On Some Identities and Generating Functions for Mersenne Numbers and Polynomials

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Abstract   In this paper, we introduce a new operator in order to derive some properties of homogeneous symmetric functions. By making use of the proposed operator, we give some new generating functions for Mersenne numbers, Mersenne numbers and product of sequences and Chebychev polynomials of second kind.

Keywords: Mersenne numbers, Generating functions, Symmetric functions

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1. Introduction

In this paper we consider one of the sequences of positive integers satisfying a recurrence relation and we give some well-known identities for this type of sequences [9]. One of the sequences of positive integers (also defined recursively) that have been studied over several years is the well-known Fibonacci (and Lucas) sequence. Many papers are dedicated to Fibonacci sequence, such as the works of Hoggatt in [10], and Koshy in [11], among others. Others sequences satisfying a second-order recurrence relations are the main topic of the research for several authors, such as the studies of the sequences $\{J_n\}_{n\in\mathbb{N}}$ and $\{j_n\}_{n\in\mathbb{N}}$ of Jacobsthal and Jacobsthal-Lucas numbers, respectively.

In this paper we do not have such kind of generalization, but we will follow closely some of these studies. About the Mersenne sequence, also some studies about this sequence have been published, such as the work of Koshy [12], where the authors investigate some divisibility properties of Catalan numbers with Mersenne numbers Mk as their subscripts, developing their work in [12]. In number theory, recall that a Mersenne number of order n, denoted by Mn, is a number of the form $2^n - 1$, where n is a nonnegative number. This identity is called as the Binet formula for Mersenne sequence and it comes from the fact that the Mersenne numbers can also be defined recursively by

$$M_{n+1} = 2M_n + 1, \qquad (1.1)$$

with initial conditions $M_0 = 0, \ M_1 = 1$. Since this recurrence

is inhomogeneous, substituting n by $n+1$, we obtain the new form

$$M_{n+2} = 2M_{n+1} + 1, \qquad (1.2)$$

Subtracting (1.1) to (1.2), we have that

$$M_{n+2} - M_{n+1} = 2M_{n+1} + 1 - (2M_n + 1)$$

and then

$$M_{n+2} = 3M_{n+1} - 2M_n,$$

other form for the recurrence relation of Mersenne sequence, with initial conditions $M_0 = 0$, $M_1 = 1$. The roots of the respective characteristic equation

$$r^2 - 3r + 2 = 0,$$

are

$$r_1 = 1 \text{ and } r_2 = 2,$$

and we easily get the Binet formula

$$M_n = 2^n - 1.$$

The main purpose of this paper is to present some results involving the Mersenne numbers using define a new useful operator denoted by $\delta_{p_1 p_2}^k$ for which we can formulate, extend and prove new results based on our previous ones [1,3,4]. In order to determine generating functions of the product of Mersenne numbers and Chebychev polynomials of first and second kind, we combine between our indicated past techniques and these presented polishing approaches.

2. Definitions and Some Properties

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results.

Definition1. [6] *Let B and P be any two alphabets. We define $S_n(B-P)$ by the following form*

$$\frac{\Pi_{p\varepsilon P}(1-pt)}{\Pi_{b\varepsilon B}(1-bt)} = \sum_{n=0}^{\infty} S_n(B-P)t^n, \qquad (2.1)$$

with the condition $S_n(B-P)=0$ for $n<0$.

Equation (2.1) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(B-P)t^n = \left(\sum_{n=0}^{\infty} S_n(B)t^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)t^n\right),$$

where

$$S_n(B-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(B). \qquad (2.2)$$

We know that the polynomial whose roots are P is written as

$$S_n(x-P) = \sum_{j=0}^{n} S_{n-j}(-P)x^n, \text{ with } card(P)=n.$$

On the other hand, if B has cardinality equal to 1, i.e., $B=\{x\}$, then (2.1) can be rewritten as follows [6]:

$$\sum_{n=0}^{\infty} S_n(x-P)t^n = \frac{\prod_{p\in P}(1-pt)}{(1-xt)}$$

$$= 1 + \cdots + S_{n-1}(x-P)t^{n-1} + \frac{S_n(x-P)}{(1-xt)}t^n,$$

where $S_{n+k}(x-P) = x^k S_n(x-P)$ for all $k \geq 0$.

