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An Alternative Proof of a Closed Formula for Central Factorial Numbers of the Second Kind

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Abstract In the short note, by virtue of several formulas and identities for special values of the Bell polynomials of the second kind, the authors provide an alternative proof of a closed formula for central factorial numbers of the second kind. Moreover, the authors pose two open problems on closed form of a special Bell polynomials of the second kind and on closed form of a finite sum involving falling factorials.

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1. Introduction

In mathematics, a closed formula is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.

The Bell polynomials of the second kind, denoted by $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge 0$, are defined [1-7] by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The central factorial numbers of the second kind $T(n,k)$ for $n\ge k\ge 0$ can be generated [8,9,10] by

$$\frac{1}{k!}\left(2\sinh\frac{x}{2}\right)^k = \sum_{n=k}^{\infty} T(n,k)\frac{x^n}{n!}. \tag{1.1}$$

The central factorial numbers of the first kind $t(n,k)$ for $n\ge k\ge 0$ can be generated [8,9] by

$$\frac{1}{k!}\left(2\operatorname{arsinh}\frac{x}{2}\right)^k = \sum_{n=k}^{\infty} t(n,k)\frac{x^n}{n!}, |x|\le 2. \tag{1.2}$$

In [[8], Proposition 2.4, (xii)], the authors established the closed formula

$$T(n,k) = \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\left(\frac{k}{2}-\ell\right)^n. \tag{1.3}$$

In this short note, by virtue of several formulas and identities for special values of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$, we will provide an alternative proof of the closed formula (1.3).

2. Lemmas

For alternatively proving the closed formula (1.3), we need the following lemmas.

Lemma 2.1 ([1,2,11,12,13,14]). *The Faà di Bruno formula can be described in terms of* $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *by*

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f\circ h(x) = \sum_{k=0}^{n} f^{(k)}(h(x))\mathrm{B}_{n,k}\left(h'(x),h''(x),\dots,h^{(n-k+1)}(x)\right). \tag{2.1}$$

For $n \ge k \ge 0$, the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ satisfy the identity

$$B_{n,k}\left(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}\right) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \tag{2.2}$$

is valid, where $a, b \in \mathbb{C}$.

Lemma 2.2 [10,11,12,14-18]. *For $n \ge k \ge 0$, the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ satisfy the closed formula*

$$\begin{aligned} B_{n,k}\left(1, 0, 1, \ldots, \frac{1-(-1)^{n-k+1}}{2}\right) &= \frac{1}{2^k k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (k-2\ell)^n \\ &= (\pm 2)^{n-k} S_{-k/2}(n,k), \end{aligned} \tag{2.3}$$

where 0^0 is regarded as 1 and $S_r(n,k)$ denote the associate Stirling numbers of the second kind or weighted Stirling numbers which can be generated by

$$\frac{(e^x-1)^k}{k!} e^{rx} = \sum_{n=k}^{\infty} S_r(n,k) \frac{x^n}{n!}.$$

3. An Alternative Proof of the Closed Formula (1.3)

The closed formula (1.3) can be rewritten in terms of the associate Stirling numbers of the second kind or weighted Stirling numbers $S_r(n,k)$ as follows.

Theorem 3.1. *For $n \ge k \ge 0$, the central factorial numbers of the second kind $T(n,k)$ satisfy*

$$\begin{aligned} T(n,k) &= \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \left(\frac{k}{2} - \ell\right)^n \\ &= (\pm 1)^{n-k} S_{-k/2}(n,k). \end{aligned} \tag{3.1}$$

Proof. The equation (1.1) implies that

$$\begin{aligned} T(n,k) &= \frac{1}{k!} \lim_{x\to 0} \frac{d^n}{dx^n}\left[\left(2\sinh\frac{x}{2}\right)^k\right] \\ &= \frac{2^k}{k!} \lim_{x\to 0} \frac{d^n}{dx^n}\left[\left(\sinh\frac{x}{2}\right)^k\right]. \end{aligned} \tag{3.2}$$

Let $v = v(x) = \sinh\frac{x}{2}$. By virtue of the Faà di Bruno formula (2.1), we obtain

$$\begin{aligned} &\frac{d^n}{dx^n}\left[\left(\sinh\frac{x}{2}\right)^k\right] \\ &= \sum_{\ell=0}^{n} \frac{d^\ell v^k}{dv^\ell} B_{n,\ell}\left(\left(\sinh\frac{x}{2}\right)', \left(\sinh\frac{x}{2}\right)'', \ldots, \left(\sinh\frac{x}{2}\right)^{(n-\ell+1)}\right) \\ &= \sum_{\ell=0}^{n} \langle k\rangle_\ell v^{k-\ell} B_{n,\ell}\left(\left(\sinh\frac{x}{2}\right)', \left(\sinh\frac{x}{2}\right)'', \ldots, \left(\sinh\frac{x}{2}\right)^{(n-\ell+1)}\right) \\ &= \sum_{\ell=0}^{n} \langle k\rangle_\ell \left(\sinh\frac{x}{2}\right)^{k-\ell} B_{n,\ell}\left(\left(\sinh\frac{x}{2}\right)', \left(\sinh\frac{x}{2}\right)'', \ldots, \left(\sinh\frac{x}{2}\right)^{(n-\ell+1)}\right), \mathrm{v} \end{aligned}$$

where the quantity

$$\langle x\rangle_n = \prod_{\ell=0}^{n-1}(x-\ell) = \begin{cases} x(x-1)\ldots(x-n+1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

is called [2,19] the falling factorial of x.

