On Generalization of Dragomir’s Inequalities

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Abstract In this paper, we establish some generalization of weighted Ostrowski type integral inequalities for functions of bounded variation.

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1. Introduction

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \((a, b)\) whose derivative \( f' : (a, b) \rightarrow \mathbb{R} \) is bounded on \([a, b]\), i.e.

\[ \|f'\|_{\infty} := \sup_{t \in [a, b]} |f'(t)| < \infty. \]

Then we have the inequality

\[
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \right| \leq \frac{1}{4} \frac{\left( x - \frac{a+b}{2} \right)^2}{(b-a)^2} (b-a) \|f'\|_{\infty},
\]

for all \( x \in [a, b] \). The constant \( \frac{1}{4} \) is the best possible. This inequality is well known in the literature as the Ostrowski inequality.

Definition 1. Let \( P : a = x_0 < x_1 < \ldots < x_n = b \) be any partition of \([a, b]\) and let \( \Delta f(x_i) = f(x_{i+1}) - f(x_i) \) Then \( f(x) \) is said to be of bounded variation if the sum

\[ \sum_{i=1}^{n} |\Delta f(x_i)| \]

is bounded for all such partitions.

Let \( f \) be of bounded variation on \([a, b]\), and \( \sum(P) \) denotes the sum \( \sum_{i=1}^{n} |\Delta f(x_i)| \) corresponding to the partition \( P \) of \([a, b]\). The number

\[ \sqrt{v}(f) := \sup \{ \sum(P) : P \in \mathcal{P}([a, b]) \}, \]

is called the total variation of \( f \) on \([a, b]\). Here \( \mathcal{P}([a, b]) \) denotes the family of partitions of \([a, b]\).

In [7], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

**Theorem 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a mapping of bounded variation on \([a, b]\). Then

\[
\left| \int_{a}^{b} f(t) \, dt - (b-a) f(x) \right| \leq \frac{1}{4} \left( b-a + \frac{x-a+b}{2} \right)^2 \sqrt{v}(f)
\]

for all \( x \in [a, b] \). The constant \( \frac{1}{4} \) is the best possible.

In [9], Dragomir gave a simple proof of following Lemma:

**Lemma 1.** Let \( f, u : [a, b] \rightarrow \mathbb{R} \). If \( f \) is continuous on \([a, b]\) and \( u \) is bounded variation on \([a, b]\), then

\[
\sqrt{v}(f) \leq \max_{t \in [a, b]} \sqrt{v}(u). \]

In [5], Dragomir obtained following Ostrowski type inequality for functions of bounded variation:

**Theorem 2.** Let \( I_k = \{ x_0 < x_1 < \ldots < x_k = b \} \) be a division of the interval \([a, b]\) and \( \alpha_i (i = 0, 1, \ldots, k+1) \) be \( k+2 \) points so that \( \alpha_0 = a, \alpha_i \in [x_{i-1}, x_i], (i = 1, \ldots, k), \alpha_{k+1} = b \). If \( f : [a, b] \rightarrow \mathbb{R} \) is of bounded variation on \([a, b]\), then we have the inequality:

\[
\left| \int_{a}^{b} f(x) \, dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \leq \frac{1}{2} \sqrt{v}(h) + \max_{t \in [a, b]} \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right| \sqrt{v}(f)
\]

for all \( x \in [a, b] \).
where $\nu(h) = \max \{ h_i \mid i = 0, \ldots, n - 1 \}$, $h_i = x_{i+1} - x_i$ ($i = 0, 1, \ldots, k - 1$) and $\nu(f)$ is the total variation of $f$ on the interval $[a, b]$.

For some recent results connected with functions of bounded variation see [1,2,3,4,6,8,10-15,17-21].

The aim of this paper is to obtain some generalization of weighted Ostrowski type integral inequalities for functions of bounded variation.

2. Main Results

Firstly, we will give the following notations which are used in main Theorem:

Let $I_n : a = x_0 < \ldots < x_n = b$ be a partition of the interval $[a, b]$. $\alpha_i$ ($i = 0, 1, \ldots, n - 1$) be $n+2$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ ($i = 1, \ldots, n$), $\alpha_n = b$. Let $w : [a, b] \to (0, \infty)$ be continuous and positive mapping on $[a, b]$, and

$\nu(h) = \max \{ h_i \mid i = 0, \ldots, n - 1 \}$,

$h_i = x_{i+1} - x_i$ ($i = 0, 1, \ldots, n - 1$),

$\nu(L) = \max \{ L_i \mid i = 0, \ldots, n - 1 \}$,

$L_i = \int_{x_i}^{x_{i+1}} w(u) du$ ($i = 0, 1, \ldots, n - 1$).

