On a Class of P-Kenmotsu Manifolds Admitting Weyl-projective Curvature Tensor of Type (1, 3)

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Abstract We study a class of para-Kenmotsu manifolds admitting Weyl-projective curvature tensor of type (1, 3). At the end, it is shown that an n-dimensional (n > 2) P-Kenmotsu manifold is Ricci semisymmetric if and only if it is an Einstein manifold.

Keywords: para kenmotsu manifold, recurrent manifold, W2 - curvature tensor, ricci tensor, einstein manifold


1. Introduction

In [1,2], Sato introduced the notions of an almost para contact Riemannian manifold. In 1977, Adati and Matsumoto defined para-Sasakian and special para-Sasakian manifolds, which are regarded as a special kind of an almost contact Riemannian manifolds [3]. Para-Sasakian manifolds have been studied by Adati and Miyazawa [4], De and Avijit [5], Matsumoto, Ianus and Mihai [6] and many others. Before Sato, Kenmotsu defined a class of almost contact Riemannian manifolds [7]. In 1995, Sinha and Sai Prasad defined a class of almost para contact metric manifolds namely para-Kenmotsu (briefly P-Kenmotsu) and special para-Kenmotsu (briefly SP-Kenmotsu) manifolds [8].

In 1970, Pokhariyal and Mishra introduced new tensor fields, called W2 and E tensor fields, on a Riemannian manifold [9]. Later, in [10], Pokhariyal studied some of the properties of these tensor fields on a Sasakian manifold. In 1986, Matsumoto, Ianus and Mihai have extended these concepts to almost para-contact structures and studied para-Kenmotsu manifolds admitting these tensor fields [6]. These results were further generalised by De and Sarkar, in [5]. Motivated by these studies, in 2015, Sai Prasad and Satyanarayana studied W2-tensor field in an SP-Kenmotsu manifold [11]. In the present work, we investigate a class of para-Kenmotsu manifolds admitting Weyl-projective curvature tensor W2 of type (1, 3). The present work is organised as follows: Section 2 is equipped with some prerequisites about P-Kenmotsu manifolds. In Section 3, we define W2-recurrent and semisymmetric para-Kenmotsu manifolds and shown that W2-recurrent para-Kenmotsu manifold is a semisymmetric manifold. Further, it is shown that the curvature of W2-semisymmetric para-Kenmotsu manifold is constant and hence we establish that a W2-recurrent para-Kenmotsu manifold is an SP-Kenmotsu manifold. Section 4 is devoted to study Ricci semisymmetric P-Kenmotsu manifold.

2. Preliminaries

Let $M_n$ be an n-dimensional differentiable manifold equipped with structure tensors $(\Phi, \xi, \eta)$ where $\Phi$ is a tensor of type (1, 1), $\xi$ is a vector field, $\eta$ is a 1-form such that

$$\begin{align*}
\Phi(X, \xi) &= 1 \\
\Phi^2(X) &= X - \eta(X)\xi; \bar{X} = \Phi X.
\end{align*}$$

(2.1)

Then the manifold $M_n$ is called an almost para-contact Riemannian manifold.

Let $g$ be a Riemannian metric such that, for all vector fields $X$ and $Y$ on $M_n$

$$\begin{align*}
g'(X, \xi) &= \eta(X) \\
\Phi \xi &= 0, n(X) = 0, rank \Phi = n - 1 \\
g(\Phi X, \Phi Y) &= g(X, Y) - \eta(X)\eta(Y).
\end{align*}$$

(2.2)

Then the manifold $M_n$ is called an almost para-contact manifold.

In addition, if $(\Phi, \xi, \eta, g)$ satisfies the conditions

$$\begin{align*}
(\nabla_X \eta)Y - (\nabla_Y \eta)X &= 0, \\
(\nabla_X \nabla_Y \eta)Z &= \left[ -g(X, Z) + \eta(X)\eta(Z) \right] \eta(Y) \\
&+ \left[ -g(X, Y) + \eta(X)\eta(Y) \right] \eta(Z),
\end{align*}$$

(2.3)

$$\begin{align*}
\nabla_X \xi &= X - \eta(X)\xi, \\
(\nabla_X \Phi)Y &= -g(X, \Phi Y)\xi - \eta(Y)\Phi X;
\end{align*}$$

The manifold $M_n$ [1] is said to admit an almost para-contact Riemannian structure $(\Phi, \xi, \eta, g)$. In the present work, we study a class of para-Kenmotsu manifolds admitting Weyl-projective curvature tensor $W_2$ of type (1, 3). The present work is organised as follows: Section 2 is equipped with some prerequisites about P-Kenmotsu manifolds. In Section 3, we define W2-recurrent and semisymmetric para-Kenmotsu manifolds and shown that W2-recurrent para-Kenmotsu manifold is a semisymmetric manifold. Further, it is shown that the curvature of W2-semisymmetric para-Kenmotsu manifold is constant and hence we establish that a W2-recurrent para-Kenmotsu manifold is an SP-Kenmotsu manifold. Section 4 is devoted to study Ricci semisymmetric P-Kenmotsu manifold.
then $\mathcal{M}_n$ is called para-Kenmotsu manifold or briefly a P-Kenmotsu manifold [8].

