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Ward's Generalized Special Functions

Anthony G. Shannon[1,*], Ömür Deveci[2]

[1]Emeritus Professor, University of Technology Sydney, NSW 2007, Australia, Fellow, Warrane College, University of New South Wales, Kensington NSW 2033
[2]Department of Mathematics, Faculty of Science & Letters, Kafkas University, 36100, Kars, Turkey
*Corresponding author: tshannon38@gmail.com

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Abstract This paper considers generalizations of Bernoulli and Euler numbers to clarify and extend some known relations studied by Morgan Ward. It does this with the Euler-Maclaurin sum formula. It relates the mappings to category theory as a means of applying the ideas further.

Keywords: *Special function, Bernoulli polynomial, Euler function, difference operator, normal sequence, divisibility sequence, difference operator, generalized integer, Fermatian numbers, Euler-Maclaurin sum-formula, commutative diagram, category*

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1. Introduction

We explore here some formal aspects of generalized Bernoulli numbers in terms of normal divisibility sequences as defined by Morgan Ward [1,2]. The generalized Bernoulli polynomials in question have the form $B_n(x)$ such that

$$\frac{tE(xt)}{E(t)-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{u_n!} \tag{1}$$

in which

$$E(t) = \sum_{n=0}^{\infty} \frac{t^n}{u_n!} \tag{2}$$

is an analogous Euler function and $\{u_n\}$ is a sequence which is normal in that $u_{(s,t)} = (u_s, u_t)$ and divisible in that $u_s \mid u_t$ whenever $s \mid t$. We shall also require that $u_0 = 0,\ u_1 = 1,\ u_n \neq 0,\ n > 1$ and $u_n! = \prod_{j=1}^{n} u_j, u_0! = 1.$

$B_n(0) = B_n$, a generalized Bernoulli number. Ward's generalized coefficients were actually rediscoveries of work by Fontené [3].

2. Associated Operators

We use the operator $D_x : D_x x^n = u_n x^{n-1}$, and forward difference operator Δ:

$$\Delta \mathrm{B_n}(\mathrm{x}) = \mathrm{B_n}(\mathrm{x}+1) - \mathrm{B_n}(\mathrm{x})\,. \tag{3}$$

Formally,

$$\sum_{n=0}^{\infty} B_n(x+1)\frac{t^n}{u_n!} = \frac{tE\big((x+1)t\big)}{E(t)-1} = \frac{tE(xt)E(t)}{E(t)-1}$$

so that

$$\begin{aligned}
\sum_{n=0}^{\infty} \Delta B_n(x)\frac{t^n}{u_n!} &= \frac{tE(xt)E(t)-tE(xt)}{E(t)-1} \\
&= \frac{tE(xt)\big(E(t)-1\big)}{E(t)-1} \\
&= tE(xt) \\
&= \sum_{n=0}^{\infty} x^n \frac{t^{n+1}}{u_n!} \\
&= \sum_{n=1}^{\infty} u_n x^{n-1} \frac{t^n}{u_n!} \\
&= \sum_{n=1}^{\infty} Dx^n \frac{t^n}{u_n!}
\end{aligned}$$

and so that on equating coefficients of t, we get the rather neat result that extends Ward [4]

$$\Delta B_n(x) = D_x x^n \tag{4}$$

3. Generalized Bernoulli Numbers

As with ordinary Bernoulli numbers we set $B_0 = 1$, $B_{2n+1} = 0,\ \ n \geq 1,$ and define

$$B_n(t) = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{t^n}{u_n!} \tag{5}$$

where

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{u_n!}{u_k! u_{n-k}!}$$

is a Fontené-Ward binomial coefficient for which Henry Gould [3] obtained some elegant results. (For other approaches to generalized integers see Graeme Cohen [5].) The generalization used here consists essentially in systematically replacing the ordinary binomial coefficient with a binomial coefficient to the base u. Thus, when $\{u_n\} = Z$, the set of integers, the two binomial coefficients are formally identical. We shall refer briefly to the mappings in Section 5. When $\{u_n) = \{\underline{q}_n)$, the set of Fermatian numbers [6] defined by

$$\underline{q}_n = \sum_{j=1}^{n} q^{j-1}$$

where q may be indeterminate with $\underline{q}_1 = 1$, we get

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix},$$

the well-known q-binomial coefficient [7]. Note that $\underline{1}_n = n$. $\underline{q}_n$ is described as the n^{th} Fermatian of index q [8]. More particularly, when $\{u_n\}$ is the sequence of Fibonacci numbers we have the Fibonacci binomial coefficients [9]. Ward has defined a sequence to be normal when

$$\begin{aligned} E(t)E(-t) &= E(t-t) \\ &= E(0) \\ &= 1. \end{aligned}$$

So

$$t\left(1 - E(t)\right) = (-t)\left(E(t) - 1\right)$$

and

$$t\left(1 - E(t)\right) = tE(t)\left(E(-t) - 1\right)$$

so that

$$\begin{aligned} \frac{tE(t)}{E(t)-1} &= \frac{(-t)}{E(-t)-1} \\ &= \sum_{n=0}^{\infty} B_n \frac{(-t)^n}{u_n!}. \end{aligned}$$

