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Generalized Optical Bloch Equations for a General Three-Level Atom Interacting with a Linearly Polarized Light in the Rotating Wave and Electric Dipole Approximations

Mesfin Woldeyohannes*

Department of Chemistry & Physics, Western Carolina University, 112 Apodaca Science Building, Cullowhee, NC, USA
*Corresponding author: woldeyohan@email.wcu.edu

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Abstract The nine (9) elements of the 3×3 density matrix of a general three-level atom interacting with a classically described linearly polarized light are parameterized by eight (8) variables related to the populations of the atomic levels, and the coherences (interference effects) between the levels. The dynamical equations of these eight parameters (that is, the optical Bloch equations for a general three-level atom) are derived under the electric dipole as well as the rotating wave approximations, and under the assumption that the matrix elements of the atomic electric dipole are all real. The general optical Bloch equations so derived are shown to contain the well-known optical Bloch equations for three-level atoms in the V , Λ , and cascade configurations as well as those for two-level atoms as special cases. Expressing the interaction of light with a three-level atom in all possible configurations as well as with a two-level atom by a single set of generalized optical Bloch equations is the main result of this work.

Keywords: *Optical Block equations, two-level atom, three-level atom, Λ-configuration, V-configuration, cascade configuration*

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1. Introduction

The optical Bloch Equations for an atom-field interaction system are the equations of motion for the elements of the density matrix of the system [1,2]. In the case of a two-level atom interacting with light, these equations are formally identical to the dynamical equations of a spin one-half particle in an oscillatory magnetic field. As a result, the spin vector formalism of Bloch developed for magnetic resonance [3] can be directly applied to optical resonance problems involving two-level atoms [1], hence the term optical Bloch equations.

In this work we derive the optical Bloch equations governing the interaction of a general three-level atom with a linearly polarized light, under the electric dipole as well as the rotating wave approximations, and under the assumption that the matrix elements of the atomic electric dipole are all real. The electric dipole approximation is invoked by assuming that the external light is in the visible region so that its wavelength is much larger than the size of the interacting atom. The rotating wave approximation is applied by transforming the 8 parameters of the density matrix of the general three-level atom into another set of 8 parameters through an 8×8 orthogonal transformation matrix representing "pure rotation" in an 8-dimensional state space, and then ignoring the rapidly varying terms. The matrix elements of the electric dipoles associated with the atomic transitions can be made real by adjusting the arbitrary phases of the wave functions of the relevant atomic levels, provided the transitions between the levels obey the $\Delta m = 0$ selection rule [1]. The optical Bloch equations for the special cases of three-level atoms in the V, Λ, and cascade configurations as well as those for two-level atoms are easily derived from the general Bloch equations by setting to zero all quantities non-relevant for each of those special cases.

The remainder of this paper is organized in three sections as follows. Section (2) gives a general description of the operators needed to describe the interaction of a general n-level atom with a classically described electromagnetic field in the electric dipole approximation. This description is then applied in section (3) to derive the optical Bloch equations for a general three-level atom with arbitrary arrangement of atomic levels. Section (4) deals with the derivation of the optical Bloch equations for the special cases of three-level atoms in the V, Λ, and cascade configurations as well as those for two-level atoms from the general Bloch equations of section (3). Finally, section (5) provides a brief conclusion of the paper.

2. General Description of Operators

2.1. Atom-field Interaction Hamiltonian in the Electric Dipole Approximation

The Hamiltonian H_A of a bare atom (with no radiation present) can be written as [2,4]

$$H_A = \hbar\omega_i |i> \Rightarrow \begin{cases} H_A = \hbar\sum_i \omega_i \sigma_{ii} \\ \sigma_{ij} = |i><j| \ , \ i,j = 1,2,3 \\ \left[\sigma_{ij}, \sigma_{lk}\right] = \sigma_{ik}\delta_{jl} - \sigma_{lj}\delta_{ki} \end{cases} \tag{2.1.}$$

where $|i\rangle$ and $\hbar\omega_i$ are, respectively, the eigenvectors and eigenvalues of H_A [2.4] satisfying the orthonormality ($\langle i|j\rangle = \delta_{ij}$) and closure ($\sum_i |i\rangle\langle i| = 1$) relations, whereas $\sigma_{ij} = |i\rangle\langle j|$ are the atomic operators satisfying the commutation relation $[\sigma_{ij}, \sigma_{lk}] = \sigma_{ik}\delta_{jl} - \sigma_{lj}\delta_{ki}$ in which δ_{ij} stands for the Kronecker delta.

Like the bare atom Hamiltonian H_A, any linear operator Q can be expressed in terms of the atomic operators σ_{ij} by applying the closure relation twice as

$$Q = \sum_{ij} Q_{ij} \ , \ \begin{cases} Q_{ij} = <i|Q|j> \\ Q_{ij} \ is \ real \ if \ i = j \\ Q_{ij} = Q_{ij}^* \ if \ i \neq j \end{cases} \tag{2.2}$$

Here, the expansion coefficients $Q_{ij} = \langle i|Q|j\rangle$ are the matrix elements of Q in the basis $\{|i\rangle\}$. If Q is a Hermitian operators (as any observable should be), it is represented by a Hermitian matrix [5], one in which all the diagonal elements Q_{ii} are real, and any two elements symmetric with respect to the diagonal are complex conjugates of each other ($Q_{ij} = Q_{ji}^*$).

The interaction of an atom with a radiation field is represented by an interaction Hamiltonian [2] H_I given by Eq.2.3. In this equation, $\vec{d}$ is the atomic dipole moment vector which is expanded in terms of the atomic operators σ_{ij} in accordance with Eq.2.2, $\vec{d}_{ij}$ are the matrix elements of $\vec{d}$, d_{ij} is the magnitude of $\vec{d}_{ij}$, and $\hat{d}_{ij}$ is the direction (unit vector) of $\vec{d}_{ij}$. The matrix elements $\vec{d}_{ij}$ are in general complex, and (since $\vec{d}$ is Hermitian) $d_{ij} = d_{ji}^*$. Moreover, since $\vec{d}$ is a vector operator, its diagonal matrix elements $\vec{d}_{ii}$ are all zero [5].

$$H_I = -\vec{d}\bullet\vec{E}\left(\vec{r},t\right) \ , \ \begin{cases} \vec{d} = \sum_{i,j} \overrightarrow{d_{ij}}\sigma_{ij} \\ \overrightarrow{d_{ij}} = <i|\vec{d}|j> = d_{ij}\widehat{d_{ij}} \\ d_{ij} = 0 \ \ if \ \ i = j \\ d_{ij} = d_{ji}^* \ \ if \ \ i \neq j \end{cases} \tag{2.3}$$

In the full quantum mechanical treatment of atom-field interaction, the atom is treated quantum mechanically, and the radiation field is taken to be a quantized field [7,8,9]. In this work, we employ the semiclassical approach in which the atom is described quantum mechanically whereas the radiation field is taken to be a purely classical field (by assuming the field to be sufficiently strong). Moreover, we assume the radiation field to be a monochromatic (or quasi-monochromatic) laser light with central frequency ω which is assumed to be in the visible region. We also assume the light to be linearly polarized in the direction of a unit vector $\hat{e}$, and propagating in the direction of the wave vector $\vec{k}$. Taking the nucleus of the atom as the origin of coordinates, the electric field $\vec{E}(\vec{r},t)$ of the applied laser light can be expressed as

$$\vec{E}\left(\vec{r},t\right) = \hat{e}\varepsilon\left(\vec{r},t\right)\left[e^{i\left(\vec{k}\bullet\vec{r}-\omega t\right)} + c.c\right] \tag{2.4}$$

Here, $\varepsilon(\vec{r},t)$ is a real amplitude of the electric field, ω is the central frequency of the field, and $k = |\vec{k}| = \omega/c$ is the corresponding wave number. If $\vec{E}(\vec{r},t)$ is truly monochromatic, $\varepsilon(\vec{r},t)$ will be constant in time, whereas, more generally, $\varepsilon(\vec{r},t)$ varies with time.

