Stability Results of the Additive and Quartic Functional Equations in Random p-Normed Spaces

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Abstract In this paper, we investigate the Hyers-Ulam stability of mixed type additive and quartic functional equations in Random p-normed spaces by direct and fixed-point method.

Keywords: Hyers-Ulam stability, additive functional equation, quartic functional equation, random p-normed spaces, fixed point method, direct method


1. Introduction

In 1940, the stability problems of functional equations about homomorphism of groups was introduced by Ulam [1]. In 1941, Hyers [2] gave an affirmative answer to Ulam’s question for additive groups (under the assumption that groups are Banach spaces). Hyers’ theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference \( ||f(x+y) - f(x) - f(y)|| \leq \varepsilon (||x||^p + ||y||^p) \) for all \( \varepsilon > 0 \) and \( p \in [0,1) \). Following the same approach as Rassias, Gajda [5] gave an affirmative solution of this problem for \( p > 1 \) and also proved that it is possible to solve the Rassias-type problem for \( p = 1 \). Also, in 1994, Rassias generalization theorem was delivered by Gavruta [6] who replaced \( \varepsilon (||x||^p + ||y||^p) \) by a control function \( \phi(x,y) \). The paper of Rassias has significantly influenced the development of what we now call the Hyers-Ulam-Rassias stability of functional equations. J.M. Rassias [4] followed the modern approach of the Th.M. Rassias [7] theorem in which he replaced the factor product of norms instead of sum of norms.

The functional equations
\[
f(x+y) = f(x) + f(y)
\]
are known as additive functional equation. Each additive solution of a functional equation must be an additive mapping. For Stability of additive, quadratic, cubic and quartic functional equations in random normed spaces, we may refer [8,9,10,11,12].

Some notions and conventions of the theory of random and random p-normed spaces are taken in our paper as in [11,12,13,14,15].

Throughout the paper \( \Delta^+ \) is the distribution functions space, that is, the space of all mappings \( V: \mathbb{R} \to \mathbb{R} \) such that \( F = \{ -\infty, \infty \} \to [0,1] \), such that \( V(0) = 0 \) and \( V(\pm \infty) = 1 \). \( \text{D}^+ \subset \Delta^+ \) consisting of all functions \( V \in \Delta^+ \) for which \( l^{-}V(\pm \infty) = 1 \), where \( l^{-}\phi(s) \) denotes \( \phi(s) = \lim_{t \to s^-} \phi(t) \). The space \( \Delta^+ \) is partially ordered by the usual point wise ordering of functions, i.e., \( V \leq W \iff V(t) \leq W(t) \forall t \in \mathbb{R} \). The maximal element for \( \Delta^+ \) in this order is the distribution function \( \delta_0 \) given by
\[
\delta_0(t) = \begin{cases} 
0 & \text{if } t \leq 0, \\
1 & \text{if } t > 0.
\end{cases}
\]

In 2012, SA Mohiuddine et al. [16] are to present a relationship between three various disciplines the theory of functional equation
\[
2f\left(\frac{x+y+z}{2}\right) = f(x) + f(y) + f(z)
\]
(1.2)

Recently, in 2019, Senthil Kumar et al. [17] proved the general solution of Hyers-Ulam stability of mixed type additive and quartic functional equations of the form
\[
f(2x+y)+f(2x-y)+f(x+2y)+f(x-2y)
\]
\[
=8\left[f(x+y)+f(x-y)\right]+f(2x)-5f(x)
\]
\[
+7f(-x)+2f(2y)-5f(-y)-9f(y),
\]
(1.3)

We focus on the ensuing mixed type functional equation derived from additive and quartic mappings. We validate the Hyers-Ulam stability of equation (1.3) in p-normed spaces. It is not hard for one for one that the mixed type function \( f(x) = ax + bx^4 \) is a solution of the equation (1.3).

In this section, we determine the generalized Hyers-Ulam stability of mixed type additive quartic functional
equations in random p-normed space, by using direct method and fixed-point method. Following definitions and notions will be used to prove our main results:

**Definition 1.1** [11] (t-norm) \( T : [0,1] \times [0,1] \rightarrow [0,1] \) is a continuous triangular norm (briefly t-norm) if \( T \) satisfies the following conditions:

i) \( T \) is commutative and associative;
ii) \( T \) is a continuous;
iii) \( T(a,1) = a \) for all \( a \in [0,1] \);
iv) \( T(a,b) \leq T(c,d) \), whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0,1] \).

Examples of continuous t-norm are \( T(x,y) = xy \), \( T(x,y) = \max\{x + y - 1,0\} \) and \( T(x,y) = \min(x,y) \).

Recall that, if \( T \) is a t-norm and \( \{x_n\} \) are given numbers in \([0,1]\), then, \( T^n_{\infty}x_i \) is defined by recursively by

\[
T^n_{\infty} = \begin{cases} x_1 & \text{if } n = 1, \\ (T^{n-1}_{\infty}x_i, x_n) & \text{if } n \geq 2, \end{cases}
\]

\( T^n_{\infty}x_i \) is defined as \( T^n_{\infty}x_{i+1} \).

**Definition 1.2.** [11] A Random Normed space (briefly RN-space) is a triple \((X, \Theta, T)\), where \( X \) is a vector space, \( T \) is a continuous t-norm and \( \Theta : X \rightarrow D^+ \) (for all \( x \in X, \Theta(x) \) is denoted by \( \Theta_x \), satisfying the following conditions:

i) \( \Theta_x(t) = e_0(t) \), for all \( t > 0 \) if and only if \( x = 0 \);
ii) \( \Theta_{ax}(t) = \Theta_x \left( \frac{t}{|a|} \right) \), for all \( x \in X, t \geq 0 \) and \( a \in \mathbb{R} \) with \( a \neq 0 \);
iii) \( \Theta_{x+y}(t + u) \geq T(\Theta_x(t), \Theta_y(u)) \) for all \( x, y \in X \) and \( t, u \geq 0 \).

**Definition 1.3.** [11] Let \((X, \Theta, T)\) be a RN-space.

RN1) A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if \( \lim_{n \to \infty} \Theta_{x_n-x}(t) = 1, t > 0 \).

RN2) A sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if \( \lim_{n \to \infty} \Theta_{x_n-x_{n+1}}(t) = 1, t > 0 \).

RN3) A RN-space \((X, \Theta, T)\) is said to be complete if every Cauchy sequence in \( X \) is convergent.

