

Solving the Boundary Value Problems for Second-Order Ordinary Differential Equations using a Class of Obrechhoff-Type Block Methods

Taiwo E. Fayode¹, Bola T. Olabode², Emmanuel A. Areo², Ezekiel O. Omole³, Oluwasegun M. Ibrahim^{4,5,6,*}

¹Bamidele Olumilua University of Education, Science and Technology, Ikere-Ekiti, Nigeria

²Federal University of Technology Akure, Nigeria

³Landmark University, Nigeria

⁴University of Texas at Austin, USA

⁵STEAM City Initiative, Nigeria

⁶African Institute for Mathematical Sciences, Rwanda

*Corresponding author: oluwasegun.micheal@aims.ac.rw, contact@steamcityinitiative.org

Received May 13, 2025; Revised June 15, 2025; Accepted June 23, 2025

Abstract: Employing a combination of collocation and interpolation techniques, this research introduces a novel set of Obrechhoff-type methods designed to address second-order boundary value problems (BVPs) characterized by Neumann and Dirichlet boundary conditions. The methodology involves placing the derivative function equation at all grid points and interpolating the basis function at only two locations, resulting in the development of a series of continuous multistep techniques with variable step numbers. For numerical implementation, our approach employs block mode techniques. We investigate the order, consistency, stability, and convergence of these algorithms to ensure their robustness and reliability. To evaluate the efficacy and precision of the proposed method, comprehensive testing is conducted using Obrechhoff-type problems. Notably, the numerical solutions demonstrate improved performance compared to traditional methods, highlighting the potential of our approach to deliver enhanced accuracy in solving second-order BVPs.

Keywords: Boundary value problems, interpolation, collocation, differential equations, numerical methods

Cite This Article: Taiwo E. Fayode, Bola T. Olabode, Emmanuel A. Areo, Ezekiel O. Omole, and Oluwasegun M. Ibrahim, "Solving the Boundary Value Problems for Second-Order Ordinary Differential Equations using a Class of Obrechhoff-Type Block Methods." *American Journal of Applied Mathematics and Statistics*, vol. 13, no. 2 (2025): 30-37. doi: 10.12691/ajams-13-2-2.

1. Introduction

Mathematics is widely utilized and often considered a prerequisite for progress in various fields, such as biology, economics, chemical kinetics, circuit theory, and others. Physical phenomena in these disciplines are frequently examined through the lens of mathematical models, which often yield equations involving derivatives of unidentified functions in one or more variables commonly known as differential equations.

Interestingly, differential equations arising from the modeling of physical events often lack analytical solutions. They form the basis for modeling a diverse range of problems across sciences, technology, engineering, social sciences, and more. Consequently, the development of numerical techniques to approximate solutions becomes imperative. Various numerical methods, including finite difference techniques, finite element techniques, and finite volume techniques, have been devised to address the nature and type of the differential equations at hand.

A boundary value problem comprises a differential

equation along with additional constraints referred to as boundary conditions. A Dirichlet boundary condition, also known as a first-type boundary condition, specifies the values of the functions themselves. On the other hand, a Neumann boundary condition provides details about the normal derivative of the function at the boundary. A solution to a boundary value problem is a solution to the associated differential equation that also satisfies the specified boundary conditions. The main goal of this study is to numerically solve the boundary value problems of the second-order ordinary differential equations (ODEs) of the type

$$y'' = f(x, y, y'), x \in [a, b], y(a) = y_0, y(b) = y_N, \quad (1)$$

Where $f: \mathcal{R} \times \mathcal{R}^{2m} \rightarrow \mathcal{R}^m$ are continuous functions $y, y', y'' \in \mathcal{R}^m$ and m is the system dimension. There are numerous engineering and applied science fields that deal with these second-order boundary value problems. Numerous authors have worked to solve equation (1) utilizing numerical approaches for resolving boundary value problems, which can essentially be divided into the

shooting method, finite element method, and finite difference method. The technique utilized can be divided into two categories: linear multistep methods (LMMs) and one-step (single-step) techniques; see [1,2,3,4,5,6,7]. Evidently, approaches of the Obreckhoff-type are effective for both start and boundary value problems. The well-known Numerov method is the simplest of the Obreckhoff-type approaches. For second-order initial value problems, authors have created a number of Obreckhoff-type approaches with higher orders in the past. Consider the category of second-order boundary value problem with Dirichlet boundary conditions in equation (1),

$$y(a) = \alpha, y(b) = \beta \quad (2)$$

and Neumann boundary conditions

$$y'(a) = \alpha, y'(b) = \beta. \quad (3)$$

Numerous branches of science, engineering, and medicine encounter second-order boundary-value problems. For instance, they can be observed in the bending of Euler-Bernoulli beams Fernandez-Sez et al. [8] and Li et al. [9], curved cantilever beams Ghuku and Saha [10] homogenous-heterogenous flow of fluids over surfaces, Malik and Khan [11], and Hayat et al. [12] among other phenomena. Numerous techniques, including the shooting technique Filipov et al. [13], Block Nystrom type method Jator and Manathunga [14], Quartic B-spline method Goh et al. [15], Boubaker polynomials expansion strategy Koak et al. [16], and the Adomian decomposition method, Aly et al. [17]. The resolution of second-order initial value problems given as follows has been studied by a number of scholars.