When the applied light is in the optical (visible) region so that the wavelength of the light is much greater than the size of the atom, the spatial variation of the electric field of the light across the atom can be ignored to a good approximation, and this is what is meant by electric dipole approximation [5]. Invoking this approximation in Eq.2.4, we ignore $\vec{k}\cdot\vec{r}$ in the exponent, and $\vec{r}$ in $E(\vec{r},t)$ to write the applied electric field as

$$\vec{E}\left(t\right) = \hat{e}\varepsilon\left(t\right)\left[e^{i\omega t} + c.c\right] = 2\hat{e}\varepsilon\left(t\right)\cos\omega t \tag{2.5}$$

Using Eq.2.5 in Eq.2.3 and expanding the result according to Eq.2.2, we obtain

$$H_I = -\hbar\sum_{i\,j} \beta_{ij}(t)\sigma_{ij} \ , \ \begin{cases} \beta_{ij}\left(t\right) = \Omega_{ij}\left(t\right)\cos\omega t \\ \Omega_{ij}\left(t\right) = \left(\hat{d}_{ij}\bullet\hat{e}\right)\left[\frac{2d_{ij}\varepsilon\left(t\right)}{\hbar}\right] \end{cases} \tag{2.6}$$

The quantity Ω_{ij} has the dimensions of frequency and is called the Rabi frequency of the external laser radiation. It quantifies the interaction of the external laser radiation with the relevant atomic transition $|i\rangle \leftrightarrow |j\rangle$ and is zero when $i = j$ (because $d_{ii} = 0$, in accordance Eq. 2.3)

$$\left.\begin{aligned} \Omega_{ij} &= 0 \quad if\ i = j \\ \Omega_{ij} &= \Omega_{ji}^{*}\ if\ i \neq j \end{aligned}\right\} \Rightarrow \begin{cases} \beta_{ij} = 0 \quad if\ i = j \\ \beta_{ij} = \beta_{ji}^{*}\ if\ i \neq j \end{cases} \tag{2.7}$$

In general, the laser field envelope $\varepsilon(t)$ defined by Eq. 2.4 depends on time, and so do the quantities Ω_{ij} and β_{ij} which are dependent on $\varepsilon(t)$ as defined by Eq.2.6. However, if the external laser field is purely monochromatic, ε is a constant independent of time so that Ω_{ij} and β_{ij} are also constants independent of time. For most applications, if ε varies sufficiently slowly within an optical period, it (and, therefore, both Ω_{ij} and β_{ij}) can be taken to be constants to a good approximation.

Combining Eq.2.1 and Eq. 2.6, we finally obtain

$$H = \hbar\sum_{i} \omega_i \sigma_{ii} - \hbar\sum_{i,j} \beta_{ij}(t)\sigma_{ij} \tag{2.8}$$

for the total Hamiltonian describing atom-field interaction in the electric dipole approximation. In connection with the coefficients β_{ij} of the interaction Hamiltonian H_I, we introduce the atomic transition frequency ω_{ij}, and the detuning frequency δ_{ij} by

$$\omega_{ij} = \omega_i - \omega_j \ , \ \delta_{ij} = \omega_{ij} - \omega \tag{2.9}$$

The atomic transition frequency ω_{ij} is the frequency difference between any two atomic levels $|i\rangle$ and $|j\rangle$ as shown in Fig.1, whereas the detuning frequency δ_{ij} is the frequency difference between ω_{ij} and the radiation frequency ω.

2.2. State Vectors and Density Operators in General

In this paper, we choose to work in the Schrödinger picture where the basis vectors $\{|i\rangle\}$ are stationary but the state vectors $|\psi(t)\rangle$ vary in time according to the time dependent Schrödinger equation. Each time dependent state vector $|\psi(t)\rangle$ can be expanded on the time independent basis $\{|i\rangle\}$ uniquely as in Eq. 2.10.

Here, the expansion coefficient $a_i(t) = \langle i|\psi(t)\rangle$ gives the probability amplitude for finding the atom in the eigenstate $|i\rangle$ and is, in general, complex. We assume that $|\psi(t)\rangle$ is normalized ($\langle\psi(t)|\psi(t)\rangle = 1$) so that the coefficients $a_i(t)$ satisfy the probability conservation condition $\sum_i |a_i(t)|^2 = 1$.

$$|\psi(t)> = \sum_i a_i(t)\,|i> \ , \ \begin{cases} a_i(t) = <i|\psi(t)> \\ \sum_i |a_i(t)|^2 = 1 \end{cases} \tag{2.10}$$

The density operator $\rho(t)$ of a system is such that, characterizing the system by a state vector $|\psi(t)\rangle$ is completely equivalent to characterizing it by the corresponding $\rho(t)$ acting in that state space, in the sense that all physical information that can be obtained from $|\psi(t)\rangle$ can also be obtained from $\rho(t)$ [5]. For each state vector $|\psi(t)\rangle$, the corresponding density operator $\rho(t)$ and its time evolution are given by,

$$\rho(t) = |\psi(t)><\psi(t)| \ , \ i\hbar\frac{d}{dt}\rho(t) = \left[H, \rho(t)\right] \tag{2.11}$$

where H is the total Hamiltonian of the atom-radiation system given by Eq. 2.8. The density operator $\rho(t)$ can be expanded in terms of the atomic operators σ_{ij} in accordance with Eq. 2.2 to obtain,

$$\rho(t) = \sum_{ij} \rho_{ij}(t)\sigma_{ij} \tag{2.12a}$$

$$\rho_{ij}(t) = <i|\rho(t)|j> = a_i(t)\,a_j^{*}(t) \tag{2.12b}$$

$$\rho_{ij}\ real\ if\ i = j \ , \ \rho_{ij} = \rho_{ji}^{*}\ if\ i \neq j \tag{2.12c}$$

$$Tr\{\rho(t)\} = \sum_i \rho_{ii} = \sum_i |a_i(t)|^2 = 1 \tag{2.12d}$$

Here $\rho_{ij}(t)$ are elements of the density matrix, and $a_i(t)$ are the expansion coefficients of the state vector $|\psi(t)\rangle$ given by Eq. 2.10. The diagonal elements of the density matrix $\rho_{ii} = |a_i|^2$ are clearly real, and the trace of the density matrix is equal to unity which follows directly from the probability condition of Eq. 2.10. The off-diagonal elements of the density matrix are in general complex, and satisfy $\rho_{ij} = \rho_{ji}^{*}$, since $\rho(t)$ is a Hermitian operator, as can be easily deduced from its definition by Eq. 2.11. From the definition of the density operator (Eq. 2.11), the orthonormality condition ($\langle i|j\rangle = \delta_{ij}$), and Eq. 2.12c for the trace of the density matrix, it readily follows that,

$$\begin{aligned} \rho^2(t) &= \rho(t) \\ Tr\{\rho^2(t)\} &= \sum_i <i|\rho^2(t)|i> = 1 \end{aligned} \tag{2.13}$$

The diagonal elements the density matrix ρ_{ii} represent the probability of finding the atom in its eigenstate state $|i\rangle$ and, therefore, are known as the populations. The non-diagonal elements $\rho_{ij}\,(i \neq j)$ express the interference effects between the states $|i\rangle$ and $|j\rangle$ when the atom is in a coherent linear superposition of these states and, therefore, are known as the coherences. According to Eq. 2.12b, $\rho_{ij} = a_i a_j^*$ is non-zero only when both a_i and a_j are non-zero. Therefore, the density matrix can have coherences (non-diagonal elements) only between populated states.