**Definition 1.4.** [13] Let \( X \) be a real linear space \( p \in R^+ \) with \( 0 < p \leq 1 \) and \( T \) be a continuous t-norm. The triple \((X, \Theta, T)\) is called a random p-normed space if a mapping \( \Theta : X \rightarrow D^+ \) (for all \( x \in X, \Theta(x) \) is denoted by \( \Theta_x \), satisfying the following conditions:

i) \( \Theta_x(t) = e_0(t) \), for all \( t > 0 \) if only if \( x = 0 \);

ii) \( \Theta_{ax}(t) = \Theta_x \left( \frac{t}{|a|} \right) \), for all \( x \in X, t \geq 0 \) and \( a \neq 0 \);

iii) \( \Theta_{x+y}(t + u) \geq T(\Theta_x(t), \Theta_y(u)) \) for all \( x, y \in X \) and \( t, u \geq 0 \).

Note that every p-normed space \((X, \| \cdot \|)\) defines a random p-normed space \((X, \Theta, T_M)\) where \( \Theta_x(t) = \frac{t}{t + p\|x\|} \), for all \( t > 0 \) and \( T_M \) is the minimum t-norm. This space is called the induced random p-normed space.

**Definition 1.5.** [13] Let \((X, \Theta, T)\) is called a random p-normed space.

1. A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if for all \( t > 0 \) and \( \lambda > 0 \), there exists a positive integer \( N \) such that \( \Theta_{x_n-x}(t) > 1 - \lambda \) whenever \( n \geq N \).

2. A sequence \( \{x_n\} \) in \( X \) is called a Cauchy convergent if for all \( t > 0 \) and \( \lambda > 0 \), there exists a positive integer \( N \) such that \( \Theta_{x_n-x_m}(t) > 1 - \lambda \) whenever \( n \geq m \geq N \).

3. The random p-normed space \((X, \Theta, T)\) is said to be complete if every Cauchy sequence is convergent to a point in \( X \).

**Theorem 1.6.** [14]. If \((X, \Theta, T)\) is a random normed space and \( \{x_n\} \) is a sequence of \( X \) such that \( x_n \rightarrow x \), then \( \lim_{n \to \infty} \Theta_{x_n-x}(t) = \Theta_x(t) \).

## 2. Stability Results for Additive Functional Equation by Using Direct Method

In this section, we investigate the generalized Hyers-Ulam stability problem of the functional equation (1.2), in random p-normed spaces under the minimum t-norm \( T_M \) by using direct method.

In this paper, let \( X \) be a linear space, \((X, \Theta, T)\) be a random p-normed space and \((X, \Theta, T_M)\) be a complete random p-normed space. We determine the stability of the additive functional equation defined by

\[
D_f(x,y,z) = 2f \left( \frac{x+y+z}{2} \right) - f(x) - f(y) - f(z),
\]

in random p-normed spaces by using Direct method.

**Theorem 2.1.** Let \( \phi : X^3 \rightarrow Z \) be an amapping such that for some \( 0 < \alpha < 2 \).

\[
\Theta_{\phi(2x,2y,2z)}(t) \geq \Theta_{\phi(x,y,z)}(t)
\]

and

\[
limit_{n \to \infty} \Theta_{\phi(2^{n+1}x,2^{n+1}y,2^{n+1}z)}(2^{n+1}t) = 1,
\]

for all \( x, y, z \in X \) and all \( t > 0 \). If \( f : X \rightarrow Y \) is an odd mapping with \( f(0) = 0 \) such that

\[
\Theta_{\phi(x,y,z)}(t) \geq \Theta_{\phi(x,y,z)}(t)
\]

for all \( x, y, z \in X \) and all \( t > 0 \), then there exists a unique additive mapping \( Q : X \rightarrow Y \) such that \( \Theta_{\phi(x,y,z)}(t) \geq \Theta_{\phi(x,y,z)}(2^p - \alpha^p)(t) \), for all \( x \in X \) and \( t > 0 \).

**Proof.** Replacing \( y \) by \( 2x \) and \( z \) by \( x \) in (2.3), we obtain

\[
\Theta_{\phi(x)}(t) \geq \Theta_{\phi(x,2x)}(2^p t),
\]

for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) by \( 2^n x \) in (2.5), we obtain

\[
\Theta_{\phi(x)}(2^n t) \geq \Theta_{\phi(x,2x)}(2^p t)
\]

and

\[
\Theta_{\phi(x,2x)}(2^n t) \geq \Theta_{\phi(x,2x)}(2^p t)
\]

for all \( x \in X \) and \( t > 0 \).

Since

\[
f(x) - f(2^n x) = \sum_{j=0}^{n-1} \left( \frac{2}{2^j} - \frac{2^{j+1}}{2^j} \right).
\]
for all $x \in X$ and all $t > 0$.

$$
\Theta f(x) = \frac{\sum_{j=0}^{n-1} \left( \frac{1}{2^p} \alpha \right)^j t}{2^n}
$$

for all $x \in X$ and all $t > 0$. Fix $x \in X$ and put $m = 0$ in (2.9). Then, we obtain

$$
\Theta f(x) = \frac{\sum_{j=0}^{n-1} \left( \frac{1}{2^p} \alpha \right)^j t}{2^n}
$$

for all $x \in X$ and all $t > 0$. For every $s > 0$,

$$
\Theta f(x) - Q(x) (s + t) \geq T_M \left( \Theta f(x) - \frac{\sum_{j=0}^{n-1} \left( \frac{1}{2^p} \alpha \right)^j t}{2^n} Q(x) \right) (s)
$$

for all $x \in X$ and all $t > 0$. Taking the limit $n \to \infty$ in (2.10), we obtain

$$
\Theta f(x) - Q(x) (s + t) \geq \Theta \phi^{(x,2,x)}(2^p - \alpha^p) t,
$$

Thus, since $s$ is arbitrary and taking $s \to 0$ in (2.11), we have

$$
\Theta f(x) - Q(x) (t) \geq \Theta \phi^{(x,2,x)}(2^p - \alpha^p) t,
$$

for all $x \in X$ and all $t > 0$. Thus, the condition (2.4) holds for all $x \in X$ and all $t > 0$. If we replace $(x,y,z)$ by $(2^x,2^y,2^z)$ in (2.3), then

$$
\Theta D_{\phi^{(x,2,x)}} (t) \geq \Theta \phi^{(x,2,x)} (2^p - \alpha^p) t,
$$

for all $x, y, z \in X$ and all $t > 0$. Letting $n \to \infty$ in (2.12), we find that $\Theta \phi^{(x,y,z)} = 1$, for all $t > 0$, which implies $D_{\phi^{(x,y,z)}} = 0$, for all $x, y, z \in X$. Therefore, the mapping $Q$ is additive. Now, we prove that the additive mapping $Q$ is unique. Let us assume that there exists another mapping $R : X \to Y$ which satisfies (2.4). For fixed $x \in X$, $Q(2^n x) = 2^n Q(x)$ and $R(2^n x) = 2^n R(x)$, for all $n \in \mathbb{Z}^+$. It follows from (2.4) that

$$
\Theta Q(x) - R(x) (t) = \Theta \phi^{(x,2,x)} (2^p - \alpha^p) t
$$

and so it converges to some point $Q(x) \in Y$.