$$y'' = f(x, y, y'), y(a) = y_0, y'(a) = y'_0. \quad (4)$$

In reality, problems of kind (4) can be found in circuit theory, biology, celestial mechanics, chemical kinetics, and celestial mechanics. The second-order problem has been widely studied; see Ibrahim and Ikhile [1], Ibrahim and Ikhile [2], Ibrahim and Ikhile [3], Ibrahim and Ikhile [4] and other references herein.

Obreckhoff [18] introduced a series of techniques for solving first-order IVPs in 1942 as

$$\sum_{j=0}^k \alpha_j y_{n+j} = \sum_{i=1}^L h^i \sum_{j=0}^k \beta_{ij} y_{n+j}^{(i)}. \quad (5)$$

Classical Obreckhoff methods $L = 1, 2, \dots, 5$ $-k, k = 1, 2, 3, 4$ were

listed by Lambert [27]. By enhancing and extending Lambert's work on stability as Boundary Value Methods (BVMs), Ghelardoni [20], expanded its scope, while some BVPs were considered in Ibrahim et al. [21]; Ibrahim and Lawrence [22]. The step length k was set at 2, and Usmani's formula [23] was used as the starting technique.

$$-y_n + y_{n+1} = \sum_{i=1}^{r'} h^i \left(\gamma_{i0} y_n^{(i)} + \gamma_{i1} y_{n+1}^{(i)} \right). \quad (6)$$

Lambert [24] observed that, for large k , it is impossible to satisfy the zero-stability of (5), as the error constant decreases more rapidly with an increase in 1 than with k . As a result, its stability domain appears to be limited. Investigators have since conducted studies in this area. Shokri [26] separately created higher-order L-stable Obreckhoff methods with varied step sizes that are precise and effective and produce a bigger stability domain. A fascinating thing about Obreckhoff methods is that they can be extended to solve higher-order ordinary differential equations. Considering this, numerous authors have expanded the Obreckhoff techniques (6) to

$$\sum_{j=0}^k \alpha_j y_{n+j} = \sum_{i=1}^L h^{2i} \sum_{j=0}^k \beta_{ij} y_{n+j}^{(2i)} \quad (7)$$

where the aforementioned equation successfully and reliably resolves second-order initial value problems. The inclusion of higher derivatives makes this family of approaches stand out as distinct. The k -step Obreckhoff method's general form using the derivative y is provided by Lambert [27]. These methods, known as Obreckhoff methods, can be particularly effective with the first few derivatives. Although Obreckhoff's original work primarily addressed numerical quadrature, it appears that Milne [28] was the first to promote the application of the Obreckhoff formula. However, before implementing the method described in (6), additional one-step numerical methods like the Euler or Trapezoidal method must be introduced. This then can be translated to the block method, Ajayi et al. [25].

2. Derivation of the Block Method

The Block Method is derived with the intention of approximating the exact solution $y(x)$ in the partition $a = X_0 < X_1 < \dots < X_{n-1} < X_n < X_{n+1} < \dots < X_N = b$ for $X \in [a, b]$ of a power series polynomial of the form,

$$y(x) = \sum_{j=0}^{2s+r-1} a_j x^j \quad (8)$$

where the numbers of collocation points and interpolating points, respectively, are r and s . The second and fourth derivatives of (8) are thus

$$y''(x) = \sum_{j=2}^{2s+r-1} j(j-1) a_j x^{j-2} = f(x, y) \quad (9)$$

$$y^{iv}(x) = \sum_{j=4}^{2s+r-1} j(j-1)(j-2)(j-3) a_j x^{j-4} = \gamma(x, y) \quad (10)$$

The interpolation, collocation, and collocation equations in equations (8), (9), and (10) will be used to develop the suggested approaches.

The system of equations arriving from (8), (9) and (10) will be expressed in the form of matrix equation as given below.

$$AX = B \quad (11)$$

where

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1s} \\ x_{21} & x_{22} & \dots & \dots \\ x_{31} & x_{32} & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ x_{s1} & x_{s2} & \dots & x_{ss} \end{bmatrix}, A = \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_n \end{bmatrix}, B = \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_n \end{bmatrix} \quad (12)$$

is resolved for the parameters $\square'$'s using the Gaussian elimination approach. Equation (8) requires the parameters to be swapped in order to produce continuous schemes. The approach is then determined by evaluating the continuous schemes at the non-interpolating point.