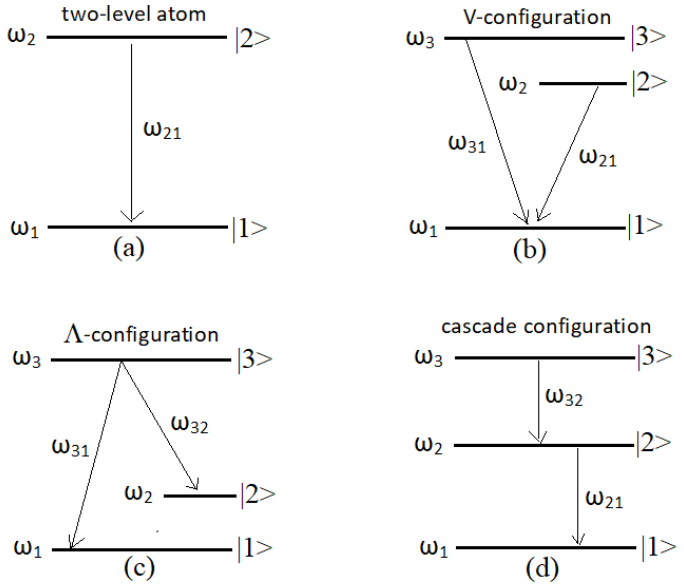

Figure 1. (a) Two-level atom, (b) three-level atom in the V-configuration, (c) three-level atom in the Λ-configuration, and (d) three-level atom in the cascade configuration. For a three-level atom in the V-configuration, the upper levels |3> and |2> are close to each other but far from the ground level |1>, and the transition |3> ←→ |2> is not dipole allowed. On the other hand, for a three-level atom in the Λ-configuration the lower levels |2> and |1> are close to each other but far from the third (highest) level |3>, and the transition |3> ←→ |2> is not dipole allowed. For a three-level atom in the cascade configuration, the transition |3> ←→ |1> is not dipole allowed

3. Optical Bloch Equations for a General Three-Level Atom

3.1. Parametric Equations of the Density Matrix Elements of a General Three-Level Atom

For a three-level atom, the state space is a 3-dimensional space spanned by the eigenvectors $|1\rangle$, $|2\rangle$ and $|3\rangle$, Figure. 1. The most general (normalized) state vector of this 3-dimensional state space is given by the linear superposition of these eigenvectors as

$$|\psi(t)\rangle = a_3(t)\,|3\rangle + a_2(t)\,|2\rangle + a_1(t)\,|1\rangle \quad (3.1a)$$

$$|a_1(t)|^2 + |a_2(t)|^2 + |a_3(t)|^2 = 1 \quad (3.1b)$$

In the three state basis $\{|3\rangle, |2\rangle, |1\rangle\}$ arranged in this order, any operator Q is represented by a 3×3 matrix in accordance with Eq. 2.2. In particular, the density operator $\rho(t)$ of Eq. 2.11 corresponding to the state vector $|\psi(t)\rangle$ of Eq. 3.1 is represented by a 3×3 density matrix whose elements are, according to Eq. 2.12b, $\rho_{ij} = a_i a_j^*$.

$$\rho(t) = \begin{pmatrix} \rho_{33} & \rho_{32} & \rho_{31} \\ \rho_{23} & \rho_{22} & \rho_{21} \\ \rho_{13} & \rho_{12} & \rho_{11} \end{pmatrix} = \begin{pmatrix} a_3 a_3^* & a_3 a_2^* & a_3 a_1^* \\ a_2 a_3^* & a_2 a_2^* & a_2 a_1^* \\ a_1 a_3^* & a_1 a_2^* & a_1 a_1^* \end{pmatrix} \quad (3.2)$$

Noting that the diagonal elements of the density matrix are real, and add up to unity (Eq. 2.12), whereas the non-diagonal elements are related by complex conjugation, the nine elements of density matrix of Eq. 3.2 can be parameterized by a new set of eight variables as

density matrix for a general three-level atom

$$\rho(t) = \frac{1}{3}\begin{pmatrix} 1+S_{33,z} & S_{32,x} - iS_{32,y} & S_{31,x} - iS_{31,y} \\ S_{32,x} + iS_{32,y} & 1+S_{22,z} & S_{21,x} - iS_{21,y} \\ S_{31,x} + iS_{31,y} & S_{21,x} + iS_{21,y} & 1 - S_{22,z} - S_{33,z} \end{pmatrix} \quad (3.3)$$

Comparing Eqs. 3.2 and 3.3, we obtain the following expressions for S_{ij} in terms of a_{ij}

$$S_{22,z} = 3|a_2|^2 - 1 \quad (3.4a)$$

$$S_{33,z} = 3|a_3|^2 - 1 \tag{3.4b}$$

$$s_{21,x} = \frac{3}{2}\left(a_1 a_2^* + a_1^* a_2\right) = 3Re\left\{a_1 a_2^*\right\} \tag{3.4c}$$

$$s_{31,x} = \frac{3}{2}\left(a_1 a_3^* + a_1^* a_3\right) = 3Re\left\{a_1 a_3^*\right\} \tag{3.4d}$$

$$s_{32,x} = \frac{3}{2}\left(a_2 a_3^* + a_2^* a_3\right) = 3Re\left\{a_2 a_3^*\right\} \tag{3.4e}$$

$$s_{21,y} = -i\frac{3}{2}\left(a_1 a_2^* - a_1^* a_2\right) = 3Im\left\{a_1 a_2^*\right\} \tag{3.4f}$$

$$s_{31,y} = -i\frac{3}{2}\left(a_1 a_3^* - a_1^* a_3\right) = 3Im\left\{a_1 a_3^*\right\} \tag{3.4g}$$

$$s_{32,y} = -i\frac{3}{2}\left(a_2 a_3^* - a_2^* a_3\right) = 3Im\left\{a_2 a_3^*\right\} \tag{3.4h}$$

where $Re\{\}$ and $Im\{\}$ denote real and imaginary parts, respectively.

From Eq. 3.4, we see that all eight parameters of the density matrix are real. The parameter $S_{33,z}$ and $S_{22,z}$ are related to the populations of the upper levels $|3\rangle$ and $|2\rangle$. When the atom is certainly in level $|3\rangle$ so that $\rho_{33} = 1$, we obtain $S_{33,z} = 2$ and $S_{22,z} = -1$. On the other hand, when the atom is certainly not in level $|3\rangle$ so that $\rho_{33} = 0$, we obtain $S_{33,z} = -1$ and $S_{22,z} = 2$. Therefore, the population parameters $S_{33,z}$ and $S_{22,z}$ vary between 2 and -1.

$$\rho_{33} = 1 \quad \Rightarrow \quad \begin{cases} S_{33,z} = 2 \\ S_{22,z} = -1 \end{cases} \tag{3.5a}$$

$$\rho_{33} = 0 \quad \Rightarrow \quad \begin{cases} S_{33,z} = -1 \\ S_{22,z} = 2 \end{cases} \tag{3.5b}$$

Apart from the population parameters $S_{33,z}$ and $S_{22,z}$, all the other six parameters of the density matrix are coherence parameters related to interference effects between the atomic levels $|1\rangle$, $|2\rangle$ and $|3\rangle$. The quantity

$$S_{33,z} - S_{22,z} = 3\left(|a_3|^2 - |a_2|^2\right) \tag{3.6}$$

is related to the population inversion between the upper levels $|3\rangle$, and is relevant for three-level atoms in the Λ configuration, as will be seen in section (4). Taking the square of the density matrix in parameter form (Eq. 3.3), and then applying Eq. 2.13 to equate the trace of this square matrix to 1, we obtain,

$$S_{21,x}^2 + S_{21,y}^2 + S_{22,z}^2 + S_{31,x}^2 + S_{31,y}^2 + S_{33,z}^2 + S_{32,x}^2 + S_{32,y}^2 + S_{22,z}S_{33,z} = 3 \tag{3.7}$$

which can also be obtained directly from Eq. 3.4.