The summation is actually limited to a finite number of terms since $S_{-k}(\cdot)=0$ for all $k>0$. In particular, we have

$$\prod_{p\in P}(x-p) = S_n(x-P)$$

$$= S_0(-P)x^n + S_1(-P)x^{n-1} + S_2(-P)x^{n-2} + \cdots,$$

where $S_k(-B)$ are the coefficients of the polynomials $S_n(x-P)$ for $0 \leq k \leq n$. These coefficients are zero for $k>n$.

For example, if all $p \in P$ are equal, i.e., $P=np$, then we have $S_n(x-np)=(x-p)^n$

By choosing $P=1$ i.e., $P=\left\{\underbrace{1,1,...1}_{n}\right\}$, we obtain

$$S_k(-n) = (-1)^k \binom{n}{k} \text{ and } S_k(n) = \binom{n+k-1}{k} \qquad (2.3)$$

By combining (2.2) and (2.3), we obtain the following expression

$$S_n(B-nx) = S_n(B) - \binom{n}{1}S_{n-1}(B)x$$

$$+ \binom{n}{2}S_{n-2}(B)x^2 - \cdots + (-1)^n \binom{n}{n}x^n.$$

Definition 2. [4] *Given a function f on R^n, the divided difference operator is defined as follows*

$$\partial_{p_i p_{i+1}}(f) = \frac{\begin{bmatrix} f(p_1,\cdots,p_i,p_{i+1},\cdots p_n) \\ -f(p_1,\cdots p_{i-1},p_{i+1},p_i,p_{i+2}\cdots p_n) \end{bmatrix}}{p_i - p_{i+1}}.$$

Definition 3. The symmetrizing operator $\delta_{e_1 e_2}^k$ is defined by

$$\delta_{p_1 p_2}^k(g) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2} \text{ for all } k \in N. \qquad (2.4)$$

Proposition 1. [5] *Let $P=\{p_1,p_2\}$ an alphabet, we define the operator $\delta_{e_1 e_2}^k$ as follows*

$$\delta_{p_1 p_2}^k g(p_1) = S_{k-1}(p_1+p_2)g(p_1) + p_2^k \partial_{p_1 p_2} g(p_1),$$
for all $k \in N$.

3. On the Generating Functions

In our main result, we will combine all these results in a unified way such that they can be considered as a special case of the following Theorem.

Theorem 1. *Let A and P be two alphabets, respectively, $\{a_1,a_2\}$ and $\{b_1,b_2\}$, then we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n+k-1}(P)z^n$$

$$= \frac{\begin{pmatrix} S_{k-1}(p_1+p_2) - (a_1+a_2)\delta_{p_1 p_2}^k(p_2)z \\ -a_1 a_2 p_1 p_2 \delta_{p_1 p_2}(p_2^{k-1})z^2 \end{pmatrix}}{\left(\sum_{n=0}^{\infty} S_n(-A)p_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)p_2^n z^n\right)}. \qquad (3.1)$$

for all $k \in \mathbb{N}$.