Theorem 3. If $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then we have the inequalities

$$
\left| \sum_{i=0}^{n} \left( \frac{a_{i+1}}{a_i} \int_{a_i}^{a_{i+1}} w(u) du \right) f(x_i) - \frac{b}{a} \int_a^b f(t) w(t) dt \right| \\
\leq \| h \|_{w,[a,b]} \left[ \frac{1}{2} \nu(h) + \max_{i \in [0, 1, \ldots, k - 1]} a_{i+1} - a_{i} \right] \sqrt{\nu(f)} \tag{3}
$$

and

$$
\left| \sum_{i=0}^{n} \left( \frac{a_{i+1}}{a_i} \int_{a_i}^{a_{i+1}} w(u) du \right) f(x_i) - \frac{b}{a} \int_a^b f(t) w(t) dt \right| \\
\leq \left[ \frac{1}{2} \nu(L) + \max_{i \in [0, 1, \ldots, n - 1]} \left[ \frac{1}{2} \int_{x_i}^{x_{i+1}} w(u) du - \frac{x_i + x_{i+1}}{2} \right] \right] \sqrt{\nu(f)} \tag{4}
$$

where $\sqrt{\nu(f)}$ is the total variation of $f$ on the interval $[a, b]$.

Proof. Let us consider the functions $K$ defined by

$$
K(t) = \begin{cases}
\frac{1}{a_1} \int_a^{a_1} w(u) du, & t \in [a, x_1) \\
\frac{1}{a_2} \int_{a_1}^{a_2} w(u) du, & t \in [x_1, x_2) \\
\vdots \\
\frac{1}{a_n} \int_{a_{n-1}}^{a_n} w(u) du, & t \in [x_{n-1}, b].
\end{cases}
$$

Integrating by parts, we obtain

$$
\frac{b}{a} \int_a^b K(t) df(t) = \sum_{i=0}^{n} \left[ \frac{1}{a_{i+1}} \int_{a_i}^{a_{i+1}} w(u) du \right] f(x_i). \tag{5}
$$

In last equality in (5), we have

$$
\sum_{i=1}^{n} \left[ \frac{1}{a_i} \int_{a_{i-1}}^{a_i} w(u) du \right] f(x_i) = \int_a^b w(u) du \int_a^b f(x) dx.
$$

Adding (6) and (7) in (5), we get the equality
\[
\frac{b}{a} \int_{a}^{b} K(t) \, df(t) = \left( \frac{b}{a} \int_{a}^{b} w(u) \, du \right) f(b) + \sum_{i=0}^{n-1} \int_{a}^{a_{i+1}} w(u) \, du \frac{f(x_i)}{a_i} + \int_{a}^{a_{i+1}} w(u) \, du \frac{f(a) - \int_{a}^{a_{i+1}} w(u) \, df(t)}{a} \int_{a}^{a_{i+1}} w(u) \, df(t) \right) \int_{a}^{a_{i+1}} w(u) \, df(t) \right) \int_{a}^{a_{i+1}} w(u) \, df(t) \right)
\]

On the other hand, taking modulus in (8) and using triangle inequality we have
\[
\left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{a_{i+1}} f(x_i) - \int_{a}^{a_{i+1}} w(u) \, df(t) \right| \leq \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{a_{i+1}} f(x_i) \right| \leq \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{a_{i+1}} f(x_i) \right|
\]

Using Lemma 1 in last inequality in (9), we have
\[
\left| \int_{x_i}^{x_{i+1}} (t - a_{i+1}) \, df(t) \right| \leq \sup_{t \in [x_i, x_{i+1}]} \left| t - a_{i+1} \right| \sqrt{(f)}
\]

Finally, for proof of inequality (4), taking modulus in (8), we have
\[
\left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{a_{i+1}} f(x_i) \right| \leq \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{a_{i+1}} f(x_i) \right|
\]

Adding (12) in last inequality in (11), we obtain the inequality (3).

This completes the proof of first inequality in (3).

On the other hand, in last inequality in (11), we have
\[
\left[ a_{i+1} - \frac{x_{i+1} + x_i}{2} \right] \sqrt{(f)}
\]

Adding (14) in (13), we obtain
\[
\sum_{i=0}^{n} \left( \int_{a_i}^{a_{i+1}} w(u)du \right) f(x_i) - \int_{a_i}^{a_{i+1}} f(t)w(t)dt \leq \frac{1}{2} v(L) + \max_{i \in \{0, 1, \ldots, n-1\}} \left( h(a_{i+1}) - x_{i} + \frac{x_{i+1}}{2} \right) \frac{b}{a} \tag{16}
\]

which completes the proof of first inequality in (4).

Using triangle inequality in last inequality in (15), we have

\[
\max_{i \in \{0, 1, \ldots, n-1\}} \left( h(a_{i+1}) - x_{i} + \frac{x_{i+1}}{2} \right) \leq \frac{1}{2} v(L) \tag{17}
\]

which was proved by Kuei-Lin Tseng et al. in [20].

Remark 3. If we choose \( w(u) = 1, \ h(u) = u \) in (16), inequality reduces inequality (2).