For a general three-level atom, the density matrix given by Eq. 3.3 is expanded in terms of the atomic operators σ_{ij} of Eq. 2.1, in accordance with Eq. 2.12a as

$$\rho(t) = \sum_{i,j=1}^{3} \rho_{ij} = \frac{1}{3}\begin{bmatrix} \left(1+S_{33,z}\right)\sigma_{33} + \left(S_{32,x} - iS_{32,y}\right)\sigma_{32} + \\ \left(S_{31,x} - iS_{31,y}\right)\sigma_{31} + \left(S_{32,x} + iS_{32,y}\right)\sigma_{23} + \\ \left(1+S_{22,z}\right)\sigma_{22} + \left(S_{21,x} - iS_{21,y}\right)\sigma_{21} + \\ \left(S_{31,x} + iS_{31,y}\right)\sigma_{13} + \left(S_{21,x} + iS_{21,y}\right)\sigma_{12} + \\ \left(1-S_{22,z}-S_{33,z}\right)\sigma_{11} \end{bmatrix} \tag{3.8}$$

Moreover, the total Hamiltonian of a general three-level atom is obtained from the general Hamiltonian of an n-level atom (Eq. 2.8) to be,

$$H = \hbar\begin{pmatrix} \omega_1\sigma_{11} + \omega_2\sigma_{22} + \omega_3\sigma_{33} - \\ \beta_{12}\sigma_{12} - \beta_{21}\sigma_{21} - \beta_{13}\sigma_{13} - \\ \beta_{31}\sigma_{31} - \beta_{23}\sigma_{23} - \beta_{32}\sigma_{32} \end{pmatrix} \tag{3.9}$$

where we have used Eq. 2.7 to set $\beta_{ij} = 0$ for $i = j$. Using Eq. 3.8 and Eq. 3.9 in Eq. 2.11, we obtain, after considerable careful accounting of terms, the following eight coupled equations for time evolutions of the eight parameters of the density matrix of a general three-level atom defined by Eq. 3.3

$$\dot{S}_{21,x} = -i\left(\frac{\beta_{21}-\beta_{12}}{2}\right)\left(2S_{22,z}+S_{33,z}\right) - i\left(\frac{\beta_{32}-\beta_{23}}{2}\right)S_{31,x} - i\left(\frac{\beta_{31}-\beta_{13}}{2}\right)S_{32,x} - \omega_{21}S_{21,y} + \left(\frac{\beta_{31}+\beta_{13}}{2}\right)S_{32,y} + \left(\frac{\beta_{32}+\beta_{23}}{2}\right)S_{31,y} \tag{3.10a}$$

$$\dot{S}_{21,y} = -i\left(\frac{\beta_{32}-\beta_{23}}{2}\right)S_{31,y} + i\left(\frac{\beta_{31}-\beta_{13}}{2}\right)S_{32,y} + \omega_{21}S_{21,x} + \left(\frac{\beta_{21}+\beta_{12}}{2}\right)\left(2S_{22,z}+S_{33,z}\right) + \left(\frac{\beta_{31}+\beta_{13}}{2}\right)S_{32,x} - \left(\frac{\beta_{32}+\beta_{23}}{2}\right)S_{31,x} \tag{3.10b}$$

$$\dot{S}_{22,z} = i(\beta_{21} - \beta_{12})S_{21,x} - i(\beta_{32} - \beta_{23})S_{32,x} - (\beta_{21} + \beta_{12})S_{21,y} + (\beta_{32} + \beta_{23})S_{32,y} \tag{3.10c}$$

$$\dot{S}_{31,x} = -i\left(\frac{\beta_{31} - \beta_{13}}{2}\right)(2S_{33,z} + S_{22,z}) + i\left(\frac{\beta_{32} - \beta_{23}}{2}\right)S_{21,x} - i\left(\frac{\beta_{21} - \beta_{12}}{2}\right)S_{32,x} - \omega_{31}S_{31,y} - \left(\frac{\beta_{21} + \beta_{12}}{2}\right)S_{32,y} + \left(\frac{\beta_{32} + \beta_{23}}{2}\right)S_{21,y} \tag{3.10d}$$

$$\dot{S}_{31,y} = i\left(\frac{\beta_{32} - \beta_{23}}{2}\right)S_{21,y} - i\left(\frac{\beta_{21} - \beta_{12}}{2}\right)S_{32,y} + \omega_{31}S_{31,x} + \left(\frac{\beta_{31} + \beta_{13}}{2}\right)(2S_{33,z} + S_{22,z}) + \left(\frac{\beta_{21} + \beta_{12}}{2}\right)S_{32,x} - \left(\frac{\beta_{32} + \beta_{23}}{2}\right)S_{21,x} \tag{3.10e}$$

$$\dot{S}_{33,z} = i(\beta_{31} - \beta_{13})S_{31,x} + i(\beta_{32} - \beta_{23})S_{32,x} - (\beta_{31} + \beta_{13})S_{31,y} - (\beta_{32} + \beta_{23})S_{32,y} \tag{3.10f}$$

$$\dot{S}_{32,x} = i\left(\frac{\beta_{31} - \beta_{13}}{2}\right)S_{21,x} + i\left(\frac{\beta_{21} - \beta_{12}}{2}\right)S_{31,x} - i\left(\frac{\beta_{32} - \beta_{23}}{2}\right)(S_{33,z} - S_{22,z}) - \omega_{32}S_{32,y} - \left(\frac{\beta_{31} + \beta_{13}}{2}\right)S_{21,y} - \left(\frac{\beta_{21} + \beta_{12}}{2}\right)S_{31,y} \tag{3.10g}$$

$$\dot{S}_{32,y} = -i\left(\frac{\beta_{31} - \beta_{13}}{2}\right)S_{21,y} + i\left(\frac{\beta_{21} - \beta_{12}}{2}\right)S_{31,y} + \omega_{32}S_{32,x} - \left(\frac{\beta_{31} + \beta_{13}}{2}\right)S_{21,x} + \left(\frac{\beta_{21} + \beta_{12}}{2}\right)S_{31,x} + \left(\frac{\beta_{32} + \beta_{23}}{2}\right)(S_{33,z} - S_{22,z}) \tag{3.10h}$$

Here, a dot over a symbol represents time derivative, and $\omega_{ij} = \omega_i - \omega_j$ is the frequency difference between the atomic levels $|i\rangle$ and $|j\rangle$, as defined by Eq. 2.9.