We can define a mapping $Q : X \to Y$ by $Q(x) = \lim_{n \to \infty} f\left( \frac{2^n x}{2^n} \right)$, for all $x \in X$ and all $t > 0$. Fix $x \in X$ and put $m = 0$ in (2.9). Then, we obtain

$$
\Theta f(x) = \frac{\sum_{j=0}^{n-1} \left( \frac{1}{2^p} \alpha \right)^j t}{2^n}
$$

for all $x \in X$ and all $t > 0$. For every $s > 0$,
which satisfies (2.3), then there exists a unique additive mapping $Q : X \to Y$ such that
\[ \Theta_{f(x)}(t) - Q(x)(t) \geq \mathcal{O}^\prime_{\delta Z_0}(t), \]
for all $x \in X$ and all $t > 0$.

**Proof.** Putting $x = z = \frac{3}{2}$ and $y = x$ in (2.3), we obtain
\[ \Theta_{f(x)}(2^m f(x)) - \mathcal{O}^\prime_{\delta Z_0}(2^m f(x)), \]
for all $x, y \in X$ and all $t > 0$.

Replacing $x$ by $\gamma^k$ in (2.15), we obtain
\[ \Theta_{f(x)}(2^m f(x)) \geq \mathcal{O}^\prime_{\delta Z_0}\left(\frac{2^m f(x)}{2^m}\right), \]
for all $x, y \in X$ and all $t > 0$.

Since
\[ f(x) - 2^m f\left(\frac{x}{2^m}\right) = 2x - 2^{m+1}f\left(\frac{x}{2^{m+1}}\right), \]
for all $x \in X$ and all $t > 0$.

From inequality (2.16) and (2.17), we get
\[ \Theta_{f(x)}(2^m f(x)) \geq \mathcal{O}^\prime_{\delta Z_0}\left(\frac{x}{2^m}\right), \]
for all $x, y \in X$ and all $t > 0$.

Replacing $x$ by $\gamma^k$ in (2.18), we get
\[ \Theta_{f(x)}(2^m f(x)) \geq \mathcal{O}^\prime_{\delta Z_0}\left(\frac{x}{2^m}\right), \]
for all $x, y \in X$ and all $t > 0$.

**Proof.** Let a mapping $\phi : X^3 \to Z$ be defined by $\phi(x, y) = \delta Z_0$. Then, the proof follows from Theorem 2.1 by $\alpha = 1$. This completes the proof.

**Corollary 2.4.** Let $X$ be a linear space, $(Z, \Theta, T_M)$ be a random $p$-normed space and $(Y, \Theta, T_M)$ be a complete random $p$-normed space. Assume $r$ is a positive real number with $r \neq 3$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies
\[ \phi_{df}(<\alpha, x, y, z_0)(t) \geq \phi_{df}(2^{mp}(2^p - 2^m))(t), \]
for all $x, y, z \in X$ and all $t > 0$.

**Proof.** Let a mapping $\phi : X^3 \to Z$ be defined by $\phi(x, y) = \mathcal{O}^\prime_{\delta Z_0}$. Then, the proof follows from Theorem 2.1 by $\alpha = 2'$. This completes the proof.

**3. Stability Results for Mixed Type Functional Equations**

In this section, we prove the generalized Hyers-Ulam stability of mixed type Additive Quartic functional equations in random p-normed space, by using direct method. For any mapping $f : X \to Y$ defined by
\[ D_{f(x, y)} = f(2x + y) + f(2x - y) + f(x + 2y) + f(x - 2y) \]
\[ -8\left[ f(x + y) + f(x - y) \right] + f(2x) + 5f(x) \]
\[ = 7f(-x) - 2f(2y) + 5f(-y) + 9f(y) \]

**Theorem 3.1.** Let $\phi : X^2 \to Z$ be a mapping such that for some $0 < \alpha < 2$.
\[ \Theta_{\phi(2x, y)}(t) \geq \Theta_{\phi(\alpha, x, y)}(t), \]
and $\lim_{n \to \infty} \Theta_{\phi(2^n x^n, y^n)}(2^{mp} f(2^x)) = 1$ for all $x, y \in X$ and all $t > 0$. If an even mapping $f : X \to Y$ with $f(0) = 0$ satisfying
\[ \Theta_{f(x)}(t) \geq \Theta_{\phi(\alpha, x, y)}(t), \]
for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q : X \to Y$ such that
\[ \Theta_{f(x)}(t) \geq \Theta_{\phi(\alpha, x, y)}(2^{mp}(2^p - 2^m)t), \]
for all $x, y \in X$ and all $t > 0$.

**Proof.** Replacing $y$ by $0$, in (3.3), we obtain
\[ \Theta_{f(x)}(t) \geq \Theta_{\phi(\alpha, x, y)}(2^{mp}t), \]
for all \( x \in X \) and all \( t > 0 \).
Replacing \( x \) by \( 2^n x \) in (3.5) we obtain
\[
\frac{\Theta f(2^n x)}{2^{4n}} \frac{f(2^{n+1} x)}{2^{4(n+1)}} (t) \geq \Theta' \phi(2^n x, 2^n x) (2^{4p} t),
\]
for all \( x \in X \) and all \( t > 0 \).