This results in the equations below

$$y_{n-1} = -h^7 a_7 + h^6 a_6 - h^5 a_5 + h^4 a_4 - h^3 a_3 + h^2 a_2 - h a_1 + a_0$$

$$a_0 = y_n$$

$$f_{n-1} = -42 h^5 a_7 + 30 h^4 a_6 - 20 h^3 a_5 + 12 h^2 a_4 - 6 h a_3 + 2 a_2$$

$$f_n = 2 a_2$$

$$f_{n+1} = 2 h^5 a_7 + 30 h^4 a_6 + 20 h^3 a_5 + 12 h^2 a_4 + 6 h a_3 + 2 a_2$$

$$\gamma_{n-1} = -840 h^3 a_7 + 360 h^2 a_6 - 120 h a_5 + 24 a_4$$

$$\gamma_n = 24 a_4$$

$$\gamma_{n+1} = 840 h^3 a_7 + 360 h^2 a_6 + 120 h a_5 + 24 a_4. \quad (13)$$

Combining (13) into (11) in matrix form gives

$$X = \begin{bmatrix} 1 & -h & h^2 & -h^3 & h^4 & -h^5 & h^6 & -h^7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -6h & 12h^2 & -20h^3 & 30h^4 & -42h^5 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6h & 12h^2 & 20h^3 & 30h^4 & 42h^5 \\ 0 & 0 & 0 & 0 & 24 & -120h & 360h^2 & -840h^3 \\ 0 & 0 & 0 & 0 & 24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 24 & 120h & 360h^2 & 840h^3 \end{bmatrix}, B = \begin{bmatrix} y_{n-1} \\ y_n \\ f_{n-1} \\ f_n \\ f_{n+1} \\ \gamma_{n-1} \\ \gamma_n \\ \gamma_{n+1} \end{bmatrix}, A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}$$

noting

$$AX = B \quad (14)$$

Solving (14) for the values of parameters a_j 's we obtain

$$\begin{aligned} a_0 &= y_n \\ a_1 &= \frac{1}{5040h} (366 h^4 \gamma_n - 25 h^4 \gamma_{n-1} + 73 h^4 \gamma_{n+1} + 2928 h^2 f_n + 216 h^2 f_{n-1} - 624 h^2 f_{n+1} \\ &\quad + 5040 y_n - 5040 y_{n-1}) \\ a_2 &= \frac{1}{2} f_n \\ a_3 &= -\frac{1}{720h} (30 h^2 \gamma_n - 7 h^2 \gamma_{n-1} + 13 h^2 \gamma_{n+1} + 72 f_n + 24 f_{n-1} - 96 f_{n+1}) \\ a_4 &= \frac{1}{24} \gamma_n \\ a_5 &= -\frac{1}{240} (\gamma_{n-1} - \gamma_{n+1}) \\ a_6 &= -\frac{1}{60h^4} (h^2 \gamma_n + 2 f_n - f_{n-1} - f_{n+1}) \\ a_7 &= \frac{1}{1680h^5} (10 h^2 \gamma_n + h^2 \gamma_{n-1} + h^2 \gamma_{n+1} + 24 f_n - 12 f_{n-1} - 12 f_{n+1}). \end{aligned} \quad (15)$$

A linear multistep technique with a continuous coefficient of this type is produced by substituting (15) into (8) to yield

$$y(x) = \alpha_{-1} y_{n-1} + \alpha_0 y_n + h^2 (\beta_0 f_n + \beta_{-1} f_{n-1} + \beta_1 f_{n+1}) + h^4 (g_{-1} \gamma_{n-1} + g_0 \gamma_n + g_1 \gamma_{n+1}) -$$

$$(16)$$

The coefficients of (16) were obtained as follows, using the transformation

$$t = \frac{x - x_{n+k-1}}{h}, k = 1 \quad (17)$$

where

$$\alpha_{-1}(t) = -t$$

$$\alpha_0(t) = 1 + t$$

$$\beta_{-1}(t) = -\frac{1}{420} h^2 t (3 t^6 - 7 t^5 + 14 t^2 - 18)$$

$$\beta_0(t) = \frac{1}{210} h^2 t (3 t^6 - 7 t^5 - 21 t^2 + 105 t + 122)$$

$$\beta_1(t) = \frac{1}{210} h^2 t (3 t^6 - 7 t^5 - 21 t^2 + 105 t + 122)$$

$$\beta_1(t) = -\frac{1}{420} h^2 t (3 t^6 - 7 t^5 - 56 t^2 + 52)$$

$$g_{-1}(t) = \frac{1}{5040} h^4 t (3 t^6 - 21 t^4 + 49 t^2 - 25)$$

$$g_0(t) = \frac{1}{840} h^4 t (5 t^6 - 14 t^5 + 35 t^3 - 35 t^2 + 61)$$

$$g_1(t) = \frac{1}{5040} h^4 t (3 t^6 + 21 t^4 - 91 t^2 + 73). \quad (18)$$

Evaluating (16) at $t=1$, gives the discrete scheme below

$$y_{n+1} + y_{n-1} - 2 y_n = \frac{h^2}{840} (52 h^2 \gamma_n + h^2 \gamma_{n-1} + h^2 \gamma_{n+1} + 808 f_n + 16 f_{n-1} + 16 f_{n+1}). \quad (19)$$