Proof By applying the operator $\partial_{p_1 p_2}$ to the series

$f(p_1) = \sum_{n=0}^{\infty} S_n(A)p_1^{n+k}z^n$, we have

$$\partial_{p_1 p_2}\left(\sum_{n=0}^{\infty} S_n(A)p_1^{n+k}z^n\right)$$

$$= \frac{\sum_{n=0}^{\infty} S_n(a_1+a_2)p_1^{n+k}z^n - \sum_{n=0}^{\infty} S_n(a_1+a_2)p_2^{n+k}z^n}{p_1 - p_2}$$

$$= \sum_{n=0}^{\infty} S_n (a_1 + a_2) \left(\frac{p_1^{n+k} - p_2^{n+k}}{p_1 - p_2} \right) z^n$$

$$= \sum_{n=0}^{\infty} S_n (A) S_{n+k-1}(P) z^n.$$

On the other hand,

$$\partial_{p_1 p_2} \left(\frac{p_1^k}{\sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n} \right)$$

$$= \frac{\dfrac{p_1^k}{\sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n} - \dfrac{p_2^k}{\sum\limits_{n=0}^{\infty} S_n(-A) p_2^n z^n}}{p_1 - p_2}$$

$$= \frac{p_1^k \sum\limits_{n=0}^{\infty} S_n(-A) p_2^n z^n - p_2^k \sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n}{(p_1 - p_2) \left(\sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n \right) \left(\sum\limits_{n=0}^{\infty} S_n(-A) p_2^n z^n \right)}$$

$$= \frac{\left[\begin{array}{l} p_1^k - p_2^k - (a_1 + a_2)\left(p_1^k p_2 - p_2^k p_1 \right) z \\ - a_1 a_2 \left(p_2^k p_1^2 - p_1^k p_2^2 \right) z^2 \end{array} \right]}{(p_1 - p_2) \left(\sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n \right) \left(\sum\limits_{n=0}^{\infty} S_n(-A) p_2^n z^n \right)}$$

$$= \frac{\left[\begin{array}{l} \dfrac{p_1^k - p_2^k}{p_1 - p_2} - (a_1 + a_2) p_1 p_2 \left(\dfrac{p_1^{k-1} - p_2^{k-1}}{p_1 - p_2} \right) z \\ - a_1 a_2 p_1 p_2 \left(\dfrac{p_1 p_2^{k-1} - p_2 p_1^{k-1}}{p_1 - p_2} \right) z^2 \end{array} \right]}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n \right) \left(\sum\limits_{n=0}^{\infty} S_n(-A) p_2^n z^n \right)}$$

$$= \frac{\left[\begin{array}{l} S_{k-1}(p_1 + p_2) - (a_1 + a_2) \delta_{p_1 p_2}^k (p_2) z \\ - a_1 a_2 p_1 p_2 \delta_{p_1 p_2} (p_2^{k-1}) z^2 \end{array} \right]}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n \right) \left(\sum\limits_{n=0}^{\infty} S_n(-A) p_2^n z^n \right)}.$$

Thus, this completes the proof.

We now derive new generating functions of the products of some well-known polynomials. Indeed, we consider Theorem 1 in order to derive Mersenne numbers and Tchebychev polynomials of second kind and the symmetric functions

• If $k=0$ and $A=\{1,0\}$ we deduce the following lemma

Lemma [1] *Given an alphabet* $P = \{p_1, p_2\}$, *we have*

$$\sum_{n=0}^{\infty} S_{n-1}(p_1 + p_2) z^n = \frac{z}{(1 - p_1 z)(1 - p_2 z)} \qquad (3.2)$$

Replacing p_2 by $(-p_2)$ in (3.2), we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2]) z^n = \frac{z}{(1 - p_1 z)(1 + p_2 z)} \qquad (3.3)$$

Choosing p_1 and p_2 such that

$$\begin{cases} p_1 p_2 = -2, \\ p_1 - p_2 = 3, \end{cases}$$

and substituting in (3.3) we end up with

$$\sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2]) z^n = \frac{z}{1 - 3z + 2z^2},$$

which represents a generating function for Mersenne numbers, such that $M_n = S_{n-1}(p_1 + [-p_2])$.

