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# Chaotic and Coexistence Attractors of Classical Complex Exotic Oscillator with Position-Dependent Mass

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**Abstract** This work analyzes the chaotic dynamics and the coexistence of attractors of the complex exotic oscillator. From the Hamiltonian of the basic system, the oscillator is obtained. The complete dynamics of the exotic oscillator is studied and the coexistence of attractors analyzed using fourth order Runge-Kutta algorithm. It is obtained for appropriate conditions the coexistence of chaotic and regular attractors. The study showed that for domains of values of the pair of parameters (a,b) of the exotic oscillator and given initial conditions, the dynamics of the system can be regular, or chaotic or can have very large amplitudes. The basins of attraction and the drawn phase portraits confirmed the PT symmetry and the breaking of this symmetry of the oscillator.

***Keywords:*** *Exotic oscillator, Hamiltonian, PT-symmetry, chaos and bifurcation, coexistence of attractors*

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## 1. Introduction

In nature almost everything is dynamic since a system, whether natural or artificial, has at least one of its parameters which change over time quickly, slowly or relatively slowly depending on the time scale considered. To understand the evolution of a system or a variable in time and space, we use the notions of dynamic systems [1,2]. To do this, it is necessary to model the system to be studied. The approach used for modeling depends on the nature of the system and the area of use of the latter. The fields often concerned are mechanics, electronics, biochemistry, biophysics, chemistry, economics, finance, engineering, nonlinear optics, mathematics, etc. [1,2,3,4,5,6]. Thus, the modeling can be based on Newton's second law for mechanical systems, the laws of chemical kinetics for chemical systems, Kirchoff's laws for electronic systems.... [1,2,3,4,5,6]. With the notable advances in research on dynamic systems in general and nonlinear dynamics in particular, the areas of intervention of the notions of dynamics are no longer limited. Indeed, the modeling of the dynamics of certain classical or quantum systems is well managed using the Lagrange and Hamilton approaches when the system is mechanical [7-23]. In this sense, several researchers have constructed Hamiltonians to study quantum systems in order to determine their quantum states and the eigenvalues of their energies. Among the research carried out by scientists in the field, it is shown that the obligation of hermiticity when one wants to construct Hamiltonians which take into account the unitary temporal evolution of quantum states and eigenvalues of energy can be replace with PT symmetry [17]. Thus, several types of Hamiltonians previously rejected for reasons of non-hermiticity are now considered thanks to this discovery. PT symmetry defined as symmetry in time and space plays several roles in non-Hermitian physical models and models with individually symmetric Hamiltonians P or T PT [17]. This is the case of the harmonic oscillator where the particles in a quartic potential present symmetrical closed trajectories PT [17,18].

According to the literature, closed but not PT-symmetric trajectories are very rare to find. This is the case of the complexified exotic oscillator studied by Gosh and Modak [17]. In their work, they considered the harmonic oscillator complexified by taking into account a specific form of position-dependent mass. Their motivation for studying this particular form of effective mass oscillator is because in the classical sense it is a PT-symmetric model. Indeed, they showed that this oscillator has an interesting form of Hamiltonian dynamics and moreover is exactly solvable as a quantum system. They also proved that it is possible to have closed trajectories that are mostly PT symmetric and closed orbits that are not PT symmetric. They therefore concluded that there is a possible link between this phenomenon and the classic analogue of exceptional.

Generally, the harmonic oscillator is the classic example often used to move from the classical oscillator to the quantum oscillator using the quantization of position and velocity variables. However, natural phenomena are complex and cannot be better explained by a harmonic oscillator. The oscillator whose mass depends on the position is a good example of a classical oscillator used to model macroscopic classical systems also observed on the microscopic scale. It is to understand the classical behaviors possibly observable at the atomic scale that several authors have studied a Classical oscillator with position-dependent mass in a complex domain [17] [21,22,23]. Since the exotic oscillator considered by Ghosh and Modak is an example of oscillators whose mass depends on position, the exotic oscillator cannot escape the existence of very complex physical phenomena.

Chaos and the coexistence of attractors are two other complex physical phenomena studied for complex systems [24-36]. Chaos is a complex phenomenon that cannot accurately predict the position and speed of the system. Depending on the domain and the objective, chaos may or may not be desired. In case it is not desired, its control or elimination is suggested [29,30,31]. Finally, the presence of the coexistence of attractors in a system makes it difficult to predict the nature of its dynamics. Indeed, a system for which the coexistence of attractors appears for the same value of one of its parameters is in two or more states or oscillates with several amplitudes, which makes it difficult to control [32-44]. This phenomenon can be bistable or megastable. Indeed, bistability reflects the coexistence of two attractors, while megastability designates the coexistence of an infinite number of attractors for the same system. Just like chaos, the coexistence of attractors has advantages and disadvantages. One of the advantages of the coexistence of attractors in a system allows it to be a good candidate for reinforcing the key to securing communication while one of its disadvantages is in the difficulty of designing control laws. a system presenting the coexistence of chaotic attractors. To help resolve this drawback, several researchers have embarked on work to predict and control the coexistence of attractors. This explains the numerous works published over the last decade on this subject [32-44]. In these different works, several techniques are used by the authors to research, analyze and control coexistence of attractors and very conclusive results are obtained. As applications, these authors showed that great flexibility in system performance is made achievable by the coexistence of different stable states without major parameter changes.