The first derivatives of (16) gives

$$\begin{aligned} hy'(x) &= \alpha'_{-1} y_{n-1} + \alpha'_0 y_n + h^2 (\beta'_1 f_{n-1} + \beta'_0 f_n + \beta'_1 f_{n+1}) \\ &\quad + h^4 (g_{-1} \gamma_{n-1} + g_0 \gamma_n + g_1 \gamma_{n+1}) \end{aligned} \quad (20)$$

where

$$\alpha'_{-1}(t) = -1$$

$$\alpha'_0(t) = 1$$

$$\beta'_{-1}(t) = -\frac{1}{420} h^2 (7 t^6 - 14 t^5 + 14 t^2 - 6)$$

$$\beta'_0(t) = \frac{1}{210} h^2 (21 t^6 - 42 t^5 - 63 t^2 + 210 t + 122)$$

$$\beta'_1(t) = \frac{1}{420} h^2 (21 t^6 - 42 t^5 - 168 t^2 + 52)$$

$$g'_{-1}(t) = \frac{1}{5040} h^4 (21 t^6 - 105 t^4 + 147 t^2 - 25)$$

$$g'_0(t) = \frac{1}{840} h^4 (35 t^6 - 84 t^5 + 140 t^3 - 105 t^2 + 61)$$

$$g'_1(t) = \frac{1}{5040} h^4 (21 t^6 + 105 t^4 - 273 t^2 + 73). \quad (21)$$

Evaluating (21) at $t=-1, 0$, and 1 gives the following first derivative schemes:

$$\begin{aligned} y'_{n-1} &= -\frac{1}{2520h} (195 h^4 \gamma_n - 19 h^4 \gamma_{n-1} + 37 h^4 \gamma_{n+1} + 1056 h^2 f_n + 522 h^2 f_{n-1} \\ &\quad - 318 h^2 f_{n+1} - 2520 y_n + 2520 y_{n-1}) \end{aligned} \quad (22)$$

$$\begin{aligned} y'_n &= \frac{1}{5040h} (366 h^4 \gamma_n - 25 h^4 \gamma_{n-1} + 73 h^4 \gamma_{n+1} + 2928 h^2 f_n + 216 h^2 f_{n-1} - 624 h^2 f_{n+1} \\ &\quad + 5040 y_n - 5040 y_{n-1}) \end{aligned} \quad (23)$$

$$\begin{aligned} y'_{n+1} &= \frac{1}{2520h} (141 h^4 \gamma_n + 19 h^4 \gamma_{n-1} - 37 h^4 \gamma_{n+1} + 2976 h^2 f_n - 18 h^2 f_{n-1} + 822 h^2 f_{n+1} \\ &\quad + 2520 y_n - 2520 y_{n-1}). \end{aligned} \quad (24)$$

2.1. Block form of One-step Obreckhoff-type Method

Combining (19) and (23) give the block method below

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y'_n - 1 \\ y'_n \end{bmatrix} \\ + h^2 \begin{bmatrix} 0 & \frac{101}{105} \\ 0 & \frac{2928}{5040} \end{bmatrix} \begin{bmatrix} f_n \end{bmatrix} + h^2 \begin{bmatrix} \frac{2}{105} & \frac{2}{105} \\ \frac{216}{5040} & \frac{-624}{5040} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n+1} \end{bmatrix} \\ + h^4 \begin{bmatrix} 0 & \frac{13}{210} \\ 0 & \frac{366}{5040} \end{bmatrix} \begin{bmatrix} \gamma_n \end{bmatrix} + h^4 \begin{bmatrix} -\frac{840}{5040} & \frac{840}{5040} \\ \frac{-25}{5040} & \frac{-73}{5040} \end{bmatrix} \begin{bmatrix} \gamma_{n-1} \\ \gamma_{n+1} \end{bmatrix} \quad (25)$$

Solving (25) using matrix inversion gives,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \end{bmatrix} + h \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y'_n - 1 \\ y'_n \end{bmatrix} \\ + h^2 \begin{bmatrix} 0 & \frac{61}{105} \\ 0 & \frac{8}{21} \end{bmatrix} \begin{bmatrix} f_n \end{bmatrix} + h^2 \begin{bmatrix} \frac{3}{70} & \frac{-13}{105} \\ -\frac{1}{42} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n+1} \end{bmatrix} \\ + h^4 \begin{bmatrix} 0 & \frac{61}{840} \\ 0 & -\frac{3}{280} \end{bmatrix} \begin{bmatrix} \gamma_n \end{bmatrix} + h^4 \begin{bmatrix} -\frac{5}{1008} & \frac{73}{5040} \\ \frac{31}{5040} & \frac{-67}{5040} \end{bmatrix} \begin{bmatrix} \gamma_{n-1} \\ \gamma_{n+1} \end{bmatrix} \quad (26)$$