Since
\[
f(x) = \frac{f(2^n x)}{2^{4n}} = \sum_{j=0}^{n-1} \left( \frac{1}{2^4} \right)^j \left( \frac{\alpha}{2} \right)^{(p-1)j} t,
\]
So,
\[
\frac{\Theta}{f(x)} \frac{f(2^n x)}{2^{4n}} \left( \sum_{j=0}^{n-1} \left( \frac{1}{2^4} \right)^j \left( \frac{\alpha}{2} \right)^{(p-1)j} t \right) \geq \Theta' \phi(x, 0)^n \Theta' \phi(x, 0) \ldots \Theta' \phi(x, 0) (t).
\]

Thus, since \( s \) is arbitrary and taking limit \( s \to 0 \) in (3.10), we have
\[
\Theta f(x) Q(x) (s + t) \geq \Theta' \phi(x, 0) \left( 2^{3p} \left( 2^p - \alpha^p \right) t \right).
\]

Defining a mapping \( Q : X \to Y \) by \( Q(x) = \lim_{n \to \infty} \left\{ f(2^n x) \right\} \), for all \( x \in X \) and all \( t > 0 \). Fix \( x \in X \) and put \( m = 0 \) in (3.8). Then, we obtain
\[
\Theta f(x) - f(2^n x) (t) \geq \Theta' \phi(x, 0) \left( \frac{2^{4p}}{\sum_{j=0}^{n-1} (\alpha/2)^{p(j+1)}} t \right),
\]
for all \( x \in X \) and all \( t > 0 \). For every \( s > 0 \),
\[
\Theta f(x) - Q(x) (s + t)
\]
for all \( x \in X \) and all \( t > 0 \). Taking the limit \( n \to \infty \) in (3.9), we obtain
\[
\Theta f(x) - Q(x) (s + t) \geq \Theta' \phi(x, 0) \left( 2^{3p} \left( 2^p - \alpha^p \right) t \right) (3.10)
\]
Thus, since \( s \) is arbitrary and taking limit \( s \to 0 \) in (3.10), we have
\[
\Theta f(x) - Q(x) (t) \geq \Theta' \phi(x, 0) \left( 2^{3p} \left( 2^p - \alpha^p \right) t \right),
\]
for all \( x \in X \) and all \( t > 0 \). Thus, the condition (3.4), holds for all \( x \in X \) and all \( t > 0 \). If we replace \( x, y \) by \( (2^n x, 2^n y) \) in (3.3), then
\[
\Theta Df(2^n x, 2^n y) (t) \geq \Theta' \phi(x, y) \left( 2^{4p} t \right),
\]
for all \( x, y \in X \) and all \( t > 0 \). Letting \( n \to \infty \) in (3.11), we find that \( \Theta DQ(x, y) = 1 \) for all \( x > 0 \), which implies \( DQ(x, y) = 0 \), for all \( x, y \in X \). Therefore, the mapping \( Q \) is quartic. Now, let us prove that the quartic mapping \( Q \) is unique. Let us assume that there exists another mapping \( R : X \to Y \) which satisfies (3.3). For fixed \( x \in X \), \( Q(2^n x) = 2^{4n} Q(x) \) and \( R(2^n x) = 2^{4n} R(x) \). All \( n \in Z^+ \). It follows from (3.3) that
\[
\Theta f(x) - R(x) (t) \geq \Theta \frac{Q(2^n x)}{2^{4n}} \frac{R(2^n x)}{2^{4n}} (t)
\]
for all \( x, y \in X \) and all \( t > 0 \). Letting \( n \to \infty \) in (3.11), we find that \( \Theta Q(x, y) = 1 \) for all \( x > 0 \), which implies \( Q(x, y) = 0 \), for all \( x, y \in X \). Therefore, the proof is completed.
Theorem 3.2. Let $\phi : X^2 \rightarrow Z$ be a mapping such that for some $2 < \alpha$,
\[
\Theta \phi \left( \frac{x}{2^n} \right)(t) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
and $\lim_{n \rightarrow \infty} \Theta \phi \left( \frac{x}{2^n} \right)(t) = 1$ for all $x, y \in X$ and all $t > 0$. If an even mapping $f : X \rightarrow Y$ with $f(0) = 0$ which satisfies (3.3), then there exists a unique quartic mapping $Q : X \rightarrow Y$ such that
\[
\Theta f(x) - Q(x) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x \in X$ and all $t > 0$.
\[\textbf{Proof.}\] Replacing $x$ by $\frac{x}{2^n}$ and $y$ by $0$ in (3.3), we obtain
\[
\Theta f(x) - \frac{x}{2^n}f \left( \frac{x}{2^n} \right) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x \in X$ and all $t > 0$.
Replacing $x$ by $\frac{x}{2^n}$ in (3.14), we obtain
\[
\Theta f(x) - \frac{x}{2^n}f \left( \frac{x}{2^n} \right) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x \in X$ and all $t > 0$.

From inequality (3.15) and (3.16), we get
\[
\Theta \left( \sum_{j=0}^{n-1} \left( \frac{2^{j+1}}{2^n} t \right) \right) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x \in X$ and all $t > 0$.

Replacing $x$ by $\frac{x}{2^n}$ in (3.17), we get
\[
\Theta \left( \sum_{j=0}^{n-1} \left( \frac{2^{j+1}}{2^n} t \right) \right) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x \in X$ and all $t > 0$.

Then, the sequence $\left\{ 2^{n}f \left( \frac{x}{2^n} \right) \right\}$ is a Cauchy in $(X, \Theta, T_M)$, and so it converges to some point $Q(x) \in Y$. Now, we can define a quartic mapping $Q : X \rightarrow Y$ by $Q(x) = \lim_{n \rightarrow \infty} \left\{ 2^{n}f \left( \frac{x}{2^n} \right) \right\}$, for all $x \in X$ and all $t > 0$. The remaining part goes through in a similar method to the corresponding Theorem 3.1.

Corollary 3.3. Let $X$ be a linear space, $(Z, \Theta, T_M)$ be a random $p$-normed space and $(Y, \Theta, T_M)$ be complete a random $p$-normed space. Assume $\delta$ is a positive real number and $z_0 \in Z$. If an even mapping $f : X \rightarrow Y$ with $f(0) = 0$ which satisfies
\[
\Theta f(x) - Q(x) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q : X \rightarrow Y$ such that
\[
\Theta f(x) - Q(x) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x \in X$ and all $t > 0$.

Proof. Let a mapping $\phi : X^2 \rightarrow Z$ be defined by $\phi(x, y) = \delta z_0$. Then, the proof follows from Theorem 3.1 by $\alpha = 1$. This completes the proof.

Corollary 3.4. Let $X$ be a linear space, $(Z, \Theta, T_M)$ be a random $p$-normed space and $(Y, \Theta, T_M)$ be a complete random $p$-normed space. Assume $r$ is a positive real number with $r \neq 2$ and $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping $f(0) = 0$ which satisfies
\[
\Theta f(x, y) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q : X \rightarrow Y$ such that
\[
\Theta f(x, y) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x, y \in X$ and all $t > 0$.