Eqs. 3.10 are general equations applicable for any three-level atom interacting with a radiation field in the optical region under the electric dipole approximation. In these equations, all coefficients containing β_{ij} and $\beta_{ji} = \beta_{ij}^*$ are of the form,

$$\pm i(\beta_{ij} - \beta_{ji}) = \pm 2\,Im\{\beta_{ij}\} \quad or \quad \pm(\beta_{ij} - \beta_{ji}) = \pm 2\,Re\{\beta_{ij}\} \tag{3.11}$$

and, therefore, are real, indicating that Eqs.3.10 are equations for real variables with real coefficients. These equations are simplified a great deal if we assume the atomic dipole matrix elements $\vec{d}_{31}$, $\vec{d}_{32}$, and $\vec{d}_{21}$ associated with the atomic transitions $|3\rangle \leftrightarrow |1\rangle$, $|3\rangle \leftrightarrow |2\rangle$, and $|2\rangle \leftrightarrow |1\rangle$, and (as defined by Eq. 2.3) are all real so that

$$\beta_{31} = \beta_{13} = \Omega_{31}\cos\omega t \quad \beta_{32} = \beta_{23} = \Omega_{32}\cos\omega t \quad \beta_{21} = \beta_{12} = \Omega_{21}\cos\omega t \tag{3.12}$$

where Ω_{31}, Ω_{32} and Ω_{21} (defined by Eq. 2.6) are now real. Using Eq. 3.12 in Eq. 3.10, we obtain

$$\dot{S}_{21,x} = -\omega_{21}S_{21,y} + \beta_{31}S_{32,y} + \beta_{32}S_{31,y} \tag{3.13a}$$

$$\dot{S}_{21,y} = \omega_{21}S_{21,x} + 2\beta_{21}S_{22,z} + \beta_{21}S_{33,z} + \beta_{31}S_{32,x} - \beta_{32}S_{31,x} \tag{3.13b}$$

$$\dot{S}_{22,z} = -2\beta_{21}S_{21,y} + 2\beta_{32}S_{32,y} \tag{3.13c}$$

$$\dot{S}_{31,x} = -\omega_{31}S_{31,y} - \beta_{21}S_{32,y} + \beta_{32}S_{21,y} \tag{3.13d}$$

$$\dot{S}_{31,y} = \omega_{31}S_{31,x} + 2\beta_{31}S_{33,z} + \beta_{31}S_{22,z} + \beta_{21}S_{32,x} - \beta_{32}S_{21,x} \tag{3.13e}$$

$$\dot{S}_{33,z} = -2\beta_{31}S_{31,y} - 2\beta_{32}S_{32,y} \tag{3.13f}$$

$$\dot{S}_{32,x} = -\omega_{32}S_{32,y} - \beta_{31}S_{21,y} - \beta_{21}S_{31,y} \tag{3.13g}$$

$$\dot{S}_{32,y} = \omega_{32}S_{32,x} - \beta_{31}S_{21,x} + \beta_{21}S_{31,x} + \beta_{32}S_{33,z} - \beta_{32}S_{22,z} \tag{3.13h}$$

These equations govern the time evolutions of the matrix elements of the density matrix of a general three-level atom (Eq. 3.3) when the matrix elements of the dipole moment of the atom are assumed to be real. They can be cast in matrix form as

$$\frac{d\vec{S}}{dt} = \Omega\vec{S} \tag{3.14}$$

where $\vec{S}$ is a vector in an 8-dimensional space given in column form as

$$\vec{S} = \left(S_{21,x}, S_{21,y}, S_{22,z}, S_{31,x}, S_{31,y}, S_{33,z}, S_{32,x}, S_{32,y}\right) \tag{3.15}$$

and Ω is an 8×8 coefficient matrix given by

$$\Omega = \begin{pmatrix} 0 & -\omega_{21} & 0 & 0 & \beta_{32} & 0 & 0 & \beta_{31} \\ \omega_{21} & 0 & 2\beta_{21} & -\beta_{32} & 0 & \beta_{21} & \beta_{31} & 0 \\ 0 & -2\beta_{21} & 0 & 0 & 0 & 0 & 0 & 2\beta_{32} \\ 0 & \beta_{32} & 0 & 0 & -\omega_{31} & 0 & 0 & -\beta_{21} \\ -\beta_{32} & 0 & \beta_{31} & \omega_{31} & 0 & 2\beta_{31} & \beta_{21} & 0 \\ 0 & 0 & 0 & 0 & -2\beta_{31} & 0 & 0 & -2\beta_{32} \\ 0 & -\beta_{31} & 0 & 0 & -\beta_{21} & 0 & 0 & -\omega_{32} \\ -\beta_{31} & 0 & -\beta_{32} & \beta_{21} & 0 & \beta_{32} & \omega_{32} & 0 \end{pmatrix} \tag{3.16}$$

According to Eq. 3.7, the norm of vector $\vec{S}$ is given by,

$$\begin{aligned} S^2 = \left\|\vec{S}\right\|^2 &= S_{21,x}^2 + S_{21,y}^2 + S_{22,z}^2 + \\ &\quad S_{31,x}^2 + S_{31,y}^2 + S_{33,z}^2 + \\ &\quad S_{32,x}^2 + S_{32,y}^2 \\ &= 3 - S_{22,z}S_{33,z} \end{aligned} \tag{3.17}$$

Therefore, $\left\|\vec{S}\right\|$ is not constant but depends on the population parameters $S_{22,z}$ and $S_{33,z}$. However, if we introduce the normalized population parameters $S'_{22,z}$ and $S'_{33,z}$ by

$$S_{22,z}^2 + S_{33,z}^2 + S_{22,z}S_{33,z} = S'^2_{22,z} + S'^2_{33,z} \tag{3.18a}$$

so that

$$S'_{22,z} = \frac{\sqrt{3}}{2}\left(S_{22,z} + S_{33,z}\right) \ , \ S'_{33,z} = \frac{1}{2}\left(S_{22,z} - S_{33,z}\right) \tag{3.18b}$$

the norm of the new vector $\vec{S}'$ defined by

$$\vec{S'} = \left(S_{21,x}, S_{21,y}, S'_{22,z}, S_{31,x}, S_{31,y}, S'_{33,z}, S_{32,x}, S_{32,y}\right) \tag{3.19}$$

will be a constant and is given by,

Therefore, there is a conservation law associated with Eq. 3.13 when the population parameters $S_{22,z}$ and $S_{33,z}$ are replaced by their normalized versions $S'_{22,z}$ and $S'_{33,z}$. This constraint on $\left\|\vec{S}\right\|$ is nothing but another way of expressing the probability conservation of Eq.2.10 in terms of the parameters of the density matrix.

$$\begin{aligned} S'^2 = \left\|\vec{S'}\right\|^2 &= S_{21,x}^2 + S_{21,y}^2 + S'^2_{22,z} + \\ &\quad S_{31,x}^2 + S_{31,y}^2 + S'^2_{33,z} + \\ &\quad S_{32,x}^2 + S_{32,y}^2 \\ &= 3 \end{aligned} \tag{3.20}$$

3.2. Rotating Wave Approximation for a General Three-Level Atom

Eq. 3.13 contain terms which vary rapidly at the optical frequencies ω_{31}, ω_{21}, ω_{32} and ω. Switching to not so rapidly varying variables will render the dynamics of the system more transparent. This can be achieved by an orthogonal transformation - a linear transformation in which the sum of the squares of the new variables is equal to the sum of the squares of the old variables [10]. The necessary and sufficient condition for a linear transformation to be orthogonal is that the transformation matrix M be orthogonal (that is, the transpose M^T of M be equal to its inverse M^{-1}). If M is orthogonal, then its determinant is necessarily ± 1. If $detM = 1$, the transformation is called proper rotation because it involves only rotation, and no reflection. On the other hand, if $detM = -1$ the transformation involves reflection on all or some of the axes in addition to rotation [10].