Writing out (26) explicitly gives

$$y_{n-1} = \frac{61}{840} h^4 \gamma_n - \frac{5}{1008} h^4 \gamma_{n-1} + \frac{73}{5040} h^4 \gamma_{n+1} + \frac{61}{105} h^2 f_n + \frac{3}{70} h^2 f_{n-1} - \frac{13}{105} h^2 f_{n+1} \\ + y_n - y'_n h \quad (27)$$

$$y_{n+1} = -\frac{3}{280} h^4 \gamma_n + \frac{31}{5040} h^4 \gamma_{n-1} - \frac{67}{5040} h^4 \gamma_{n+1} + \frac{8}{21} h^2 f_n - \frac{1}{42} h^2 f_{n-1} + \frac{1}{7} h^2 f_{n+1} \\ + y'_n h + y_n \quad (28)$$

Substituting (27) and (28) into (22) and (24) gives

$$y'_{n-1} = -\frac{3}{20} h^3 \gamma_n - h f_n - \frac{1}{4} h f_{n-1} + \frac{1}{4} h f_{n+1} + \frac{1}{80} h^3 \gamma_{n-1} - \frac{7}{240} h^3 \gamma_{n+1} + y'_n \quad (29)$$

$$y'_{n+1} = -\frac{1}{60} h^3 \gamma_n + \frac{1}{80} h^3 \gamma_{n-1} - \frac{7}{240} h^3 \gamma_{n+1} + \frac{3}{5} h f_n - \frac{1}{20} h f_{n-1} + \frac{9}{20} h f_{n+1} + y'_n \quad (30)$$

3. Analysis of the Method

This section presents the analysis of the properties of the method as follows.

3.1. Local Truncation Error and Order

The developed system order was investigated using the methodology described in Lambert (1973) and Fatunla (1991). Equation (31) displays the error constants as having order $p = 5$ and

$$C_{p+2} = -\frac{31}{302400} \quad (31)$$

Exact Solution:

$$y(x) = \frac{-3\cosh(x) + 3\sinh(x) + \cos(x) + 2}{4\sinh(x)} e^x + \frac{3\cosh(x) + 3\sinh(x) - \cos(x) - 2}{4\sinh(x)} e^{-x} - \frac{\cos x}{2}.$$

While the block method's order was looked at, it has order $p = 5$, and the error constants are represented in Equation (32) as

$$C_{p+2} = \left[-\frac{31}{15120}, \frac{31}{15120} \right]^T \quad (32)$$

3.2. Consistency

If a linear multistep approach converges with an order greater than 2, it is considered to be consistent. That is, ($p \geq 2$).

3.3. Zero Stability

For any well-behaved starting value problem, a linear multistep approach is zero-stable if all roots of the first characteristics polynomial lies in the unit disk, and any roots on the unit circle are simple,

$$P(Z) = \det \left[z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right] z \\ = \det \begin{bmatrix} z & -1 \\ 0 & z \end{bmatrix} \quad (33)$$

Equation (33) produces $z = 1$ when set to zero, proving that the approach is zero stable.

3.4. Convergence

Consistency and zero stability are the required and sufficient conditions for our method to be convergent.

3.5. Region of Absolute Stability

As defined by Jator and Manathunga [14] and Ajayi et al. [25], the region of absolute stability of the established schemes is taken into consideration.

4. Numerical Application on Dirichlet Boundary Conditions

In order to evaluate the effectiveness of the produced methods, we consider the following problems and their sources, Majid et al. [29] and Chew et al. [30].

Problem 1: One-Step Obreckhoff Block Method (1SOBM):

$$y'' = y' - e^{x-1} - 1, y(0) = 0, y(1) = 0, h = 0.1.$$

Exact solution: $y(x) = x(1 - e^{x-1})$.

Problem 2: Direct Adams-Moulton Method (DAMD)

$$y'' = y + \cos(x), y(0) = 0, y(1) = 1, h = 0.125.$$

Table 1. Showing the comparison of error in the method (1SOMBM) with Chew et al. [30] for Problem 1

x	y-exact	y-computed	Error in (1SOMBM) p=5, k=1	Error in Chew et al (2012) p=5, k=1
0.1	0.05934303403	0.05934303458	5.500000E-10	1.690000E-06
0.2	0.1101342072	0.1101342095	2.300000E-09	2.430000E-06
0.3	0.1510244089	0.1510244144	5.500000E-09	2.470000E-06
0.4	0.1804753456	0.1804753562	1.060000E-08	2.300000E-06
0.5	0.1967346702	0.1967346880	1.780000E-08	2.090000E-06
0.6	0.1978079724	0.1978080001	1.770000E-08	1.830000E-06
0.7	0.1814272455	0.1814272862	4.070000E-08	1.500000E-06
0.8	0.1450153975	0.1450154550	5.075000E-08	1.090000E-06
0.9	0.08564632380	0.08564640236	7.856000E-08	6.000000E-07
1.0	0.00000000000	1.0494000E-07	1.0494000E-07	-