The exotic oscillator studied by Ghosh and Modak [17] is a 4-dimensional nonlinear system whose complexity induces gauge invariance. According to these authors, the existence of PT symmetry made it possible to obtain closed trajectories. Sensitivity to initial conditions was also proven by these authors. Finally, the authors also found that the nature of the possible trajectories of the complex exotic oscillator depends on the values of the latter's parameters.

From these motivations and the important results obtained by Ghosh and Modak, we believe that it is necessary to find the different domains of the system parameters for the different dynamics of the exotic four-dimensional oscillator. It would also be important to study the chaotic dynamics and the coexistence of the attractors of this oscillator. Thus, the present work specifically concerns the determination of the different dynamics of the exotic oscillator, the determination of the different parameters of this oscillator for which there is each type of dynamic, the determination of the different initial conditions for which the oscillator has finite and infinite amplitudes and the determination of the different initial conditions for which the oscillator presents a phenomenon of coexistence of attractors.

The paper is structured as follows: section 2 gives the mathematical modeling of dynamics of an Elastical Exotic Oscillator. In section 3, the numerical study of the oscillator and the analysis of the different results are carried out. Finally, we provide conclusion in the last section.

## 2. Model of Classical Complex Exotic Oscillator

In this work, we study the classical Complex Exotic Oscillator (CEO) proposed by Gosh and Modak [17]. Indeed, we consider the CEO in a considerably simplified dimension constituting an oscillator with a function depending on the mass position. The Hamiltonian of the system is:

$$\mathbf{H} = \frac{\mathbf{p}^2}{2}\left(1-\mathbf{bx}^2\right)+\frac{\mathbf{a}}{2}\mathbf{x}^2 \quad (1)$$

where a is a stabilizing parameter of the exotic oscillator; b a control parameter; x and p designate the position and the associated conjugate moment. The equation of motion highlighting the PT-symmetry of the Hamiltonian is:

$$\ddot{x}+2\left[bH-a\left(bx^2-\frac{1}{2}\right)\right]x=0 \quad (2)$$

By defining complex variables

$$\mathbf{z}=\mathbf{x}+\mathbf{iy}$$

and

$$\boldsymbol{\pi}=\mathbf{p}-\mathbf{iq}$$

the complex Hamiltonian is written:

$$\mathbf{H} = \frac{\boldsymbol{\pi}^2}{2}\left(1-\mathbf{bz}^2\right)+\frac{\mathbf{a}}{2}\mathbf{z}^2=0 \quad (3)$$

By separating the real and imaginary parts of **H**, we obtain respectively

$$\mathbf{H}=\frac{1}{2}\left[\mathbf{a}\left(\mathbf{x}^2-\mathbf{y}^2\right)+\left(\mathbf{p}^2-\mathbf{q}^2\right)\left(1-\mathbf{b}\left(\mathbf{x}^2-\mathbf{y}^2\right)\right)-4\mathbf{bxypq}\right] \quad (4)$$

and

$$\mathbf{G}=\mathbf{xy}\left(\mathbf{a}-\mathbf{b}\left(\mathbf{p}^2-\mathbf{q}^2\right)\right)-\mathbf{pq}\left(1-\mathbf{b}\left(\mathbf{x}^2-\mathbf{y}^2\right)\right) \quad (5)$$

We consider the case where $G=0$ because in one dimension $G$ is the only first-class constraint inducing gauge invariance since $\{G,H\}=0$. So the equations of motion are:

$$\begin{cases} \dot{x} = p\left[1-b\left(x^2-y^2\right)\right]-2bxyq, \\ \dot{p} = bx\left(p^2-q^2\right)+2bypq-ax, \\ \dot{y} = -q\left[1-b\left(x^2-y^2\right)\right]-2bxyq, \\ \dot{q} = by\left(p^2-q^2\right)+2bxpq+ay. \end{cases} \tag{6}$$

## 3. Numerical Study of Complex Dynamics

Complex systems generally have several possible dynamics, the prediction of which is sometimes not easy.