A P-Kenmotsu manifold admitting a 1-form $\eta$ satisfying
\[
\begin{align*}
(\nabla_X \eta) Y &= g(X,Y) - \eta(X)\eta(Y), \\
(\nabla_X \eta) Y &= \varphi(X,Y);
\end{align*}
\]

where $\varphi$ is an associate of $\Phi$, is called special para-Kenmotsu manifold or briefly SP-Kenmotsu manifold [8].

Let $(\mathcal{M}_n,g)$ be an $n$-dimensional, $n \geq 3$, differentiable manifold of class $C^\infty$ and let $\nabla$ be its Levi-Civita connection. Then the Riemannian Christoffel curvature tensor $R$ of type $(1, 3)$ is given by:
\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.
\]

The Ricci operator $S$ and the $(0, 2)$-tensor $S^2$ are defined by
\[
g(SX,Y) = S(X,Y); \tag{2.6}
\]
and
\[
S^2(X,Y) = S(SX,Y). \tag{2.7}
\]

It is known [8] that in a P-Kenmotsu manifold the following relations hold:
\[
S(X,\xi) = -(n-1)\eta(X), \\
g[R(X,Y)Z,\xi] = \eta[R(X,Y,Z)], \\
R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \\
R(X,Y,\xi) = \eta(X)Y - \eta(Y)X;
\]

when $X$ is orthogonal to $\xi$.

An $n$-dimensional $(n > 2)$ Riemannian manifold $\mathcal{M}_n$ is said to be Einstein manifold if the Ricci curvature tensor $S(X,Y)$ of the Levi-Civita connection satisfies the condition
\[
S(X,Y) = \lambda g(X,Y) \tag{2.9}
\]
where $\lambda$ is a constant.

### 3. $W_2$-Recurrent P-Kenmotsu Manifolds

The Weyl-projective curvature tensor $W_2$ of type $(1, 3)$ of a Riemannian manifold $\mathcal{M}_n$ with respect to Riemannian connection is given by [9]:
\[
W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}g(X,Z)S(Y,U) - \frac{1}{n-1}g(Y,Z)S(X,U). \tag{3.1}
\]

Now, we define a $W_2$-semisymmetric para-Kenmotsu manifold as:

**Definition 3.1:** An $n$-dimensional para-Kenmotsu manifold is called $W_2$-semisymmetric if its $W_2$-curvature tensor satisfies the condition
\[
R(X,Y)W_2 = 0, \tag{3.2}
\]

where $R(X,Y)$ is considered to be a derivation of the tensor algebra at each point of the manifold for tangent vectors $X$ and $Y$.

It can be easily shown that on a P-Kenmotsu manifold the $W_2$-curvature tensor satisfies the condition
\[
W_2(X,Y,Z,\xi) = 0. \tag{3.3}
\]

Further, we define a $W_2$-recurrent para-Kenmotsu manifold as:

**Definition 3.2:** An $n$-dimensional para-Kenmotsu manifold with respect to the Levi-Civita connection is called $W_2$-recurrent manifold if its $W_2$-curvature tensor satisfies the condition
\[
(V_U W_2)(X,Y)Z = A(U)W_2(X,Y)Z, \tag{3.4}
\]

where $A$ is some non-zero 1-form.

Now, let us establish a relation between $W_2$-recurrent and $W_2$-semisymmetric para-Kenmotsu manifolds.

For that, let us suppose that $W_2 \neq 0$. Now, we define a function by
\[
f^2 = g(W_2,W_2). \tag{3.5}
\]

Using the fact that $V U g = 0$, from (3.5) we get
\[
2f(Uf) = 2f^2(A(U)).
\]

Since $f \neq 0$, we have
\[
Uf = f(A(U)). \tag{3.6}
\]

Then, from (3.6), we get
\[
X(Uf) = \frac{1}{f}(Xf)(Uf) + (XA(U))f, \tag{3.7}
\]

and hence, we have
\[
X(Uf) - U(Xf) = \left[XA(U) - UA(X)\right]f. \tag{3.8}
\]

Therefore,
\[
\begin{align*}
\left(\nabla_X V_U - \nabla_U V_X - \nabla_{[X,U]}\right)f \\
= \left[XA(U) - UA(X) - A([X,U])\right]f \\
= 2\left[dA(U,X)\right]f \tag{3.9}
\end{align*}
\]

Since the left hand side of (3.9) is zero and $f \neq 0$, we deduce that $dA(X,Y) = 0$ and it shows that the 1-form $A$ is closed.