Proof. Let a mapping $\phi : X^2 \rightarrow Z$ be defined by $\phi(x, y) = (\|x\|^r + \|y\|^r)z_0$. Then, the proof follows from Theorem 3.1 and Theorem 3.2 by $\alpha = r'$. This completes the proof.

Theorem 3.5. Let $\phi : X^2 \rightarrow Z$ be a mapping such that for some $0 < \alpha < 2$,
\[
\Theta \phi \left( \frac{x}{2^n} \right)(t) \geq \Theta \phi \left( \frac{x}{2^n} \right)(t),
\]
and $\lim_{n \rightarrow \infty} \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t) = 1$ for all $x, y \in X$ and all $t > 0$. If $f : X \rightarrow Y$ is an odd mapping with $f(0) = 0$ such that
\[
\Theta f(x) - Q(x) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x, y \in X$ and all $t > 0$, then there exists a unique additive mapping $Q : X \rightarrow Y$ such that
\[
\Theta f(x) - Q(x) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x, y \in X$ and all $t > 0$.

Proof. Replacing $y$ by $0$ in (3.18), we obtain
\[
\Theta f(x) - \frac{x}{2^n}f \left( \frac{x}{2^n} \right) \geq \Theta \phi \left( \frac{x}{2^n} \right)(2^{3p}t),
\]
for all $x \in X$ and all $t > 0$.

Replacing $x$ by $2^n x$ in (3.26), we obtain
\[ \Theta \left( \frac{f(2^n x)}{2^n} \right) \leq \Theta' \left( \frac{f(2^{n+1} x)}{2^{n+1}} \right) \] (3.27)

for all \( x \in X \) and all \( t > 0 \). Since

\[ f(x) - \frac{f(2^n x)}{2^n} = \sum_{j=0}^{n-1} \left( \frac{1}{2^j} \right) \left( \frac{\alpha}{2} \right)^j t \]

(3.28)

for all \( x \in X \) and all \( t > 0 \).

Replacing \( x \) by \( 2^m x \) in (3.29), we get

\[ \Theta' \left( \frac{f(2^m x)}{2^m} \right) \leq \Theta' \left( \frac{f(2^{m+n} x)}{2^{m+n}} \right) \] (3.30)

for all \( x \in X \) and all \( m, n \in Z \) with \( n > m \geq 0 \). It follows from

\[ \lim_{n,m \to \infty} \Theta' \left( \frac{f(2^n x)}{2^n} \right) = 1, \]

that the sequence \( \left\{ \frac{f(2^n x)}{2^n} \right\} \) is Cauchy in \( (X, \Theta, T_M) \), and so it converges to some point \( Q(x) \in Y \). We can define a mapping \( Q : X \to Y \) by

\[ Q(x) = \lim_{n \to \infty} \left( \frac{f(2^n x)}{2^n} \right), \]

for all \( x \in X \) and all \( t > 0 \). Fix \( x \in X \) and put \( m = 0 \) in (3.30), Then, we obtain

\[ \Theta' \left( \frac{f(2^n x)}{2^n} \right) \geq \Theta' \left( \frac{(2^p t)}{\sum_{j=0}^{n-1} \left( \frac{\alpha}{2} \right)^j} \right) \]

(3.31)

for all \( x \in X \) and all \( t > 0 \). For every \( s > 0 \),

\[ \Theta' \left( f(x) - Q(x) \right)(s + t) \geq \Theta' \left( (2^p - \alpha^p) t \right), \] (3.32)

Thus, since \( s \) is arbitrary and taking \( s \to 0 \) in (3.32), we have

\[ \Theta' \left( f(x) - Q(x) \right)(t) \geq \Theta' \left( (2^p - \alpha^p) t \right), \]

(3.33)

for all \( x, y \in X \) and all \( t > 0 \). Letting \( n \to \infty \) in (3.33), we find that \( \Theta'_{Q(x,y)} = 1 \), for all \( t > 0 \), which implies \( DQ(x,y) = 0 \), for all \( x, y \in X \). Therefore, the mapping \( Q \) is additive. Now, we prove that the additive mapping \( Q \) is unique. Let us assume that there exists another mapping \( R : X \to Y \) which satisfies (3.25). For fixed \( x \in X \), \( Q(2^n x) = 2^n Q(x) \) and \( R(2^n x) = 2^n R(x) \), all \( n \in Z^+ \). It follows from (3.25) that

\[ \Theta'_{Q(x)-R(x)}(t) = \Theta' \left( \frac{f(2^n x)}{2^n} \right) = \Theta' \left( \frac{f(2^n x)}{2^n} \right) \]

(3.34)

for all \( x, y \in X \) and all \( t > 0 \). Letting \( n \to \infty \) in (3.34), we find that \( \Theta'_{Q(x,y)} = 1 \), for all \( t > 0 \), which implies \( DQ(x,y) = 0 \), for all \( x, y \in X \). Therefore, the mapping \( Q \) is additive. Now, we prove that the additive mapping \( Q \) is unique. Let us assume that there exists another mapping \( R : X \to Y \) which satisfies (3.25). For fixed \( x \in X \), \( Q(2^n x) = 2^n Q(x) \) and \( R(2^n x) = 2^n R(x) \), all \( n \in Z^+ \). It follows from (3.25) that
as \( \lim_{n \to \infty} (2^n - \alpha^n) \left( \frac{3}{n} \right)^{\alpha^n} (t) = \infty \), we have \( \Theta_{Q(x)-R(x)}(t) = 1 \) for all \( t > 0 \). Thus, \( Q(x) = R(x) \), for all \( x \in X \). Hence, the proof is complete.

**Theorem 3.6.** Let \( \phi : X^2 \to Z \) be a mapping such that for some \( Z < \alpha \),

\[
\Theta_{\phi(x,y)}(t) \geq \Theta_{\phi(x,y)}(\alpha^n t) \quad (3.34)
\]

and \( \lim_{n \to \infty} \Theta_{\phi(x,y)}(t) = 1 \), for all \( x, y \in X \) and all \( t > 0 \). If \( f : X \to Y \) is an odd mapping with \( f(0) = 0 \), which satisfies (3.24), then there exists a unique additive mapping \( Q : X \to Y \) such that

\[
\Theta_{f(x)-Q(x)}(t) \geq \Theta_{\phi(x,y)}((2^n - \alpha^n)t), \quad (3.35)
\]

for all \( x, y \in X \) and all \( t > 0 \).

**Proof.** Replacing \( x \) by \( \frac{x}{2} \) and \( y \) by 0 in (3.24), we obtain

\[
\Theta_{f(x)-2f(x/2)}(t) \geq \Theta_{\phi(x,0)}(\alpha^n t), \quad (3.36)
\]

for all \( x, y \in X \) and all \( t > 0 \).