Chaos is one of those behaviors that is determined from Lyapunov's statement defined by [45]:

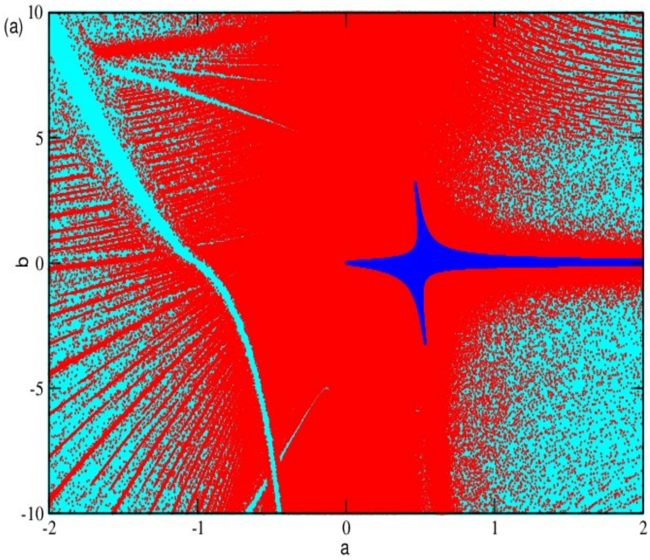
$$Lya = \lim_{t\to\infty} \frac{\ln\sqrt{(dx)^2+(dp)^2+(dy)^2+(dq)^2}}{t} \tag{7}$$

where dx, dp, dy and dq are the variations of x, p, y and q, respectively.

The system is chaotic when $Lya > 0$; periodic if $Lya < 0$ and quasi-periodic when $Lya = 0$.

The objective of this section is to numerically research and analyze the different dynamics and coexistence of attractors for the classical exotic complex oscillator. Thus, the system of equations (6) is solved using Runge-Kutta algorithms of order 4 in the Fortran language and parameter basins, basins of attraction, bifurcation diagrams and phase space are represented. Since $a$ and $b$ are the two parameters of the system, the parameter pool is plotted in the plane $(a,b)$. Figure 1 (a) and Figure 1 (b) represents the parameter pool respectively for the initial conditions.

$(x_0,p_0,y_0,q_0)$ = (0.2,0.1,0.1,0.1) and $(x_0,p_0,y_0,q_0)$ = (0.8,0.1,0.1,0.1). In this figure the domain of regular behavior (corresponding to $Lya < 0$) is represented in blue, that of chaotic behavior (corresponding to $Lya > 0$) in red and that of very large amplitude oscillations in cyan. We see through Figure 1 that for $-2 \le a \le 2$ and $-10 \le b \le 10$ the domain of existence of large amplitude oscillations is larger than that of periodic behaviors and this is larger than the domain of existence of chaotic behavior. Furthermore, we notice that the domain of chaotic behavior is more abundant for (0.8,0.1,0.1,0.1) than for (0.2,0.1,0.1,0.1). To check the different dynamics obtained in Figure 1, the phase space is plotted for different values of $a$ and $b$ chosen in each domain. Thus Figure 2 is obtained for the chosen values of the couple $(a,b)$. This figure confirms the regular and chaotic dynamics in the $(x,p)$ and $(y,q)$ planes. These two parameters pools therefore show that the dynamics of the exotic classical oscillator depends on the values of $(a,b)$ and its sensitivity to initial conditions. To collect the different initial conditions favorable to each type of dynamics, basins of attractions are drawn. The results obtained for ($a$ = 1,$b$ = 1.24), are represented in Figure 3 (a) and Figure 3 (b) respectively for ($y_0$ = 0.1,$q_0$ = 0.1) and ($x_0$ = 0.2,$p_0$ = 0.1). In this Figure 3, the blue color designates the domain of regular behavior, the red color the domain of chaotic behavior and the cyan color the large amplitude oscillations. These figures confirm the sensitivity to initial conditions of the exotic classical oscillator studied. As proof, we note that regular oscillations do not exist for ($x_0$ = 0.2,$p_0$ = 0.1) in the plane $(y,q)$ while they exist for ($y_0$ = 0.1,$q_0$ = 0.1) in the $(x,p)$. In addition to these figures, it is necessary to notice the symmetry of these basins of attraction around the origin (0,0). Through these figures, we also note that the exotic classical oscillator studied presents the coexistence of attractors. All these predictions of Figure 3 are confirmed by the phase portraits of Figure 4. Indeed, on these phase portraits obtained in the $(x,p)$ and $(y,q)$ planes, we note the coexistence of chaotic and periodic attractors corresponding to each initial conditions thus confirming the periodic and chaotic dynamics predicted by Figure 3 (a). We observe once again for the exotic oscillator the PT symmetry and its breaking through these phase portraits. In order to highlight all the coexisting attractors, the bifurcation diagram is drawn by varying the parameter $b$ from $-10$ to 10. The result obtained is that of Figure 5 where the blue curve corresponds to the increasing variation of $b$ from -6 to 10 while the red curve corresponds to the decreasing variation of $b$ from 10 to -6. It is clearly noted that the evolution in the decreasing direction largely shows chaotic dynamics while the evolution in the increasing direction shows mono-periodic behavior. We can therefore conclude that chaotic and mono-periodic attractors coexist. This result is confirmed by Figure 6 which shows mono-periodic phase portraits (the blue curve) and chaotic phase portraits (red curve). By taking a as a control parameter, we note that the existence of periodic, quasi-periodic and chaotic behavior as well as the existence of several attractors strongly depend on the parameter a (see Figure 7). These different dynamics and this coexistence of multiple attractors are confirmed by the phase portraits of the oscillators (Figure 8). In short, for a pair of fixed values of $(a,b)$, the exotic oscillator studied can have chaotic or periodic dynamics for well-chosen initial conditions. This therefore does not make it possible to say with accuracy the nature and amplitude of the oscillations and then the position of the exotic oscillator studied, thus confirming its complexity.