Since

$$(\sinh x)^{(m)} = \begin{cases} \cosh x, & m = 2i-1 \\ \sinh x, & m = 2i \end{cases} \to \begin{cases} 1, & m = 2i-1 \\ 0, & m = 2i \end{cases}$$

for $i, m \in \mathbb{N}$ as $x \to 0$, it follows that

$$\begin{aligned} &\lim_{x\to 0} \frac{d^n}{dx^n}\left[\left(\sinh\frac{x}{2}\right)^k\right] \\ &= \langle k\rangle_k \lim_{x\to 0} B_{n,k}\left(\left(\sinh\frac{x}{2}\right)', \ldots, \left(\sinh\frac{x}{2}\right)^{(n-k+1)}\right) \\ &= k! B_{n,k}\left(\frac{1}{2}\cdot 1, \frac{1}{2^2}\cdot 0, \ldots, \frac{1}{2^{n-k+1}} \frac{1-(-1)^{n-k+1}}{2}\right) \\ &= \frac{k!}{2^n} B_{n,k}\left(1, 0, 1, 0, 1, 0, \ldots, \frac{1-(-1)^{n-k+1}}{2}\right) \\ &= \frac{1}{2^{n+k}} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (k-2\ell)^n, \end{aligned}$$

where we used the identity (2.2) and the closed formula (2.3). Consequently, the formula (3.1) follows immediately. The proof of Theorem 3.1 is complete.

4. Two Open Problems

In this section, we pose two open problems.

4.1. First Open Problem

For alternatively and similarly finding a closed formula for the central factorial numbers of the first kind $t(n,k)$ generated in (1.2), we need to solve the following open problem.

Open Problem 4.1. *Can one find a closed formula of the Bell polynomials of the second kind*

$$B_{n,k}\left(\begin{aligned} &((-1)!!)^2, 0, -(1!!)^2, 0, \ldots \\ &, \frac{1-(-1)^{n-k+1}}{2} (-1)^{(n-k)/2} ((n-k-1)!!)^2 \end{aligned} \right)$$

for $n \ge k \ge 0$?

4.2. Second Open Problem.

For $\alpha \in \mathbb{C}$ and $n \in \mathbb{N} \cup \{0\}$, the falling factorial $\langle\alpha\rangle_n$ is defined by

$$\langle\alpha\rangle_n = \prod_{k=0}^{n-1}(\alpha-k) = \begin{cases} \alpha(\alpha-1)\ldots(\alpha-n+1), & n \ge 1; \\ 1 & n = 0. \end{cases} \tag{4.1}$$

In [12,20] and closely related references therein, the following conclusions were obtained.

Theorem 4.1 ([[20], Theorem 3.1] and [[12], Section 1.4]). *For* $n \ge k \ge 0$, *we have*

$$\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\langle 2\ell\rangle_n=\frac{n!}{2^{n-2k}}\binom{k}{n-k}. \tag{4.2}$$

Theorem 4.2 ([[20], Theorem 3.2] and [[12], Section 1.5]). *For* $n \ge k \ge 0$, *we have*

$$\begin{aligned}&\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\left\langle\frac{\ell}{2}\right\rangle_n\\&=(-1)^n\frac{k![2(n-k)-1]!!}{2^n}\binom{2n-k-1}{2(n-k)},\end{aligned} \tag{4.3}$$

where the double factorial of negative odd integers $-(2n+1)$ *for* $n \ge 0$ *is defined by*

$$(-2n-1)!!=\frac{(-1)^n}{(2n-1)!!}=(-1)^n\frac{2^n n!}{(2n)!}.$$

The formula (4.3) has been applied in [[21], Theorem 1.1].

Motivated by the above conclusions, one can naturally pose the following open problem.

Open Problem 4.2. *For* $n \ge k \ge 0$ *and* $\alpha \in \mathbb{R}$, *can one find an explicit, elementary, simple, and general formula of the type as in* (4.2) *and* (4.3) *for the finite sum*

$$\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle\alpha\ell\rangle_n\,?$$

In particular, how about special cases $\alpha = m^{\pm 1}$ *for* $m \in \mathbb{N}\setminus\{2\}$ *in* (4.4)?

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