Replacing \( x \) by \( \frac{x}{2^n} \) in (3.36), we obtain

\[
\Theta_{f(x)-2^n f(x/2^n)} \geq \Theta_{\phi(x/2^n,0)}(\alpha^n t), \quad (3.37)
\]

for all \( x \in X \) and all \( t > 0 \).

Since

\[
f(x) - 2^n f\left( \frac{x}{2^n} \right) = \sum_{j=0}^{n-1} \left( \frac{x}{2^n} \right)^{\alpha^n} \left( 2^n \right)^{-\alpha^n} \left( \alpha^n \right) f\left( \frac{x}{2^{n+1}} \right), \quad (3.38)
\]

for all \( x \in X \) and all \( t > 0 \).

From inequality (3.37) and (3.38), we get

\[
\Theta_{f(x)-2^n f}\left( \frac{x}{2^n} \right) \geq \Theta_{\phi\left( \frac{x}{2^n}, 0 \right)}(t), \quad (3.39)
\]

for all \( x \in X \) and all \( t > 0 \).

Replacing \( x \) by \( \frac{x}{2^n} \) in (3.39), we get

\[
\Theta_{2^n f}\left( \frac{x}{2^n} \right) \geq \Theta_{\phi\left( \frac{x}{2^n}, 0 \right)}(t), \quad (3.40)
\]

for all \( x \in X \) and all \( m, n \in Z \) with \( n + m \geq 0 \). Then, the sequence \( \left\{ 2^n f\left( \frac{x}{2^n} \right) \right\} \) is a Cauchy in \((X, \Theta, \Gamma)\), and so it converges to some point \( Q(x) \in Y \). Now, we can define a mapping \( Q : X \to Y \) by \( Q(x) = \lim_{n \to \infty} \left\{ 2^n f\left( \frac{x}{2^n} \right) \right\} \), for all \( x \in X \) and all \( t > 0 \). The remaining part goes through in a similar method to the corresponding Theorem 3.5.

**Corollary 3.7.** Let \( X \) be a linear space, \((Z, \Theta, \Gamma_M)\) be a random \( p \)-normed space and \((Y, \Theta, \Gamma_M)\) be a completerandom \( p \)-normed space. Assume \( \delta \) is a positive real number and \( z_0 \in Z \). If \( f : X \to Y \) is a mapping with \( f(0) = 0 \) which satisfies

\[
\Theta_{f(x)-Q(x)}(t) \geq \Theta_{\phi\left( \frac{x}{2^n}, 0 \right)}((2^n - \alpha^n)t), \quad (3.41)
\]

for all \( x, y \in X \) and all \( t > 0 \), then there exists a unique mapping \( Q : X \to Y \) such that

\[
\Theta_{f(x)-Q(x)}(t) \geq \Theta_{\phi\left( \frac{x}{2^n}, 0 \right)}((2^n - \alpha^n)t), \quad (3.42)
\]

for all \( x, y \in X \) and all \( t > 0 \).

**Proof.** Let a mapping \( \phi : X^2 \to Z \) be defined by \( \phi(x, y) = \delta z_0 \). Then, the proof follows from Theorem 3.5 by \( \alpha = 1 \). This completes the proof.

**Corollary 3.8.** Let \( X \) be a linear space, \((Z, \Theta, \Gamma_M)\) be a random \( p \)-normed space and \((Y, \Theta, \Gamma_M)\) be a completerandom \( p \)-normed space. Assume \( \rho \) is a positive real number with \( \rho \neq 3 \) and \( z_0 \in Z \). If \( f : X \to Y \) is an odd mapping with \( f(0) = 0 \) which satisfies

\[
\phi_{\phi(x,y)}(t) \geq \phi_{\left( \|x\| \right)}((2^n - \alpha^n)t), \quad (3.43)
\]

for all \( x, y \in X \) and all \( t > 0 \), then there exists a unique additive mapping \( Q : X \to Y \) such that

\[
\Theta_{f(x)-Q(x)}(t) \geq \Theta_{\phi\left( \frac{x}{2^n}, 0 \right)}((2^n - \alpha^n)t), \quad (3.44)
\]

for all \( x \in X \) and all \( t > 0 \).

**Proof.** Let a mapping \( \phi : X^2 \to Z \) be defined by \( \phi(x, y) = (\|x\|^3 + \|y\|)^2 z_0 \). Then, the proof follows from Theorem 3.5 and Theorem 3.6 by \( \alpha = 2^n \). This completes the proof.

### 4. Stability Results by Fixed Point Method

In this section, we give the generalized Hyers-Ulam stability of mixed type additive quartic functional equations in random \( p \)-normed spaces. Let us recall that a mapping \( d : X^2 \to [0, \infty) \) is a called a metric on a non-empty set \( X \) if

i) \( d(x, y) = 0 \) if and only if \( x = y \),

ii) \( d(x, y) = d(y, x) \),

iii) \( d(x, y) \leq d(x, z) + d(y, z) \),

for all \( x, y, z \in X \). Before proceeding to the main results in this section, we give the fixed-point theorem which plays an important role in proving our theorems.

**Theorem 4.1.** [18]. (Alternative fixed-point theorem)

Let \((E, d)\) be a generalized complete metric space and \( \Gamma : E \to E \) be a strictly contractive function with Lipschitz constant \( L < 1 \). Then, for each \( x \in E \), either \( d(\Gamma^n x, \Gamma^n x) = \infty \) for all non-negative integer \( n \geq 0 \) or there exists a natural number \( n_0 \) such that...
i) $d(\Gamma^n x, \Gamma^n x) < \infty$, for all $n \geq n_0$;

ii) the sequence $[\Gamma^n x]_{n=1}^\infty$ converges to a fixed-point $y \in E$ of $\Gamma$;

iii) $y$ is the unique fixed point of $\Gamma$ in the set $\mathcal{F} = \{ q \in E : d(\Gamma^n a, q) < \infty \}$;

iv) $d(q, y) \leq \frac{1}{1-\alpha} d(q, \Gamma q), q \in \mathcal{F}$.