But

$$\begin{aligned} \frac{tE(t)}{E(t)-1} &= E(t) \sum_{n=0}^{\infty} B_n \frac{t^n}{u_n!} \\ &= \sum_{m=0}^{\infty} \frac{t^m}{u_m!} \sum_{n=0}^{\infty} B_n \frac{t^n}{u_n!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} B_{n-k} \frac{t^n}{u_n!}, \end{aligned}$$

and so

$$(-1)^n B_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} B_{n-k}. \tag{6}$$

Since $B_{2n+1} = 0$, $n \geq 1$, when $\{u_n\} = Z$, these reduce to the ordinary Bernoulli numbers [10].

4. Euler-Maclaurin Formula

The importance of the ordinary Bernoulli numbers comes primarily from the Euler-Maclaurin sum-formula for $\sum m^k$. A generalization of this is

$$\sum_{j=1}^{n-1} j^k = \sum_{j=0}^{k} \frac{n^{k-j+1}}{u_{k-j+1}} \begin{Bmatrix} k \\ j \end{Bmatrix} B_n \tag{7}$$

the proof of which follows:

$$\begin{aligned} u_k! x \sum_{j=0}^{n-1} E(jx) &= u_k! \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} j^i \frac{x^{i+1}}{u_i!} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} \frac{u_k!}{u_i!} j^i x^{i+1} \end{aligned}$$

and the coefficients of x^{k+1} are $\sum_{j=0}^{n-1} j^k$; since

$$E(x)E(mx) = E\left((m+1)x\right)$$

$$\sum_{j=0}^{n-1} E(jx) = \frac{E(nx) - E(0)}{E(x) - 1}.$$

Thus

$$\begin{aligned} u_k! x \sum_{j-0}^{n-1} E(jx) &= u_k! \frac{x\left(E(nx) - 1\right)}{E(x) - 1} \\ &= u_k! \sum_{j=0}^{\infty} B_j \frac{x^j}{u_j!} \sum_{i=0}^{\infty} n^{i+1} \frac{x^{i+1}}{u_{i+1}!} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} \frac{n^{i-j+1} u_k!}{u_j! u_{i-j+1} 1} B_j x^{i+1} \end{aligned}$$

and the coefficients of x^{k+1} are as required in equation (7).

Hence, these generalized Bernoulli numbers B_n and the ordinary Bernoulli numbers b_n are actually related by

$$\sum_{j=0}^{k} \frac{n^{k-j+1}}{k-j+1} \binom{k}{j} b_j = \sum_{j=0}^{k} \frac{n^{k-j+1}}{u_{k-j+1}} \begin{Bmatrix} k \\ j \end{Bmatrix} B_n. \tag{8}$$

The Euler-Maclaurin formula is also of interest within the context of this paper because it is used in the classic proof of the Staudt-Clausen Theorem by Richard Rado [10,11] and its extensions [12,13].

5. Concluding Comments

Further related research can be carried out with applications of the q-umbral calculus [14] or with category theory. For an example of the latter, let u be the sequence function $u: Z \to Z$ so that $n \xrightarrow{u} u_n$; let h represent the highest common factor function $h: Z \times Z \to Z$ and then we have $n, m \xrightarrow{h} (n, m)$. The divisibility sequences mentioned in this paper can then be represented in terms of a commutative diagram:

$$\begin{tikzcd} ZxZ \arrow[r, "uxu"] \arrow[d, "h"'] & ZxZ \arrow[d, "h"] \\ Z \arrow[r, "u"] & Z \end{tikzcd}$$

For example,

$$\begin{tikzcd} n,m \arrow[r, "uxu"] \arrow[d, "h"'] & u_n, u_m \arrow[d, "h"] \\ (n,m) \arrow[r, "u"] & u_{(n,m)}=(u_n, u_m) \end{tikzcd}$$

Now a commutative diagram of four sets and functions with such a composition of functions determines a concrete category. It would be of interest to seek a natural transformation of functions between this and other categories either to generalize or to clarify the structure of the number theoretic results. The theory of categories has been applied to the problem in psychology of abstraction from the senses to the intellect through perception [15] by one of the present writers with Anthony Allen.

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