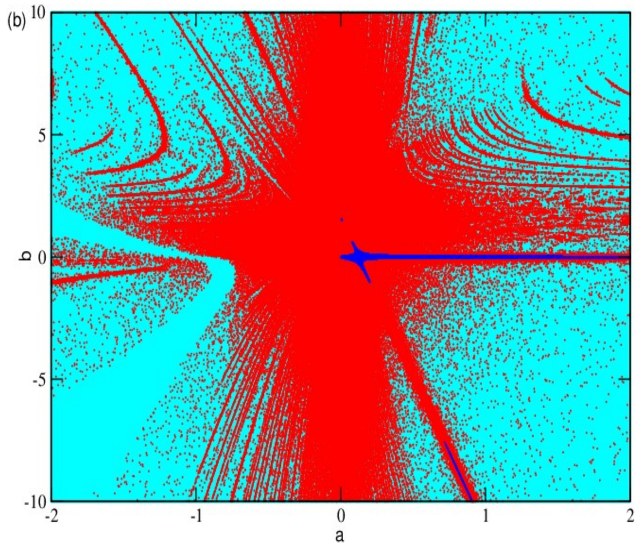



**Figure 1.** (Color online) Basin of parameters with (*a*) (0.2,0.1,0.1,0.1) and (*b*) (0.8,0.1,0.1,0.1)

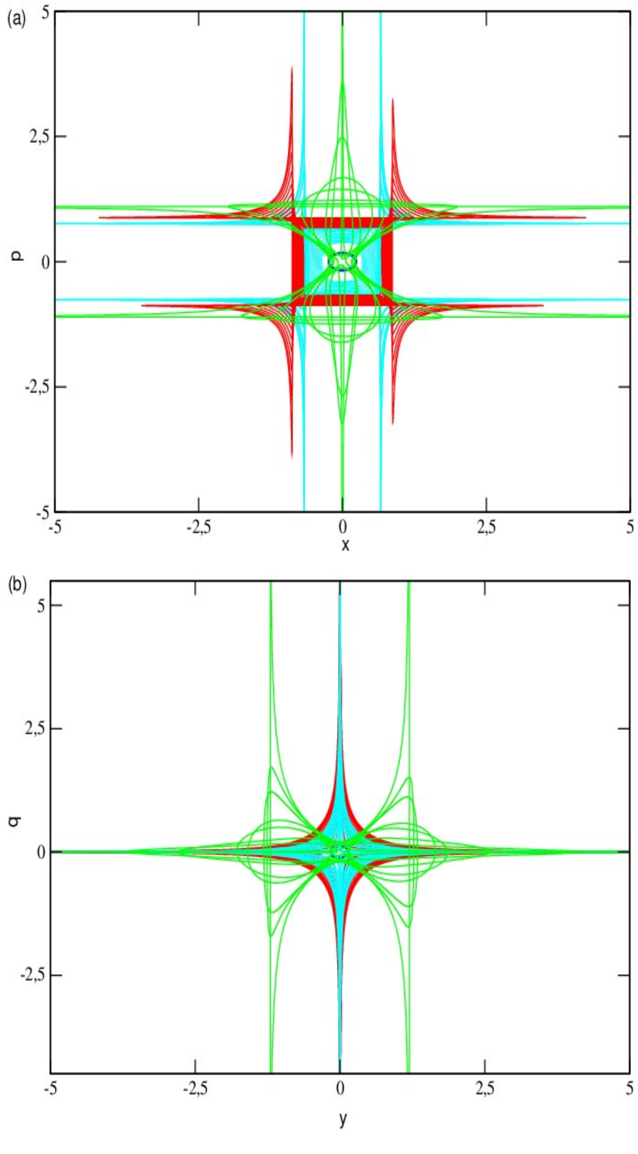



**Figure 2.** (Color online) Phase portraits with parameters of Figure 1(a) in (a)(x,p) and (b)(y,q). Blue curve: a = −0.85, b = −0.7; red curve a = 1,b = 1.3; cyan curve a = 1.3,b = 2.283 and green curve a = 0.7,b = 1.24