**Theorem 4.2.** Let $\phi : X^3 \rightarrow D^+$ ($\phi(x, y, z)$ is denoted by $\phi_{x,y,z}$) be a mapping such that, for some $0 < \alpha < 2$.

\[
\theta_{\phi(x, y, z)}(t) \geq \theta_{\phi(x, y, z)}(t),
\]

for all $x, y, z \in X$ and all $t > 0$. If $f : X \rightarrow Y$ is an odd mapping with $f(0) = 0$ such that

\[
\theta_{\phi(x, y, z)}(t) \geq \phi_{x, y, z}(t),
\]

for all $x, y, z \in X$ and all $t > 0$, then there exists a unique mapping $Q : X \rightarrow Y$ such that

\[
\theta_{\phi(x, y, z)}(t) \geq \phi_{x, y, z}(2 \alpha t),
\]

for all $x \in X$ and all $t > 0$.

**Proof.** Replacing $y$ by $x$ in (4.2), we obtain

\[
\theta_{\phi(x, y, z)}(t) \geq \phi_{x, y, z}(2t),
\]

for all $x \in X$ and all $t > 0$.

Consider a general metric $d$ on $\Lambda$, here $\Lambda$ be a set of all mappings from $X$ into $Y$ and introduce a generalized metric on $\Lambda$ as follows:

\[
d((g, h), (k, l)) = \inf\left\{ c \in (0, \infty) : \theta_{g(x) - h(x)}(ct) \geq \phi_{x, y, z}(t), x \in X, t > 0 \right\}
\]

whereas $\inf \phi = +\infty$. It is easy to show that $(\Lambda, d)$ is a complete metric space [10]. Now, let us consider a mapping $J : \Lambda \rightarrow \Lambda$ defined by

\[
J_g(x) = \frac{1}{2} g(2x),
\]

for all $g \in \Lambda$ and for all $x \in X$. Let $g, h \in \Lambda$ and $c \in (0, \infty)$ be an arbitrary constant with $d((g, h), c) < c$. Then, we have

\[
\theta_{g(x) - h(x)}(ct) \geq \phi_{x, y, z}(t),
\]

for all $x \in X$ and all $t > 0$, hence

\[
\theta_{J_g(x) - J_h(x)} \left( \frac{\alpha^p}{2} ct \right) \geq \theta_{g(x) - h(x)} \left( \frac{\alpha^p}{2} ct \right) \geq \phi_{x, y, z}(t),
\]

for all $x \in X$ and all $t > 0$, and so, if $d((g, h), c) < c$, then

\[
d(J_g, J_h) < \frac{\alpha^p}{2} d((g, h), c)
\]

for all $g, h \in \Lambda$. Then $J$ is a strictly contractive self-mapping on $\Lambda$ with Lipschitz constant $L = \frac{\alpha^p}{2p} < 1$.

Also, it follows from (4.2) that

\[
\Theta_{J_g(x) - J_h(x)}(t) \geq \Theta_{J_g(x) - J_h(x)} \left( \frac{t}{\alpha^p} \right) \geq \phi_{x, y, z}(t),
\]

for all $x \in X$ and all $t > 0$, which implies that

\[
d(J_g, J_h) < \frac{\alpha^p}{2p}.
\]

Using Theorem 4.1, there exists a mapping $Q : X \rightarrow Y$, which is a unique fixed point of $J$ in the set $\Lambda_1 = \{ g \in \Lambda : d((g, h), c) < c \}$ such that

\[
Q(x) = \lim_{n \rightarrow \infty} \frac{f(x_2^n, x_2^n, x_2^n)}{2^n},
\]

for all $x \in X$, since $\lim_{n \rightarrow \infty} d((f^n, f), c) = 0$. Again, it follows from Theorem 4.1 that

\[
d(f, J) \leq \frac{1}{1-L} d((f, J), Q) \leq \frac{1}{(1-L)} = 1 \left( 2 - \alpha^p \right)
\]

which implies

\[
\Theta_{J_g(x) - J_h(x)}(t) \geq \phi_{x, y, z}(2 \alpha t),
\]

for all $x \in X$ and all $t > 0$. Replacing $x$ and $y$ by $2^n x$ and $2^n y$ in (4.2), respectively,

\[
\Theta_{J_g(x) - J_h(x)}(t) \geq \lim_{n \rightarrow \infty} \Theta_{J_g(x) - J_h(x)} \left( \frac{2^n t}{2^n} \right)
\]

\[
\geq \lim_{n \rightarrow \infty} \phi_{x, y, z}(2^n t) = \phi_{x, y, z}(t),
\]

for all $x \in X$ and all $t > 0$. It follows from $\lim_{n \rightarrow \infty} \left( \frac{2^n t}{2^n} \right) = 1$ that $d(Q(x), y) = 0$. Hence, the mapping $Q$ is additive. Now, we show that mapping $Q$ is unique. To prove this, we assume that there exists an additive mapping $R : X \rightarrow Y$, which satisfies (4.3). Then, $R$ is a fixed point of $J$ in $\Lambda_1$. However, it follows from Theorem 4.1 that $J$ has only one fixed point in $\Lambda_1$. Hence, we deduce that $Q = R$.

**Theorem 4.3.** Let $\phi : X^3 \rightarrow D^+$ be a mapping such that, for some $2 < \alpha$.

\[
\phi_{x, y, z}(t) \geq \phi_{x, y, z}(2 \alpha t),
\]

for all $x \in X$ and all $t > 0$. If $f : X \rightarrow Y$ is a mapping with $f(0) = 0$ which satisfies (4.2), then there exists a unique mapping $Q : X \rightarrow Y$ such that

\[
\Theta_{J_g(x) - J_h(x)}(t) \geq \phi_{x, y, z}(2 \alpha t),
\]

for all $x, y \in X$ and all $t > 0$.

**Proof.** Let $\Lambda$ and $d$ be as in the proof of Theorem 4.2. Then $(\Lambda, d)$ becomes a complete metric space and the mapping $J : \Lambda \rightarrow \Lambda$ defined by

\[
J_g(x) = \frac{1}{2} g \left( \frac{x}{2} \right),
\]

for all $x \in X$ and $g \in \Lambda$. Then,
Consider a general metric $d$ on $\Lambda$, here $\Lambda$ be a set of all mappings from $X$ into $Y$ and introduce a generalized metric on $\Lambda$ as follows:

\[
d(g,h) = \inf \left\{ c \in (0,\infty) \mid \Theta_{g(x)-h(x)}(ct) \geq \phi_{x,0}(t), x \in X, t > 0 \right\},
\]

whereas $\inf \phi = +\infty$ It is easy to show that $(\Lambda,d)$ is a complete metric space [10]. Now, let us consider a mapping $J : \Lambda \to \Lambda$ defined by

\[
J_g(x) = \frac{1}{2^p} \Phi(2x),
\]

for all $g \in \Lambda$ and for all $x \in X$. Let $g, h$ in $\Lambda$ and $c \in (0,\infty)$ be an arbitrary constant with $d(g,h) < c$. Then, we have

\[
\Theta_{g(x)-h(x)}(ct) \geq \phi_{x,0}(t),
\]

for all $x \in X$ and all $t > 0$, hence

\[
\Theta_{J_g(x)-J_h(x)} \left( \left( \frac{c}{2^p} \right) ct \right) \geq \Theta_{g(x)-h(x)}(\alpha^p ct)
\]