An orthogonal matrix representing pure (proper) rotation in an 8-dimensional state space is given by,

$$M = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\omega t & \cos\omega t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\omega t & -\sin\omega t & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\omega t & \cos\omega t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\omega t & -\sin\omega t \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\omega t & \cos\omega t \end{pmatrix} \tag{3.21}$$

For the matrix M given by Eq. 3.21, direct multiplication shows that $M^T M = MM^T = 1$ (where 1 is the 8×8 identity matrix) so that $M^T = M^{-1}$. Moreover, $detM = 1$, proving our assertion that M is orthogonal, and, therefore, represents proper rotation. Under M, the 8-dimensional vector $\vec{S}$ of Eq. 3.15 orthogonally transforms to another 8-dimensional vector $\vec{Q}$ which, by the very definition of orthogonal transformation, has the same norm as $\vec{S}$

$$\vec{S} = M\vec{Q} \tag{3.22a}$$

$$\vec{S} = \left(S_{21,x}, S_{21,y}, S_{22,z}, S_{31,x}, S_{31,y}, S_{33,z}, S_{32,x}, S_{32,y}\right) \tag{3.22b}$$

$$\vec{Q} = \left(Q_{21,x}, Q_{21,y}, Q_{22,z}, Q_{31,x}, Q_{31,y}, Q_{33,z}, Q_{32,x}, Q_{32,y}\right) \tag{3.22c}$$

$$\begin{aligned} Q^2 = \left\|\vec{Q}\right\|^2 &= Q_{21,x}^2 + Q_{21,y}^2 + Q_{22,z}^2 + \\ &\quad Q_{31,x}^2 + Q_{31,y}^2 + Q_{33,z}^2 + \\ &\quad Q_{32,x}^2 + Q_{32,y}^2 \\ &= \left\|\vec{S}\right\|^2 \end{aligned} \tag{3.22d}$$

Differentiating both sides of the transformation equation $\vec{S} = M\vec{Q}$ with respect to time, rearranging results, and then using $d\vec{S}/dt = \Omega\vec{S}$ from Eq. 3.14, we obtain

$$M\frac{d\vec{Q}}{dt} = \frac{d\vec{S}}{dt} - \frac{\partial M}{\partial t}\vec{Q} = \Omega M\vec{Q} - \frac{\partial M}{\partial t}\vec{Q} \tag{3.23}$$

Multiplying both sides of Eq. 3.23 from the right by M^T, we finally obtain

$$\frac{d\vec{Q}}{dt} = M^T\left[\Omega M - \frac{\partial M}{\partial t}\right]\vec{Q} \tag{3.24}$$

This is the equation of motion for the components of the transformed vector $\vec{Q}$ obtained from the original vector $\vec{S}$ by the orthogonal transformation $\vec{S} = M\vec{Q}$. It is a general equation valid for any orthogonal transformation of the form $\vec{S} = M\vec{Q}$, independent of the dimension of vector $\vec{S}$, and the corresponding sizes of the coefficient matrix Ω and the transformation matrix M.

Using Eq. 3.16 for the coefficient matrix Ω, and Eq. 3.21 for the transformation matrix M in Eq. 3.24, we obtain (after considerable algebra) the following eight equations for the time evolution of the eight components of vector $\vec{Q}$.

$$\begin{aligned} \dot{Q}_{21,x} &= -\delta_{21}Q_{21,y} + \Omega_{21}\left(\sin 2\omega t\right)Q_{22,z} \\ &\quad + \Omega_{32}\left(\cos\omega t\right)Q_{31,y} + \frac{1}{2}\Omega_{21}\left(\sin 2\omega t\right)Q_{33,z} \\ &\quad + \frac{1}{2}\Omega_{31}\left(\sin\omega t + \sin 3\omega t\right)Q_{32,x} \\ &\quad + \Omega_{31}\left(\cos 2\omega t\right)Q_{32,y} \end{aligned} \tag{3.25a}$$

$$\begin{aligned} \dot{Q}_{21,y} &= \delta_{21}Q_{21,x} + \Omega_{21}\left(1+\cos 2\omega t\right)Q_{22,z} \\ &\quad - \Omega_{32}\left(\cos\omega t\right)Q_{31,x} + \frac{1}{2}\Omega_{21}\left(1+\cos 2\omega t\right)Q_{33,z} \\ &\quad + \frac{1}{2}\Omega_{31}\left(\cos\omega t + \cos 3\omega t\right)Q_{32,x} \\ &\quad - \frac{1}{2}\Omega_{31}\left(\sin\omega t + \sin 3\omega t\right)Q_{32,y} \end{aligned} \tag{3.25b}$$

$$\begin{aligned} \dot{Q}_{22,z} &= -\Omega_{21}\left(\cos 2\omega t\right)Q_{21,x} \\ &\quad - \Omega_{21}\left(1+\cos 2\omega t\right)Q_{21,y} \\ &\quad + \Omega_{32}\left(\sin 2\omega t\right)Q_{32,x} \\ &\quad + \Omega_{32}\left(1+\cos 2\omega t\right)Q_{32,y} \end{aligned} \tag{3.25c}$$

$$\begin{aligned} \dot{Q}_{31,x} &= \Omega_{32}\left(\cos\omega t\right)Q_{21,y} \\ &\quad + \frac{1}{2}\Omega_{31}\left(\sin 2\omega t\right)Q_{22,z} - \delta_{31}Q_{31,y} \\ &\quad + \Omega_{31}\left(\sin 2\omega t\right)Q_{33,z} - \Omega_{21}\left(\cos\omega t\right)Q_{32,y} \end{aligned} \tag{3.25d}$$

$$\begin{aligned} \dot{Q}_{31,y} &= -\Omega_{32}\left(\cos\omega t\right)Q_{21,x} \\ &\quad + \frac{1}{2}\Omega_{31}\left(1+\cos 2\omega t\right)Q_{22,z} + \delta_{31}Q_{31,x} \\ &\quad + \Omega_{31}\left(1+\cos 2\omega t\right)Q_{33,z} + \Omega_{21}\left(\cos\omega t\right)Q_{32,x} \end{aligned} \tag{3.25e}$$

$$\begin{aligned} \dot{Q}_{33,z} &= -\Omega_{31}\left(\sin 2\omega t\right)Q_{31,x} \\ &\quad - \Omega_{31}\left(1+\cos 2\omega t\right)Q_{31,y} \\ &\quad - \Omega_{32}\left(\sin 2\omega t\right)Q_{32,x} \\ &\quad - \Omega_{32}\left(1+\cos 2\omega t\right)Q_{32,y} \end{aligned} \tag{3.25f}$$

$$\begin{aligned} \dot{Q}_{32,x} &= -\frac{1}{2}\Omega_{31}\left(\sin\omega t + \sin 3\omega t\right)Q_{21,x} \\ &\quad - \frac{1}{2}\Omega_{31}\left(\cos\omega t + \cos 3\omega t\right)Q_{21,y} \\ &\quad - \frac{1}{2}\Omega_{32}\left(\sin 2\omega t\right)Q_{22,z} - \Omega_{21}\left(\cos\omega t\right)Q_{31,y} \\ &\quad + \frac{1}{2}\Omega_{32}\left(\sin 2\omega t\right)Q_{33,z} - \delta_{32}Q_{32,y} \end{aligned} \tag{3.25g}$$

$$\dot{Q}_{32,y} = -\frac{1}{2}\Omega_{31}\left(\cos\omega t + \cos 3\omega t\right)Q_{21,x} + \frac{1}{2}\Omega_{31}\left(\sin\omega t + \sin 3\omega t\right)Q_{21,y} - \frac{1}{2}\Omega_{32}\left(1+\cos 2\omega t\right)Q_{22,z} + \Omega_{21}\left(\cos\omega t\right)Q_{31,x} + \frac{1}{2}\Omega_{32}\left(1+\cos 2\omega t\right)Q_{33,z} + \delta_{32}Q_{32,x} \quad (3.25h)$$