Table 2. Showing the comparison of error in the method (1SOMBM) with Chew et al. [30] for Problem 2

x	y-exact	y-computed	Error in (1SOMBM) p=5, k=1	Error in Chew et al (2012) p=5, k=1
0.1250	0.0609853492	0.06098534883	3.700000E-10	1.460000E-06
0.2500	0.1384279351	0.1384279333	1.800000E-09	1.790000E-06
0.3750	0.2331755418	0.2331755380	3.800000E-09	1.530000E-06
0.5000	0.3461104542	0.3461104475	6.700000E-09	1.200000E-06
0.6250	0.4781726245	0.4781726139	1.060000E-08	8.840000E-07
0.7500	0.6303872830	0.6303872674	1.560000E-08	5.840000E-07
0.8750	0.8038972219	0.8038971991	2.280000E-08	5.910000E-07
1.0000	1.0000000000	0.9999999699	3.010000E-08	-
1.1250	1.220190365	1.220190324	4.100000E-08	-
1.2500	1.466208177	1.466208126	5.100000E-08	-

Table 3. Showing the exact solutions, the computed results and the error from the 1SOMBM for Problem 1.

x	y-exact	y-computed	Error in (1SOMBM)
0.1	0.05934303403	0.05934303458	5.500000E-10
0.2	0.1101342072	0.1101342095	2.300000E-09
0.3	0.1510244089	0.1510244144	5.500000E-09
0.4	0.1804753456	0.1804753562	1.060000E-08
0.5	0.1967346702	0.1967346880	1.780000E-08
0.6	0.1978079724	0.1978080001	1.770000E-08
0.7	0.1814272455	0.1814272862	4.070000E-08
0.8	0.1450153975	0.1450154550	5.075000E-08
0.9	0.08564632380	0.08564640236	7.856000E-08
1.0	0.00000000000	1.0494000E-07	1.0494000E-07

Table 4. Showing the exact solutions, the computed results and the error from the 1SOMBM for Problem 2.

x	y-exact	y-computed	Error in (1SOMBM)
0.1250	0.0609853492	0.06098534883	3.700000E-10
0.2500	0.1384279351	0.1384279333	1.800000E-09
0.3750	0.2331755418	0.2331755380	3.800000E-09
0.5000	0.3461104542	0.3461104475	6.700000E-09
0.6250	0.4781726245	0.4781726139	1.060000E-08
0.7500	0.6303872830	0.6303872674	1.560000E-08
0.8750	0.8038972219	0.8038971991	2.280000E-08
1.0000	1.0000000000	0.9999999699	3.010000E-08
1.1250	1.220190365	1.220190324	4.100000E-08
1.2500	1.466208177	1.466208126	5.100000E-08