(4.15)

\[
\geq \phi_{x,0}(\alpha^p t),
\]

for all $x \in X$ and all $t > 0$, and so, if $d(g,h) < c$, then

\[
d(J_g,J_h) < \frac{\alpha^p}{2^p} d(g,h),
\]

(4.16)

for all $g, h \in \Lambda$. Then $J$ is a strictly contractive self-mapping on $\Lambda$ with Lipschitz constant $L = \frac{\alpha^p}{2^p} < 1$.

Also, it follows from (4.12) that

\[
\Theta_{J_f(x)-J_f(x)}(\frac{c}{2^p} t) \geq \Theta_{f(x)-f(x)}(\frac{\alpha^p}{2^p} t) \geq \Phi_{x,0}(t),
\]

(4.16)

for all $x \in X$ and all $t > 0$, which implies that

\[
d(J_f,J_f) < \frac{\alpha^p}{2^p}.
\]

Using Theorem 4.1, there exists a mapping $Q : X \to Y$, which is a unique fixed point of $J$ in the set $\Lambda_1 = \{g \in \Lambda : d(g,h) < \infty\}$ such that

\[
Q(x) = \lim_{n \to \infty} f\left( \frac{2^n x}{2^n} \right),
\]

for all $x \in X$, since $\lim_{n \to \infty} d(Q^n f, Q) = 0$. Again, it follows from Theorem 4.1 that

\[
d(f,Q) \leq \frac{1}{1-L} d(f,Jf)
\]

\[
\leq \frac{1}{\left( \frac{\alpha^p}{2^p} \right) (1-L)} = \frac{2^p}{\alpha^p} \left( \frac{2^p - \alpha^p}{2^p} \right),
\]

which implies

\[
\Theta_{f(x)-Q(x)}(t) \geq \phi_{x,0}(2^p - \alpha^p t),
\]
for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) and \( y \) by \( 2^n x \) and \( 2^n y \) in (4.12), respectively,

\[
\Theta_{D_f(x,y)}(t) = \lim_{n \to \infty} \Theta_{D_f(2^n x, 2^n y)}(2^{4n} t) \geq \lim_{n \to \infty} \Phi_{4^n x, 4^n y}(2^{4n} t),
\]

for all \( x, y \in X \) and all \( t > 0 \). It follows from

\[
\lim_{n \to \infty} \left( \frac{1}{2^n} \right) = 1 \quad \text{that} \quad DQ(x, y) = 0. \quad \text{Hence, the mapping} \quad Q \quad \text{is quartic. Now, we show that mapping} \quad Q \quad \text{is unique.}
\]

To prove this, we assume that there exists a quartic mapping \( R : X \to Y \), which satisfies (4.13). Then, \( R \) is a fixed point of \( J \) in \( \Lambda_1 \). However, it follows from Theorem 4.1 that \( J \) has only one fixed point in \( \Lambda_1 \). Hence, we deduce that \( Q = R \).

**Theorem 4.6.** Let \( \phi : X^2 \to D^+ \) be a mapping such that, for some \( 2 \leq \alpha \)

\[
\phi_{x,y}(t) \geq \Phi_{2x,2y}(\alpha^p t), \quad (4.17)
\]

for all \( x \in X \) and all \( t > 0 \). If \( f : X \to Y \) is a mapping with \( f(0) = 0 \) which satisfies (4.12), then there exists a unique mapping \( Q : X \to Y \) such that

\[
\Theta_{f(x)-Q(x)}(t) \geq \phi_{0,0}(\frac{2^p}{\alpha^p} (\alpha^p - 2^p t), \quad (4.18)
\]

for all \( x \in X \) and all \( t > 0 \).

**Proof.** Let \( \Lambda \) and \( d \) be as in the proof of Theorem 4.5. Then \( (\Lambda, d) \) becomes a complete metric space and the mapping \( J : \Lambda \to \Lambda \) defined by

\[
J_g(x) = 2^n g \left( \frac{x}{2^n} \right)
\]

for all \( x \in X \) and \( g \in \Lambda \). Then,

\[
d(J_g, J_h) < \frac{2^p}{\alpha^p} d(g, h),
\]

for all \( g, h \in \Lambda \). Then, \( J \) is a strictly contractive self-mapping on \( \Lambda \) with Lipschitz constant \( L = \frac{2^p}{\alpha^p} < 1 \).

It follows from (4.12) that \( d(f,Jf) < \frac{1}{(2\alpha)^p} \), we get

\[
d(f, Q) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{2} \left( \frac{2^p}{\alpha^p} \left( \alpha^p - 2^p \right) \right),
\]

which implies the inequality (4.12) holds for all \( x, y \in X \) and all \( t > 0 \). The remaining assertion goes through in a similar method to the corresponding part of Theorem 4.5.

**Corollary 4.7.** Let \( X \) be a real \( p \)-Banach spaces, and define \( \Theta_x(t) = \frac{t}{t+\|x\|} \), for all \( x, y \in X \) and all \( t > 0 \).

Then, \( (X, \Theta, T_M) \) is a complete random \( p \)-normed space. Define

\[
\Phi_{x,y}(t) = \frac{t}{t+\|x\|+\|y\|},
\]

for all \( x, y \in X \) and all \( t > 0 \) in which \( 0 < r < \alpha \). Assume that \( f : X \to Y \) is a mapping \( f(0) = 0 \) which satisfies (4.12), then there exists a unique mapping \( Q : X \to Y \) such that

\[\Theta_{f(x)-Q(x)}(t) \geq \frac{2^p}{\alpha^p} \left( \frac{2^p}{\alpha^p} - 2^p \right), \quad (4.19)\]

for all \( x, y \in X \) and all \( t > 0 \). Where \( \alpha = 2^p \). Hence, we have

\[\|f(x) - Q(x)\| \leq \frac{2^p}{\alpha^p} \left( \frac{2^p}{\alpha^p} - 2^p \right), \quad (4.20)\]

for all \( x, y \in X \).

**References**