In these equations, $\delta_{ij} = \omega_{ij} - \omega$ is the detuning of the atomic transition frequency ω_{ij} from the applied laser field frequency ω, as defined by Eq. 2.9

In Eqs. 3.25, the terms containing ωt, $2\omega t$, and $3\omega t$ represent high frequency evolutions, and, therefore, average out to zero over the evolution of the slowly varying terms containing the detuning frequencies $\delta_{ij} = \omega_{ij} - \omega$. For this reason, these rapidly varying terms can be ignored to a good approximation, and this is what is meant by the rotating wave approximation (RWA) [1]. Therefore, when the RWA is invoked in Eqs. 3.25, we obtain,

$$\dot{Q}_{21,x} = -\delta_{21}Q_{21,y} \quad (3.26a)$$

$$\dot{Q}_{21,y} = \delta_{21}Q_{21,x} + \frac{1}{2}\Omega_{21}\left(2Q_{22,z} + Q_{33,z}\right) \quad (3.26b)$$

$$\dot{Q}_{22,z} = -\Omega_{21}Q_{21,y} + \Omega_{32}Q_{32,y} \quad (3.26c)$$

$$\dot{Q}_{31,x} = -\delta_{31}Q_{31,y} \quad (3.26d)$$

$$\dot{Q}_{31,y} = \delta_{31}Q_{31,x} + \frac{1}{2}\Omega_{31}\left(Q_{22,z} + 2Q_{33,z}\right) \quad (3.26e)$$

$$\dot{Q}_{33,z} = -\Omega_{31}Q_{31,y} - \Omega_{32}Q_{32,y} \quad (3.26f)$$

$$\dot{Q}_{32,x} = -\delta_{32}Q_{32,y} \quad (3.26g)$$

$$\dot{Q}_{32,y} = \delta_{32}Q_{32,x} + \frac{1}{2}\Omega_{32}\left(Q_{33,z} - Q_{22,z}\right) \quad (3.26h)$$

Eqs. 3.26h-h are the principal equations of this paper. They are the optical Bloch equations for a general three-level atom interacting with a linearly polarized light in the visible region under the electric dipole and rotating wave approximations, and under the assumption that the matrix elements of the atomic dipole are all real. They reduce to the Bloch equations for all possible special cases of a general three-level atom as well as to the Bloch equations for a two-level atom as shown in the next section.

4. Optical Bloch Equations for Special Cases of a General Three-Level Atoms

4.1. Optical Bloch Equations for Two-Level Atom

A general three-level atom with three atomic transitions $|3\rangle \leftrightarrow |1\rangle$, $|3\rangle \leftrightarrow |2\rangle$, and $|2\rangle \leftrightarrow |1\rangle$ reduces to a two-level atom [Fig. 1(a)] when only one of these three transitions is dipole allowed. The density matrix of such a two-level atom can be written in terms of the parameters of the density matrix of a three-level atom (Eq. 3.3) as follows.

density matrix for a two-level atom with levels |3> *and* |1>

$$\rho(t) = \frac{1}{2}\begin{pmatrix} 1+S_{33,z} & S_{31,x} - iS_{31,y} \\ S_{31,x} + iS_{31,y} & 1-S_{33,z} \end{pmatrix} \quad (4.1a)$$

density matrix for a two-level atom with levels |2> *and* |1>

$$\rho(t) = \frac{1}{2}\begin{pmatrix} 1+S_{22,z} & S_{21,x} - iS_{21,y} \\ S_{21,x} + iS_{21,y} & 1-S_{22,z} \end{pmatrix} \quad (4.1b)$$

density matrix for a two-level atom with levels |3> *and* |2>

$$\rho(t) = \frac{1}{2}\begin{pmatrix} 1+S_{33,z} & S_{32,x} - iS_{32,y} \\ S_{32,x} + iS_{32,y} & 1+S_{22,z} \end{pmatrix} \quad (4.1c)$$

From Eq. 4.1c we see that, when a three-level atom is considered as a two-level atom consisting of levels $|3\rangle$ and $|2\rangle$, the parameters $S_{33,z}$ and $S_{22,z}$ must be equal and opposite so that the trace of the density matrix is equal to unity, in accordance with Eq. 2.12d.

for two-level atom with levels |3> *and* |2>

$$S_{22,z} = -S_{33,z} \quad \Rightarrow \quad S_{33,z} - S_{22,z} = 2S_{33,z} \quad (4.2)$$

For a two-level atom consisting of levels $|3\rangle$ and $|1\rangle$, the only relevant quantities in Eqs. 3.26a-h are those related to the transition $|3\rangle \leftrightarrow |1\rangle$, namely $Q_{31,x}$, $Q_{31,y}$, $Q_{33,z}$, δ_{31} and Ω_{31}. Similarly, for a two-level atom consisting of levels $|2\rangle$ and $|1\rangle$, the only relevant quantities in Eqs. 3.26a-h are those related to the transition $|2\rangle \leftrightarrow |1\rangle$, namely $Q_{21,x}$, $Q_{21,y}$, $Q_{22,z}$, δ_{21} and Ω_{21}. Likewise, for a two-level atom consisting of levels $|3\rangle$ and $|2\rangle$, the only relevant quantities in Eqs. 3.26a-h are those related to the transition $|3\rangle \leftrightarrow |2\rangle$, namely $Q_{32,x}$,

$Q_{32,y}$, $Q_{33,z}$, $Q_{22,z}$, δ_{32} and Ω_{32} where, according to Eq. 4.2, $Q_{22,z} = -Q_{33,z}$. The general Bloch equations for a general three-level atom (Eqs. 3.26a-h) reduce to those of a two-level atom when all quantities non-relevant to the atom are set to zero. As a result, we obtain,

optical Bloch equations for a two-level atom with levels |3> and |1>

$$\begin{aligned}\dot{Q}_{31,x} &= -\delta_{31} Q_{31,y} \\ \dot{Q}_{31,y} &= \delta_{31} Q_{31,x} + \Omega_{31} Q_{33,z} \\ \dot{Q}_{33,z} &= -\Omega_{31} Q_{31,y}\end{aligned} \tag{4.3a}$$

optical Bloch equations for a two-level atom with levels |2> and |1>

$$\begin{aligned}\dot{Q}_{21,x} &= -\delta_{21} Q_{21,y} \\ \dot{Q}_{21,y} &= \delta_{21} Q_{21,x} + \Omega_{21} Q_{22,z} \\ \dot{Q}_{22,z} &= -\Omega_{21} Q_{21,y}\end{aligned} \tag{4.3b}$$

optical Bloch equations for a two-level atom with levels |3> and |12 >

$$\begin{aligned}\dot{Q}_{32,x} &= -\delta_{32} Q_{32,y} \\ \dot{Q}_{32,y} &= \delta_{32} Q_{32,x} + \Omega_{32} Q_{33,z} \\ \dot{Q}_{33,z} &= -\Omega_{32} Q_{32,y}\end{aligned} \tag{4.3c}$$

These optical Bloch equations are identical to the optical Bloch equations derived in Allen and Eberly [1] specifically for two-level atoms, showing that our general Bloch equations for three-level atoms contain those for two-level atoms as special cases.