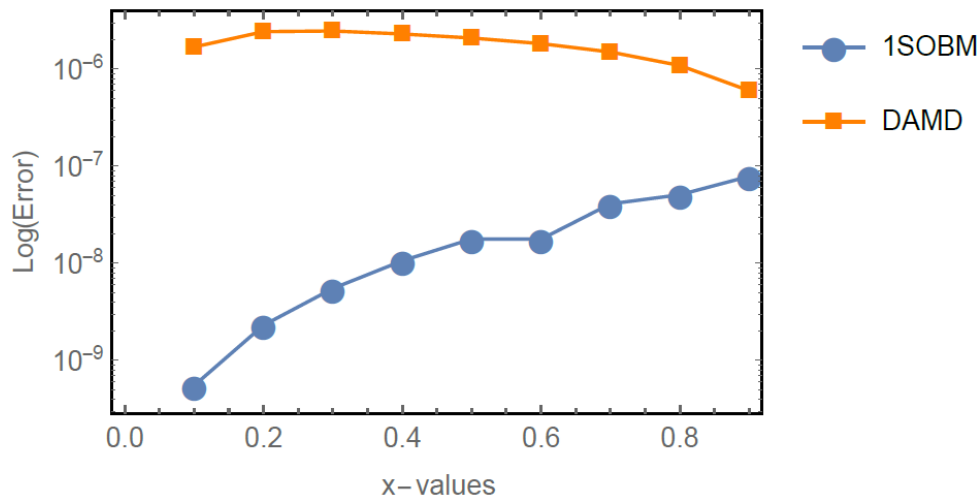


Figure 1. Comparison profile for problem 1

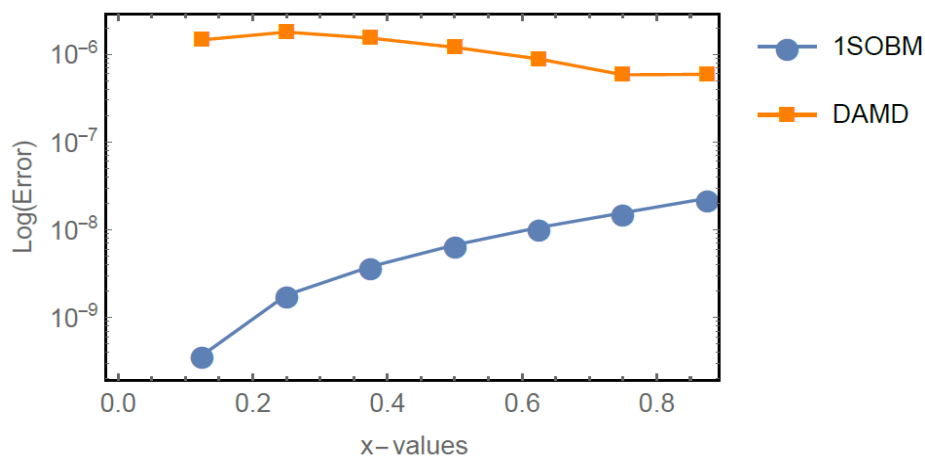


Figure 2. Comparison profile for problem 2

5. Summary and Conclusion

Through the development, rigorous analysis, and effective application of a sophisticated class of Obrechhoff-type block methods have achieved direct solutions for second-order boundary value problems 1 and 2. Our investigation delved into the comparison between Dirichlet and Neumann boundary conditions for second-order boundary value problems within ordinary differential equations. Leveraging continuous coefficients and ensuring exceptional consistency, our collocation approach has given rise to implicit methods of notable accuracy. Notably, these methods demonstrate a remarkable feature: they are implemented seamlessly without the necessity for the development of predictors, eliminating the need for additional methods to generate starting values. The comparison profiles for problems 1 and 2 are visually presented in Figures 1 and 2 above, providing a clear representation of the methods' performance.

What sets our methods apart is their high order, accuracy, and low error constants. This intrinsic superiority contributes to the methods' exceptional proficiency, allowing them to achieve a level of perfection that surpasses conventional approaches in Chew et al. [30]. The culmination of these attributes establishes our

Obrechhoff-type block methods as an impressive and reliable solution for addressing second-order boundary value problems [31].

Funding

This work did not receive any funding.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Availability of Data and Material

No specific data or unique material was used for this work

References

- [1] Ibrahim, O. M, and Ikile, M.N.O. (2020a): A generalized family of symmetric multistep methods with minimal phase-lag for initial value problems in ordinary differential equations. Mediterranean