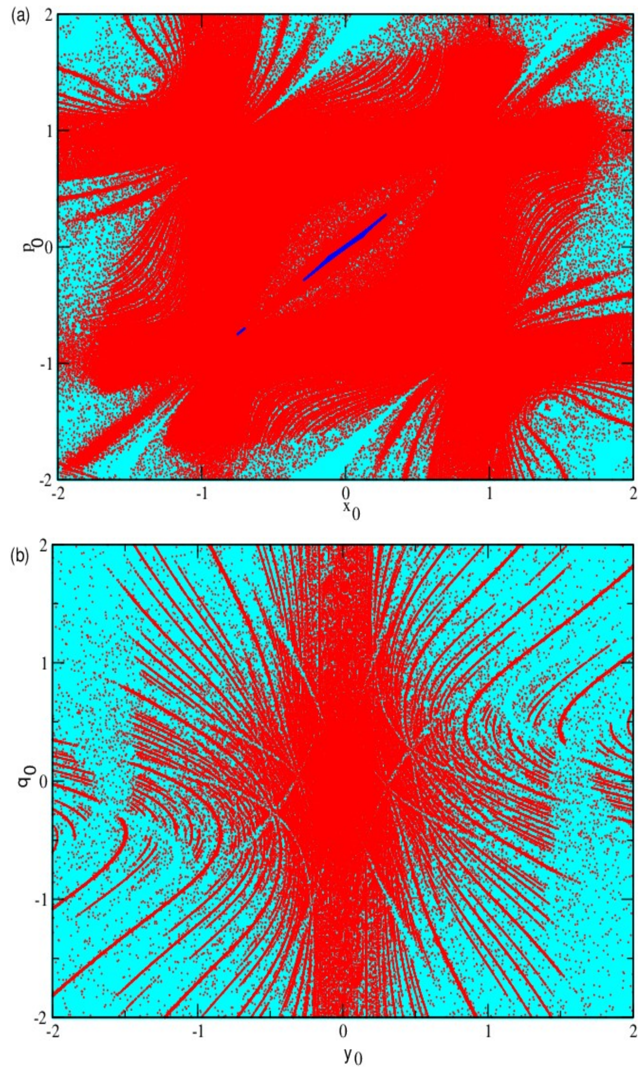
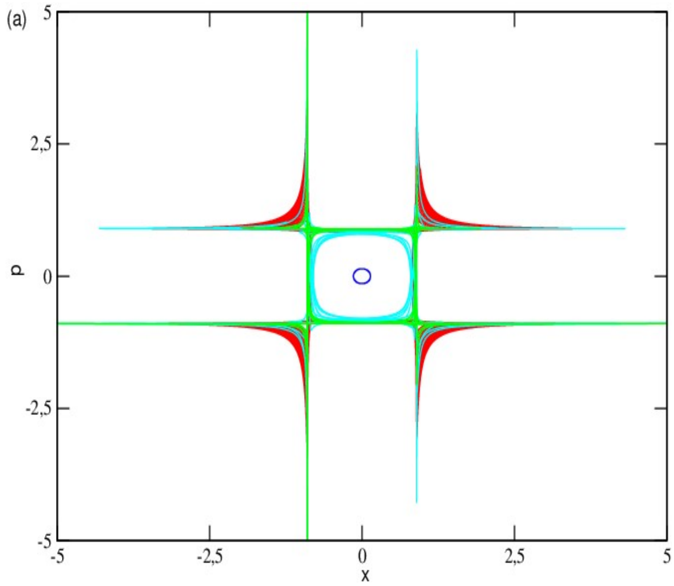