4.2. Optical Bloch Equations for Three-Level Atoms in the V, Λ, and Cascade Configurations

A three-level atom is said to be in the so called V configuration, if the upper levels $|3\rangle$ and $|2\rangle$ are close to each other but far from the ground level $|1\rangle$, and if these upper levels are of the same symmetry so that the atomic transition $|3\rangle \leftrightarrow |2\rangle$ is not dipole allowed, Figure 1(b). Therefore, for a three-level atom in the V configuration, the only relevant quantities in Eqs. 3.26a-h are $Q_{31,x}$, $Q_{31,y}$, $Q_{33,z}$, δ_{31} and Ω_{31} (associated with the atomic transition $|3\rangle \leftrightarrow |1\rangle$) as well as $Q_{21,x}$, $Q_{21,y}$, $Q_{22,z}$, δ_{21} and Ω_{21} (associated with the atomic transition $|2\rangle \leftrightarrow |1\rangle$). As a result, Eqs. 3.26a-h for a general three-level atom reduce (as they should) to those of a three-level atom in the V configuration when all quantities non-relevant to the configuration are set to zero.

optical Bloch equations for a three-level atom in V configuration

$$\begin{aligned}\dot{Q}_{21,x} &= -\delta_{21} Q_{21,y} \\ \dot{Q}_{21,y} &= \delta_{21} Q_{21,x} + \frac{1}{2}\Omega_{21}\left(2Q_{22,z} + Q_{33,z}\right) \\ \dot{Q}_{22,z} &= -\Omega_{21} Q_{21,y} \\ \dot{Q}_{31,x} &= -\delta_{31} Q_{31,y} \\ \dot{Q}_{31,y} &= \delta_{31} Q_{31,x} + \frac{1}{2}\Omega_{31}\left(Q_{22,z} + 2Q_{33,z}\right) \\ \dot{Q}_{33,z} &= -\Omega_{31} Q_{31,y}\end{aligned} \tag{4.4}$$

In the case of a three-level atom in the Λ configuration, the lower levels $|2\rangle$ and $|1\rangle$ are close to each other but far from the third (highest) level $|3\rangle$, and these lower levels are of the same symmetry so that the atomic transition $|2\rangle \leftrightarrow |1\rangle$ is not dipole allowed, Figure 1(c). Therefore, in such a case, the only relevant quantities in Eqs. 3.26a-h are $Q_{31,x}$, $Q_{31,y}$, $Q_{33,z}$, δ_{31} and Ω_{31} (associated with the atomic transition $|3\rangle \leftrightarrow |1\rangle$) as well as $Q_{32,x}$, $Q_{32,y}$, $Q_{22,z}$, δ_{32} and Ω_{32} (associated with the atomic transition $|3\rangle \leftrightarrow |2\rangle$). As a result, Eqs. 3.26a-h for a general three-level atom reduce to those of a three-level atom in the Λ configuration when all quantities non-relevant to the configuration are set to zero.

optical Bloch equations for a three-level atom in Λ configuration

$$\begin{aligned}\dot{Q}_{31,x} &= -\delta_{31} Q_{31,y} \\ \dot{Q}_{31,y} &= \delta_{31} Q_{31,x} + \frac{1}{2}\Omega_{31}\left(Q_{22,z} + 2Q_{33,z}\right) \\ \dot{Q}_{33,z} &= -\Omega_{31} Q_{31,y} - \Omega_{32} Q_{32,y} \\ \dot{Q}_{32,x} &= -\delta_{32} Q_{32,y} \\ \dot{Q}_{32,y} &= \delta_{32} Q_{32,x} + \frac{1}{2}\Omega_{32}\left(Q_{33,z} - Q_{22,z}\right) \\ \dot{Q}_{22,z} &= \Omega_{32} Q_{32,y}\end{aligned} \tag{4.5}$$

The main difference between the Bloch equations for V and Λ configurations is that, in addition to the

individual populations $Q_{33,z}$ and $Q_{22,z}$ of the upper levels $|3\rangle$ and $|2\rangle$, the Λ configuration (because of its allowed $|3\rangle \leftrightarrow |2\rangle$ transition) involves the population difference $Q_{33,z} - Q_{22,z}$ between these upper levels, as indicated by the second term on the RHS of Eq. 4.5e.

For a three-level atom in the cascade configuration, there is no direct transition from $|3\rangle$ to $|1\rangle$, Figure 1(d). Therefore, in such a case, the only relevant quantities in Eqs. 3.26a-h are $Q_{31,x}$, $Q_{31,y}$, $Q_{33,z}$, δ_{31} and Ω_{31} (associated with the atomic transition $|3\rangle \leftrightarrow |1\rangle$) as well as $Q_{21,x}$, $Q_{21,y}$, $Q_{22,z}$, δ_{21} and Ω_{21} (associated with the atomic transition $|2\rangle \leftrightarrow |1\rangle$). As a result, Eqs. 3.26a-h for a general three-level atom reduce to those of a three-level atom in the cascade configuration when all quantities non-relevant to the configuration are set to zero.

optical Bloch equations for a three-level atom in cascade configuration

$$\begin{aligned}
\dot{Q}_{32,x} &= -\delta_{32} Q_{32,y} \\
\dot{Q}_{32,y} &= \delta_{32} Q_{32,x} + \frac{1}{2}\Omega_{32}\left(Q_{33,z} - Q_{22,z}\right) \\
\dot{Q}_{33,z} &= -\Omega_{32} Q_{32,y} \\
\dot{Q}_{21,x} &= -\delta_{21} Q_{21,y} \\
\dot{Q}_{21,y} &= \delta_{21} Q_{21,x} + \frac{1}{2}\Omega_{21}\left(2Q_{22,z} + Q_{33,z}\right) \\
\dot{Q}_{22,z} &= -\Omega_{21} Q_{21,y} + \Omega_{32} Q_{32,y}
\end{aligned} \tag{4.6}$$

Eqs. 4.4, 4.5, and 4.6 are identical to the optical Bloch equations derived separately for V-configuration, Λ-configuration, and cascade configuration [7,8], showing once again that our general Bloch equations for three-level atoms contain those for V, Λ, and cascade configurations as special cases.

5. Conclusions

We derived a single set of optical Bloch equations for a general three-level atom from which the optical Bloch equations for three-level atoms in the V, Λ, and cascade configurations as well as those for two-level atoms can be easily obtained as special cases. These general optical Bloch equations govern the interaction of a general three-level atom (and all its special cases) with linearly polarized light under the electric dipole and rotating wave approximations and under the assumption that all relevant electric dipoles are real.

References

[1] L. Allen and J.H. Eberly, *Optical Resonance and Two-Level Atoms*, (Wiley, New York, 1975), available as a reprinted version by Dover Publishing.

[2] Cohen-Tannoudji C., Dupont-Roc J., and Grynberg G., *Atom-photon Interactions: Basic processes and applications* (Wiely, New York, 1992).

[3] F. Bloch, *Nuclear Induction*, Phys. Rev. 70, 460-474 (1946).

[4] Woldeyohannes, M and John, S, *Coherent control of spontaneous emission near a photonic band edge, PhD Tutorial*, J. Opt. B, 5, R43-R82 (2003).

[5] Cohen-Tannoudji C., Diu B., and Lalö F., *Quantum Mechanics, Volume 1*, (Wiely, Toronto, 1977).

[6] Loudon R, *The Quantum Theory of Light*, (Oxford, Calendron, 1983).

[7] Meystre P. and Sargen M., *Elements of Quantum Optics*, (Springer, New York, 1991).

[8] Scully M. O. and Zubary S., *Quantum Optics*, (Cambridge University Press, 1997).

[9] Walls D. F. and Milburn G. J., *Quantum Optics*, (Springer, New York, 1995).

[10] Boas M. L, *Mathematical Methods in the Physical Sciences*, (Wiely, New York, 1991).

[11] Woldeyohannes, M and John, S, *Coherent control of spontaneous emission near a photonic band edge*, Phys. Rev. A, 60, 5046-66 (1999).

[12] Woldeyohannes M., Idehenre I, and Hardin T., *Coherent control of cooperative spontaneous emission from two identical three-level atoms in a photonic crystal*, J. Opt. 17 085105(19pp) (2015).