- Journal of Mathematics 17, 1-30. [https:// link.springer.com/article/10.1007/s00009-020-01507-5](https://link.springer.com/article/10.1007/s00009-020-01507-5).
- [2] Ibrahim, O.M., and Ikile, M.N.O. (2020b): Inverse hybrid linear multistep methods for solving the second order initial value problems in ordinary differential equations. *International Journal of Applied and Computational Mathematics*, 6(6), 158. [https:// link.springer.com/article/10.1007/s40819-020-00910-6](https://link.springer.com/article/10.1007/s40819-020-00910-6).
 - [3] Ibrahim, O.M., and Ikile, M.N.O (2017a): On the construction of high accuracy symmetric super-implicit hybrid formulas with phase-lag properties. *Transaction of the Nigerian Association of Mathematical Physics* 4, 101-108. [https:// www.researchgate.net/publication/ 377330540](https://www.researchgate.net/publication/377330540).
 - [4] Ibrahim, O.M., and Ikile, M.N.O. (2017a): Highly stable super-implicit hybrid methods for special second-order IVPs. *American Journal of Applied Scientific Research*, 3(3), 21-27.
 - [5] Omole, E.O., Adeyefa, E.O., Ayodele, V.I., Shokri, A., Wang, Y. (2023) Ninth-order Multistep Collocation Formulas for Solving Models of PDEs Arising in Fluid Dynamics: Design and Implementation Strategies. *Axioms* 2023, 12, 891.
 - [6] E. Aeao and M. T. Omojola (2015): A new one-twelfth step continuous block method for the solution of modeled problems of ordinary differential equation, *American Journal of Mathematics*. 5(04), 447-450.
 - [7] Rufai, M.A., Shokri, A., Omole, E.O. A (2023): One-Point Third-Derivative Hybrid Multistep Technique for Solving Second-Order Oscillatory and Periodic Problems. *Hindawi J. Math.* 2023, 2023, 2343215.
 - [8] Fernandez-Saez, J., Zaera, R., Loya, J. A., & Reddy, J. (2016). Bending of Euler-Bernoulli beams using Eringen's integral formulation: a paradox resolved. *International Journal of Engineering Science*, 99, 107-116.
 - [9] Li, S. R., Cao, D. F., & Wan, Z. Q. (2013). Bending solutions of FGM Timoshenko beams from those of the homogenous Euler-Bernoulli beams. *Applied Mathematical Modelling*, 37(10-11), 7077-7085.
 - [10] Ghuku S, K.N Saha (2016): A theoretical and experimental study on geometric nonlinearity of initially curved cantilever beams. *Engineering Science and Technology, an International Journal* 19 (1): 135-146.
 - [11] Malik R, M. Khan (2018): Numerical study of homogenous heterogeneous reactions in Sisko fluid flow past a stretching cylinder. *Results in Physics* 8 (2018) 415 121.
 - [12] Hayat T, K. Muhammed, A. Alsaedi, and S. Asghar (2018): Numerical study for melting heat transfer and homogenous-heterogenous reactions in flow involving carbon nanotubes, *Results in Physics* 8 (2018) 415 121.
 - [13] Filipov, S. M., Gospodinov, I. D., & Faragó, I. (2017). Shooting-projection method for two-point boundary value problems. *Applied Mathematics Letters*, 72, 10-15.
 - [14] Jator, S. N., & Manathunga, V. (2018). Block Nyström type integrator for Bratu's equation. *Journal of Computational and Applied Mathematics*, 327, 341-349.
 - [15] Goh, J., Majid, A. A., & Ismail, A. I. M. (2012). A quartic B-spline for second-order singular boundary value problems. *Computers & Mathematics with Applications*, 64(2), 115-120.
 - [16] Koçak, H., Yıldırım, A., Zhang, D. H., & Mohyud-Din, S. T. (2011). The Comparative Boubaker Polynomials Expansion Scheme (BPES) and Homotopy Perturbation Method (HPM) for solving a standard nonlinear second-order boundary value problem. *Mathematical and Computer Modelling*, 54(1-2), 417-422.
 - [17] Aly E. H., Ebaid A, Rach R. (2012): Advances in the Adomian decomposition method for solving two-point nonlinear Boundary Value Problems with Neumann boundary conditions. *Computers and Mathematics with Applications*. 2012; 63(6): 1056–65.
 - [18] Obrechhoff N (1942): On Mechanical quadrature (Bulgarian French summary). *Spisanie Bulgar Akad. Nauk.* 65, 191-289.
 - [19] Lambert J.D. (1973): *Computational methods in ordinary differential equations*. New York.
 - [20] Ghelardoni. P. Marzulli (1995): Stability of some boundary value methods for IVPs, *Ap plied Numerical Mathematics* 18 (1) (1995) 144-153.
 - [21] Ibrahim, O.M., and Ikile, M.N.O (2017): Spectral Collocation Method for the Numerical Solution of Ordinary Differential Equations. *Transaction of the Nigerian Association of Mathematical Physics* 4, 95-100. <https://www.researchgate.net/profile/Oluwasegun-Ibrahim/publication/319763706>.
 - [22] Ibrahim, O.M., and Lawrence, P. W. (2019): Spectral rectangular collocation formula: an approach for solving oscillatory initial value problems and/or boundary value problems in ordinary differential equations. *Turkish J. Anal. Number Theory*, 7(1), 11-17.
 - [23] Usmani R.A (1966): Boundary value techniques for the numerical solution of certain initial value problems in ordinary differential equations *J. ACM* 13 (1966) 287 295.
 - [24] Lambert J. D. and Mitchel A. R. (1962): On the solution of $y' = f(x,y)$ by a class of high-accuracy differential formulae of low order, *Z. Angew. Math. Phys.* 13, 223-232.
 - [25] Ajayi, S. A., Muka, K. O., and Ibrahim, O. M. (2019): A family of stiffly stable second derivative block methods for initial value problems in ordinary differential equations. *Earthline Journal of Mathematical Sciences*, 1(2), 221-239.
 - [26] Shokri A (2013): The new class of implicit 1-stable hybrid Obrechhoff method for the numerical solution of first-order initial value problems. *Computer Physics Communications* 184 (3) (2013) 529-531.
 - [27] Lambert J.D (1973): *Computational methods in ordinary differential equations*. John Wiley.
 - [28] Milne, W.E. (1949): A note on the numerical integration of differential equations, *J. Res. Nat. Bur. Standards*, 43, 537-542 (1949).
 - [29] Majid Z.A., M.M. Hasni, and N. Senu (2013): "Solving second-order linear dirichlet and Neumann boundary value problems by block method," *IAENG International Journal of Applied Mathematics*, vol. 43, no. 2, pp. 71-76.
 - [30] Chew K. T., Zanariah A. M., Mohamed S. and Norazak S. (2012): Solving Linear Two-Point Boundary Value Problems by Direct Adams Moulton Method, *Applied Mathematical Sciences*, Vol. 6, no. 99, 4921-4929.
 - [31] Ibrahim, O. M. (2017). High order symmetric super-implicit hybrid LMMs with minimal phase-lag: Solving differential equations using the numerically stable Super-Implicit Hybrid formula. LAMBERT Academic Publishing. ISBN 978-6202023795. <https://www.amazon.ca/Symmetric-Super-Implicit-Hybrid-Minimal-Phase-Lag/dp/6202023791>