**Figure 3.** (Color online) Basin of attraction with $a$ = 1 ; $b$ = 1.24: ($a$)($x_0$,$p_0$) and ($b$)($y_0$,$q_0$)



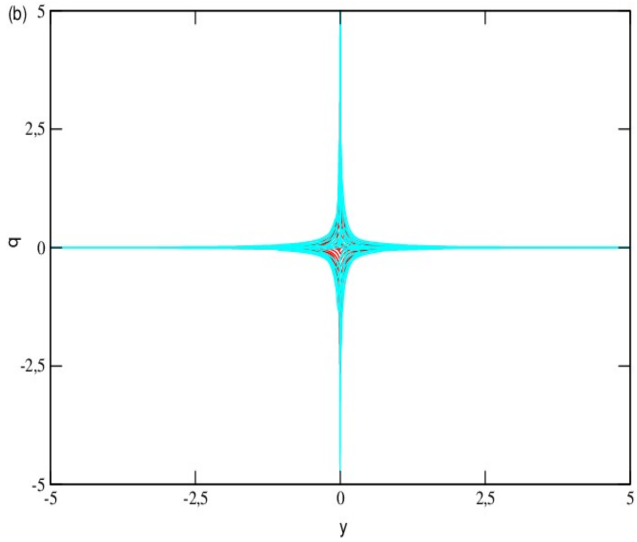



**Figure 4.** (Color online) Coexistence of attractors with parameters of Figure 3: (a) (x,p) where blue curve correspond to (0.1,0.1,0.1,0.1), red curve correspond to (0.3,0.34,0.1,0.1), green curve correspond to (0.8,0.1,0.1,0.1), cyan curve correspond to (1,0.9,0.1,0.1); (b) (y,q) red curve correspond to (0.8,0.1,0.001,0.03), cyan curve correspond to (0.8,0.1,0.1,0.3)

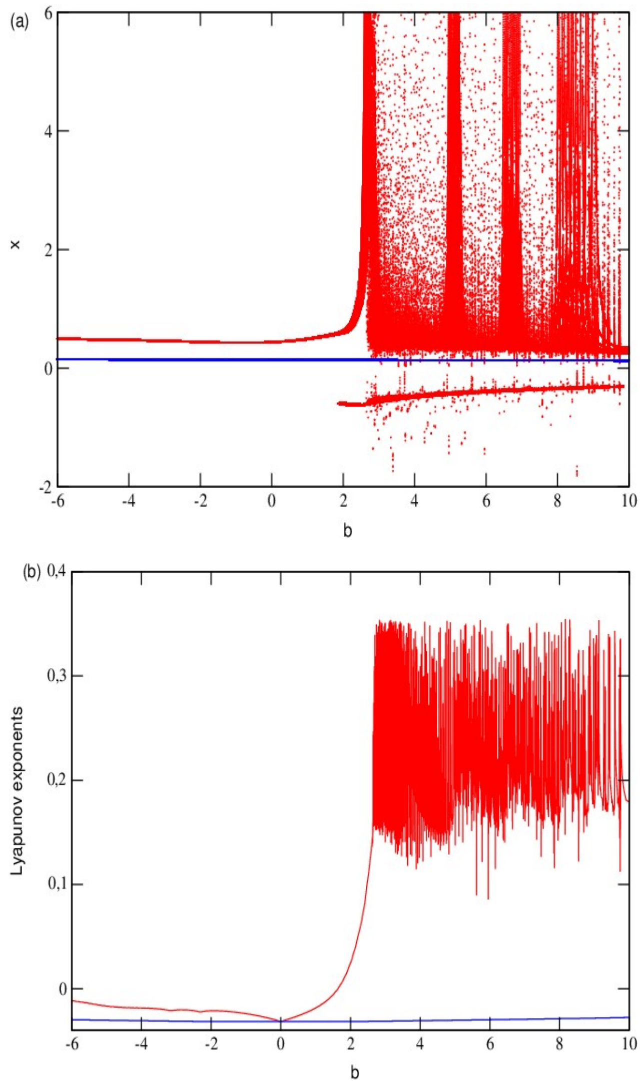



**Figure 5.** (Color online) Bifurcation diagram and its corresponding Lyapunov exponents for a = 1: where blue curve correspond to (0.1,0.1,0.1,0.1) and red curve correspond to (0.3,0.32,0.1,0.1)

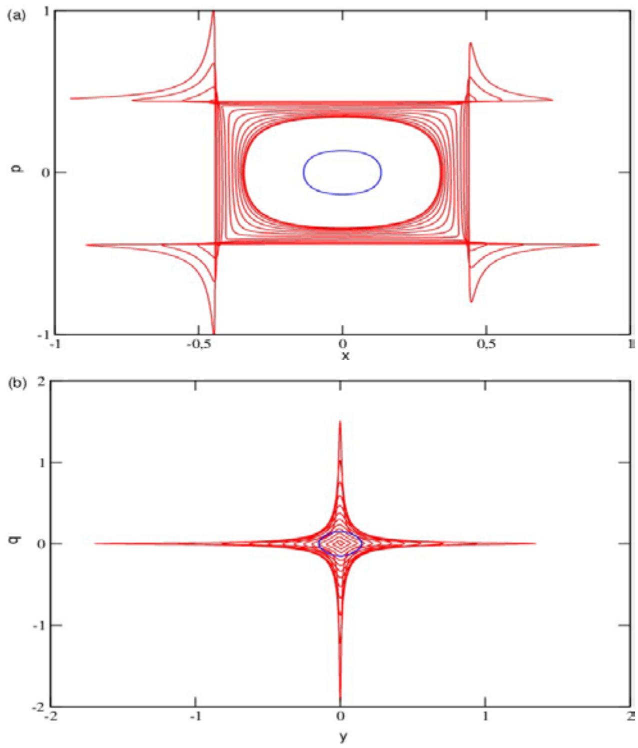



**Figure 6.** (Color online) Coexistence of attractors with parameters of Figure 5 for b = 5 where blue curve correspond to (0.1,0.1,0.1,0.1) and red curve correspond to (0.8,0.1,0.1,0.1)