If $k = 0$, $k = 1$ and $A = \{a_1, a_2\}$ we deduce the following theorems

Theorem 2. [7] *Given two alphabets* $A = \{a_1, a_2\}$ *and* $P = \{p_1, p_2\}$ *we have*

$$\sum_{n=0}^{\infty} S_n(A) S_{n-1}(P) z^n$$

$$= \frac{(a_1 + a_2) z - a_1 a_2 (p_1 + p_2) z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n \right) \left(\sum\limits_{n=0}^{\infty} S_n(-A) p_2^n z^n \right)}. \qquad (3.4)$$

Theorem 3. [8] *Given two alphabets* $A = \{a_1, a_2\}$ *and* $P = \{p_1, p_2\}$ *we have*

$$\sum_{n=0}^{\infty} S_n(A) S_n(P) z^n$$

$$= \frac{1 - a_1 a_2 p_1 p_2 z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n \right) \left(\sum\limits_{n=0}^{\infty} S_n(-A) p_2^n z^n \right)}. \qquad (3.5)$$

From (3.5) we can deduce

$$\sum_{n=0}^{\infty} S_{n-1}(A) S_{n-1}(P) z^n$$

$$= \frac{z - a_1 a_2 p_1 p_2 z^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) p_1^n z^n \right) \left(\sum\limits_{n=0}^{\infty} S_n(-A) p_2^n z^n \right)}. \qquad (3.6)$$

Case 1: Replacing p_2 by $(-p_2)$ and a_2 by $(-a_2)$ in (3.4) and (3.6) yields

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2]) z^n$$

$$= \frac{(a_1 - a_2) z + a_1 a_2 (p_1 - p_2) z^2}{(1 - a_1 p_1 z)(1 + a_2 p_1 z)(1 + a_1 p_2 z)(1 - a_2 p_2 z)} \qquad (3.7)$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2]) z^n$$

$$= \frac{z - a_1 a_2 p_1 p_2 z^3}{(1 - a_1 p_1 z)(1 + a_2 p_1 z)(1 + a_1 p_2 z)(1 - a_2 p_2 z)} \qquad (3.8)$$

This case consists of four related parts.

Firstly, the substitutions of

$$\begin{cases} a_1 - a_2 = 1, \\ a_1 a_2 = 1, \end{cases} \text{ and } \begin{cases} p_1 - p_2 = 3, \\ p_1 p_2 = -2, \end{cases}$$

in (3.7) give

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2]) z^n$$

$$= \frac{z + 3z^2}{1 - 3z - 3z^2 + 6z^3 + 4z^4},$$

which represents a new generating function for product of Fibonacci numbers with Mersenne numbers, such that $F_n M_n = S_n(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2])$.

Secondly, the substitution of

$$\begin{cases} a_1 - a_2 = 3, \\ a_1 a_2 = -2, \end{cases} \text{ and } \begin{cases} p_1 - p_2 = 3, \\ p_1 p_2 = -2, \end{cases}$$

in (3.8) give

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2]) z^n$$

$$= \frac{z - 4z^3}{1 - 9z + 28z^2 - 36z^3 + 16z^4},$$

which represents a new generating function for Mersenne numbers of second order, such that

$$M_n^2 = S_{n-1}(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2]).$$

Thirdly, the substitution of

$$\begin{cases} a_1 - a_2 = 1, \\ a_1 a_2 = 2, \end{cases} \text{ and } \begin{cases} p_1 - p_2 = 3, \\ p_1 p_2 = -2, \end{cases}$$

in (3.8) give

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2]) z^n$$

$$= \frac{z + 4z^3}{1 - 3z - 8z^2 + 12z^3 + 4z^4},$$

which represents a new generating function for product of Jacobsthal numbers with Mersenne numbers, such that $J_n M_n = S_{n-1}(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2])$.

