le_1(n+1)-(n+1)-1\right]$$
$$\sum_{k=0}^{n}GLe_p(k)=\left[Le_1(n+2)+iLe_1(n+1)\right]-(n+2)(1+i)$$

Now we introduce the matrices R_p and Q_p that play the role of the Q-matrix.

Theorem 2.5

Let $n\geq 1$.Then

$$R_pQ_p^n=$$

$$\begin{bmatrix} GLe_{2n+p-2} & \cdots & \cdots & GLe_{n+p} & GLe_{n+p-1} \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ GLe_{n+1} & \cdots & \cdots & GLe_3 & GLe_2 \\ GLe_n & \cdots & \cdots & GLe_2 & GLe_1 \end{bmatrix}\times\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & & & & 0 \\ 1 & 0 & 0 & \ddots & & & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & 1 & 0 \\ -1 & 0 & \cdots & & \cdots & 0 & 1 \\ 0 & 0 & \cdots & & \cdots & 0 & 0 \end{bmatrix}^n$$

$$= \begin{bmatrix} GLe_{2n+p+1} & \cdots & \cdots & GLe_{n+p+1} & GLe_{n+p} \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ GLe_{n+2} & \cdots & \cdots & GLe_{3} & GLe_{2} \\ GLe_{n+1} & \cdots & \cdots & GLe_{2} & GLe_{1} \end{bmatrix}$$

Proof 2.5

Theorem can be proved by mathematical induction on n.

Conclusion

This paper explores the properties and applications of Gaussian Leonardo p- numbers. Gaussian Leonardo p-numbers can be extended to Gaussian Lenardo p-polynomials.

Conflicts of Interest

The authors declare no conflict of interest.

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References

[1] C. Gauss, Theoria residuorum biquadraticorum, Commentario Prima (1828) [Werke, Vol. II, pp. 65--92].

[2] A. F. Horadam, Complex Fibonacci numbers and Fibonacci quaternions, Amer. Math. Monthly 70 (1969) 289—291.

[3] J. H. Jordan, Gaussian Fibonacci and Lucas numbers, Fibonacci Quart. 3 (1965) 315--318.

[4] S. Pethe, A. F. Horadam, Generalized Gaussian Fibonacci numbers, Bull. Aust. Math. Soc. 33 (1986) 37--48.

[5] G. Berzsenyi, Gaussian Fibonacci numbers, Fibonacci Quart. 15 (1977) 233--236.

[6] Stakhov AP, Rozin B. "Theory of Binet formulas for Fibonacci and Lucas p-numbers". Chaos, Solitions & Fractals 2006; 27: 1162-77.

[7] D. Tasci, Gaussian Balancing numbers and Gaussian Lucas-Balancing numbers, J. Sci. Arts 44 (2018) 661--666.

[8] D. Tasci, Gaussian Padovan and Gaussian Pell-Padovan sequences, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. 67 (2018) 82--88.

[9] D. Tasci, On Gaussian Mersenne numbers, J. Sci. Arts 57 (2021) 1021--1028.

[10] D. Tasci, F. Yalcin, Complex Fibonacci p-numbers, Commun. Math. Appl. 4 (2013) 213--218.

[11] Koshy T. "Fibonacci and Lucas Numbers with Applications," A Wiley-Interscience Publication, (2001)

[12] P. Catarino, A. Borges, A note on Gaussian modified Pell numbers, J. Inf. Optim. Sci. 39 (2018) 1363—1371.

[13] P. Catarino, A. Borges, A note on incomplete Leonardo numbers, Integers 20 (2020) #A43.

[14] P. Catarino, A. Borges, On Leonardo numbers, Acta Math. Univ. Comenian. 89 (2020) 75--86.

[15] D. Tasci On Gaussian Leonardo numbers. Contrib. Math. 2023, 7, 34--40.

[16] Stakhov AP. "Introduction into algorithmic measurement theory". Soviet Radio, Moscow (1977) [in Russian].

[17] M. Asci, E. Gurel "Gaussian Fibonacci and Gaussian Lucas p-Numbers" Ars Combin. 132, (2017), 389-402.

[18] Tan, E.; Leung, H.H. On Leonardo p-numbers. Integers 2023, 23, 1--11.