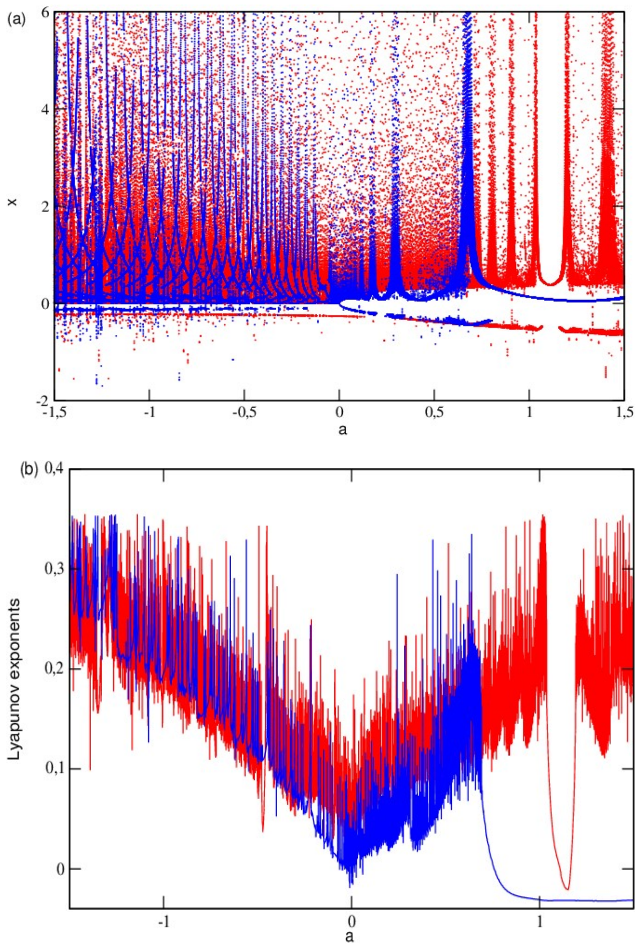
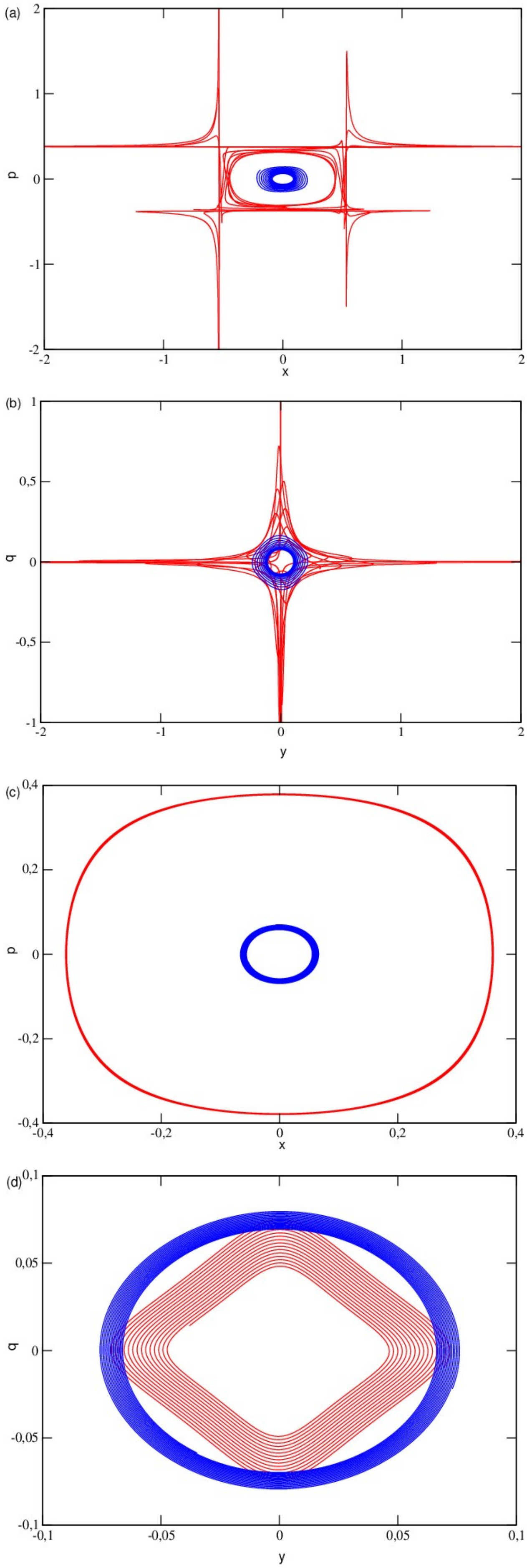