Finally, the substitution of

$$\begin{cases} a_1 - a_2 = 2, \\ a_1 a_2 = 1, \end{cases} \text{ and } \begin{cases} p_1 - p_2 = 3, \\ p_1 p_2 = -2, \end{cases}$$

in (3.8) give

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2]) z^n$$

$$= \frac{z + 2z^3}{1 - 6z + 3z^2 + 12z^3 + 4z^4}.$$

which represents a new generating function for product of Pell numbers with Mersenne numbers, such that

$$P_n M_n = S_{n-1}(a_1 + [-a_2]) S_{n-1}(p_1 + [-p_2]).$$

Case 2: Replacing p_2 by $(-p_2)$ and a_1 by $2a_1$ and a_2 by $(-2a_2)$ in (3.4) yields

$$\sum_{n=0}^{\infty} S_n(2a_1 + [-2a_2]) S_{n-1}(p_1 + [-p_2]) z^n$$

$$= \frac{2(a_1 - a_2)z + 4a_1 a_2(p_1 - p_2)z^2}{(1 - 2a_1 p_1 z)(1 + 2a_2 p_1 z)(1 + 2a_1 p_2 z)(1 - 2a_2 p_2 z)} \tag{3.9}$$

The substitution of

$$\begin{cases} p_1 - p_2 = 3, \\ p_1 p_2 = -2, \\ 4a_1 a_2 = -1, \end{cases}$$

in (3.9) and set for ease on notations $x = a_1 - a_2$, we reach

$$\sum_{n=0}^{\infty} S_n(2a_1 + [-2a_2]) S_{n-1}(p_1 + [-p_2]) z^n = \sum_{n=0}^{\infty} M_n U_n(x) z^n$$

$$= \frac{2xz - 3z^2}{1 - 6xz + (5 + 8x)z^2 - 12xz^3 + 4z^4},$$

which corresponds to a new generating function for the combined Mersenne numbers and Tchebychev polynomials of the second kind.

Theorem 4. For $n \in \mathbb{N}$, the new generating function of the product of Mersenne numbers M_n and Tchebychev polynomials of first kind is given by

$$\sum_{n=0}^{\infty} M_n T_n(x) z^n = \frac{xz - 3z^2 + 2xz^3}{1 - 6xz + (5 + 8x)z^2 - 12xz^3 + 4z^4}.$$

Proof We see that

$$\sum_{n=0}^{\infty} M_n T_n(x) z^n$$

$$= \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2]) \left(\begin{array}{c} S_n(2a_1 + [-2a_2]) \\ -xS_{n-1}(2a_1 + [-2a_2]) \end{array} \right) z^n$$

$$= \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2]) S_n(2a_1 + [-2a_2]) z^n$$

$$- x \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2]) S_{n-1}(2a_1 + [-2a_2]) z^n$$

$$= \sum_{n=0}^{\infty} M_n U_n(x) z^n$$

$$- \frac{x}{2(a_1 + a_2)} \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2])((2a_1)^n - (-2a_2)^n) z^n$$

$$= \sum_{n=0}^{\infty} M_n U_n(x) z^n$$

$$- \frac{x}{2(a_1 + a_2)} \left(\begin{array}{c} \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2])(2a_1 z)^n \\ - \sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2])(-2a_2 z)^n \end{array} \right).$$

On the other hand, we know that

$$\sum_{n=0}^{\infty} S_{n-1}(p_1 + [-p_2])z^n = \frac{z}{1 - 3z + 2z^2},$$

from which it follows

$$\sum_{n=0}^{\infty} M_n T_n(x) z^n$$

$$= \frac{2xz - 3z^2}{1 - 6xz + (5 + 8x)z^2 - 12xz^3 + 4z^4}$$

$$- \frac{x}{2(a_1 + a_2)} \left(\frac{2a_1 z}{1 - 6a_1 z + 8a_1^2 z^2} + \frac{2a_2 z}{1 + 6a_2 z + 8a_2^2 z^2} \right),$$

Therefore

$$\sum_{n=0}^{\infty} M_n T_n(x) z^n = \frac{xz - 3z^2 + 2xz^3}{1 - 6xz + (5 + 8x)z^2 - 12xz^3 + 4z^4}.$$

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