Then from (3.4), we get that
\[
(V_U V_U W_2)(X,Y)Z = \left[V_A(U) + A(V)A(U)\right]W_2(X,Y)Z, \tag{3.10}
\]

and hence, we get that
\[
\begin{align*}
(V_U V_U W_2)(X,Y)Z - (V_U V_U W_2)(X,Y)Z \\
= 2dA(V,U)W_2(X,Y)Z \tag{3.11}
\end{align*}
\]
i.e., \( R(V, U)W_2 = 0 \), where \( R(V, U) \) is considered to be a 
derivation of tensor algebra at each point of the manifold 
for the tangent vectors \( V \) and \( U \).

This shows that a \( W_2 \)-recurrent P-Kenmotsu manifold is 
\( W_2 \)-semisymmetric and hence we state that:

**Theorem 3.1:** A \( W_2 \)-recurrent para-Kenmotsu manifold is 
\( W_2 \)-semisymmetric.

Further we determine the curvature value of 
\( W_2 \)-semisymmetric P-Kenmotsu manifold.

From (3.2), we have

\[
R(X, Y)W_2(Z, U)V - W_2(R(X, Y)Z, U)V - W_2(Z, R(X, Y)U)V - W_2(Z, U)R(X, Y)V = 0, 
\]

which implies

\[
g(R(X, Y)W_2(Z, U)V, \xi) - g(W_2(R(X, Y)Z, U)V, \xi) - g(W_2(Z, R(X, Y)U)V, \xi) - g(W_2(Z, U)R(X, Y)V, \xi) = 0. 
\]  

By putting \( X = \xi \) in the above equation, we get

\[
R((\xi, Y)W_2(Z, U)V, \xi) - W_2(R(\xi, Y)Z, U)V, \xi) - W_2(Z, R(\xi, Y)U)V, \xi) - W_2(Z, U)R(X, Y)V, \xi) = 0. 
\]  

Now, by using (2.8) and (3.3), the above equation reduces to:

\[
\eta(Y)\eta(W_2(Z, U)V) - g(Y, W_2(Z, U)V) = 0. 
\]  

Again on using (3.3), we get that \( W_2(Z, U, V, Y) = 0 \).

Therefore, from (3.1) we have

\[
R(X, Y, Z, V) = \frac{1}{n-1} \left[ g(Y, Z)S(X, V) - g(X, Z)S(Y, V) \right]. 
\]  

On contracting the above equation, we get

\[
S(Y, Z) = \frac{r}{n} g(Y, Z). 
\]  

Then, from equations (3.16) and (3.17), we have

\[
R(X, Y, Z, V) = \frac{r}{n(n-1)} \left[ g(Y, Z)g(X, V) - g(X, Z)g(Y, V) \right]. 
\]  

This shows that the curvature of \( W_2 \)-semisymmetric 
P-Kenmotsu manifold is constant.

As it is known [8] that a P-Kenmotsu manifold with 
constant curvature is an SP-Kenmotsu manifold and using 
the above shown result, we state that:

**Theorem 3.2:** A \( W_2 \)-semiisymmetric P-Kenmotsu 
manifold is an SP-Kenmotsu manifold.

Therefore, form theorems (3.1) and (3.2), we have the 
following result:

**Theorem 3.3:** A \( W_2 \)-recurrent P-Kenmotsu manifold is an 
SP-Kenmotsu manifold.

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**4. Ricci Semisymmetric Para-Kenmotsu Manifolds**

**Definition 4.1:** An n-dimensional Riemannian manifold is 
said to be Ricci semisymmetric if its Ricci tensor \( S(X, Y) \) 
of the Levi-Civita connection satisfies the condition

\[
R(X, Y)S = 0. 
\]  

**Theorem 4.1:** An n-dimensional \((n > 2)\) P-Kenmotsu 
manifold \( M_n \) is Ricci semisymmetric if and only if it is an 
Einstein Manifold.

**Proof:** Let us suppose that a P-Kenmotsu manifold be 
Ricci semisymmetric. Then from (4.1), we have

\[
S(R(X, Y)U, V) + S(U, R(X, Y)V) = 0. 
\]

By putting \( X = \xi \) in (4.2), we get

\[
S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0. 
\]

Now by using the equations (2.8) (a) and (2.8) (c), the 
above equation reduces to

\[
\eta(U)\eta(Y) + (n-1)\eta(V)g(Y, U) + (n-1)\eta(U)g(Y, V) = 0. 
\]

Again by putting \( X = \xi \) in (4.4), we get

\[
S(Y, V) = -(n-1)g(Y, V). 
\]

This proves that the manifold \( M_n \) is an Einstein 
manifold.

As an every Einstein manifold is Ricci semisymmetric, 
the converse of the theorem is trivial. 
This completes the proof.

**Statement of Competing Interests**

The authors declare that there is no conflict of interests 
regarding the publication of this paper.

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**References**

[1] Sato, I. On a structure similar to the almost contact structure, 

[2] Sato, I. On a structure similar to the almost contact structure II, 

[3] Adati, T. and Matsumoto, K, On conformally recurrent and 
conformally symmetric P-Sasakian Manifolds, 

some parallel and recurrent tensors, *Tensor (N.S).* 33, 287-292, 
1979.

[5] De, U. C. and Avijit Sarkar, On a type of P-Sasakian manifolds, 