**Figure 7.** (Color online) Bifurcation diagram and its corresponding Lyapunov exponents for b = 3.5: where blue curve correspond to (0.1,0.1,0.1,0.1) and red curve correspond to (0.3,0.32,0.1,0.1)



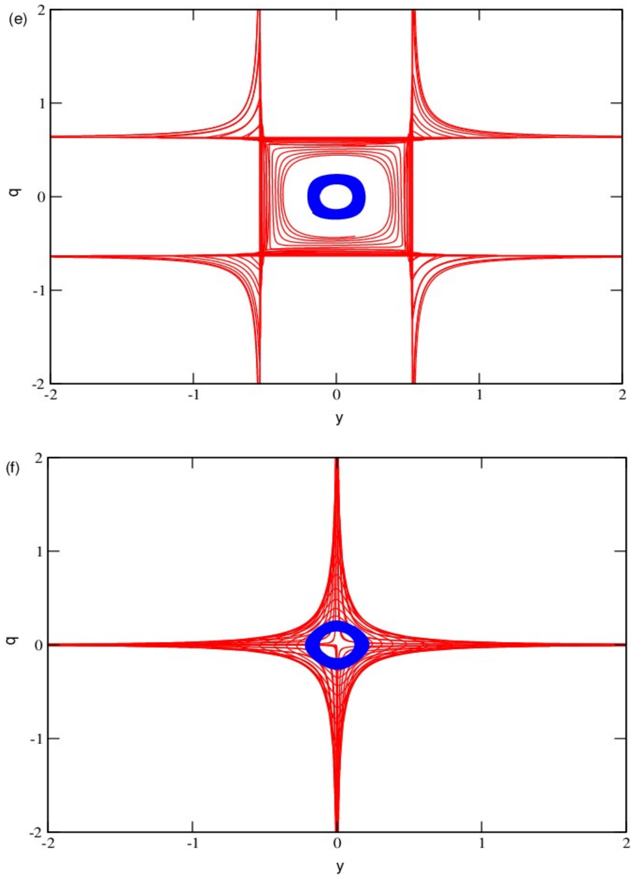




**Figure 8.** (Color online) Coexistence of attractors with parameters of Fig.7 where blue curve correspond to (0.1,0.1,0.1,0.1) and red curve correspond to (0.3,0.32,0.1,0.1). (a),(b) : a = 0.5; (c),(d) : a = 1.1 and (d),(e) : a = 1.4

# 4. Conclusion

In this work, the exotic classical oscillator is numerically studied. Using numerical simulations of the parameter diagram, the basins of attraction, the bifurcation diagram and phase spaces made it possible to identify the different dynamics and conditions of coexistence of attractors of the exotic classical oscillator. Indeed, the domains of values of *a* and *b* for which the exotic oscillator studied has a regular, chaotic behavior or a large amplitude oscillation are obtained. Sensitivity to the initial conditions of the oscillator was obtained as well as the coexistence of chaotic and mono-periodic attractors. These different results confirm the complexity of the exotic oscillator and complement the results obtained by Ghosh and Modak. This complexity is linked to the two phenomena obtained: chaos and the coexistence of attractors. They do not allow an exact prediction of the position, conjugate moment, amplitude and nature of the oscillations of the exotic oscillator. This could be a disadvantage or an advantage for systems modeled by the exotic oscillator. A disadvantage if this oscillator is used in systems where the vibration must be periodic or where position precision is mandatory. But an advantage if this oscillator could be quantized to be used in quantum systems used for securing data or information because this would strengthen the security key by synchronization. In this sense, it would be necessary to study the dynamics of the quantum equivalent of this exotic oscillator, to achieve its chaotic synchronization in order to use it in the coding of information. Finally, it would also be important to study the conditions in which the exotic oscillator studied will be hyperchaotic.

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# Author Contribution Statement

**J. G. Houeto** and **B. N. Tokpohozin** contributed to the modeling, to the various calculations, to the drafting of the paper.

**C. H. Miwadinou** contributed to the modeling, the various calculations and the analyzes of the results of the paper.

**A. A. Koukpemedji and A. V. Monwanou** contributed to the analysis of the results and supervised this work.

# Data Availability Statement

All data are included in text directly.

# Declarations

The authors declare that there are no conflicts of interest.

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