Corollary 1. Under assumption Theorem 3, choosing \( x_0 = a, x_1 = b, \ \alpha_0 = a, \ \alpha_1, \alpha_2 = b \) in inequality (4) we obtain the inequality

\[
\int_{a}^{b} w(u)du f(a) + \frac{h}{a} \int_{a}^{b} f(b) - f(t)w(t)dt \leq \frac{b}{a} \left( \int_{a}^{b} w(u)du \right) \frac{b}{a} \tag{18}
\]

Remark 4. 1) In (17), if we take \( \alpha = b \), then we have the "weighted left rectangle inequality"

\[
\int_{a}^{b} w(u)du f(a) - \frac{h}{a} \int_{a}^{b} f(t)w(t)dt \leq \frac{b}{a} \left( \int_{a}^{b} w(u)du \right) \frac{b}{a} \tag{19}
\]

2) If we take \( \alpha = a \) in (17) then we have the "weighted right rectangle inequality"

\[
\int_{a}^{b} w(u)du f(b) - \frac{h}{a} \int_{a}^{b} f(t)w(t)dt \leq \frac{b}{a} \left( \int_{a}^{b} w(u)du \right) \frac{b}{a} \tag{20}
\]

3. Applications for Quadrature Rule

Let us consider the arbitrary division

\[ I_n : a = x_0 < x_1 < \ldots < x_n = b \]

and let \( w : [a, b] \rightarrow (0, \infty) \) be continuous function with

\[ \nu(L) = \max \{ L_i \ | \ i = 0, 1, \ldots, n-1 \}, \]

\[ L_i = \int_{x_i}^{x_{i+1}} w(u)du \ (i = 0, 1, \ldots, n-1). \]

Then the following theorem holds.

Theorem 4. Let \( f : Q \rightarrow \mathbb{R} \) is of bounded variation on \( Q \) and \( \xi_i \in [x_i, x_{i+1}] \) \( (i = 0, 1, \ldots, n-1) \). Then we have the quadrature formula:
\[
\begin{align*}
\int_{a}^{b} f(t)w(t)dt &= \sum_{i=0}^{n-1} \frac{x_{i}}{\xi_{i}} f(x_{i}) \\
+ \sum_{i=0}^{n-1} \int_{x_{i}}^{x_{i+1}} w(u)du \int_{x_{i}}^{x_{i+1}} f(x_{i+1}) + R_{w}(I_{n}, f, w, \xi).
\end{align*}
\]

The remainder term \( R_{w}(I_{n}, f, w, \xi) \) satisfies
\[
R_{w}(I_{n}, f, w, \xi) \leq \frac{1}{2} \nu(L) + \max_{i \in \{0, 1, \ldots, n-1\}} \frac{1}{2} \int_{x_{i}}^{x_{i+1}} w(u)du - \int_{x_{i}}^{x_{i+1}} f(t)w(t)dt \int_{x_{i}}^{x_{i+1}} (f).
\]

Proof. Applying Corollary 1 to interval \([x_{i}, x_{i+1}]\), we have the inequality
\[
\begin{align*}
\left| \frac{x_{i+1}}{\xi_{i}} f(x_{i}) + \int_{x_{i}}^{x_{i+1}} w(u)du \int_{x_{i}}^{x_{i+1}} f(x_{i+1}) + \int_{x_{i}}^{x_{i+1}} w(u)du \right| \\
\leq \left[ \frac{1}{2} \int_{x_{i}}^{x_{i+1}} w(u)du \int_{x_{i}}^{x_{i+1}} f(x_{i+1}) + \int_{x_{i}}^{x_{i+1}} w(u)du \right] \vee (f).
\end{align*}
\]

Summing the inequality (19) over \( i \) from 0 to \( n-1 \), then we have
\[
\begin{align*}
|R_{w}(I_{n}, f, w, \xi)| &\leq \max_{i \in \{0, 1, \ldots, n-1\}} \left[ \frac{1}{2} \int_{x_{i}}^{x_{i+1}} w(u)du \int_{x_{i}}^{x_{i+1}} f(x_{i+1}) + \int_{x_{i}}^{x_{i+1}} w(u)du \right] \vee (f) \\
&\leq \frac{1}{2} \nu(L) + \max_{i \in \{0, 1, \ldots, n-1\}} \left[ \frac{1}{2} \int_{x_{i}}^{x_{i+1}} w(u)du - \int_{x_{i}}^{x_{i+1}} w(u)du \right] \vee (f).
\end{align*}
\]

This completes proof of the first inequality in (18). Also, we have
\[
\begin{align*}
\left| \frac{x_{i+1}}{\xi_{i}} f(x_{i}) + \int_{x_{i}}^{x_{i+1}} w(u)du \right| \int_{x_{i}}^{x_{i+1}} f(t)w(t)dt \\
\leq \left[ \frac{x_{i+1}}{\xi_{i}} w(u)du + \int_{x_{i}}^{x_{i+1}} w(u)du \right] \int_{x_{i}}^{x_{i+1}} (f) + \left[ \frac{x_{i+1}}{\xi_{i}} w(u)du + \int_{x_{i}}^{x_{i+1}} w(u)du \right] \int_{x_{i}}^{x_{i+1}} (f).
\end{align*}
\]

which completes the proof.

Remark 5.

1) If we choose \( \xi_{i} = x_{i+1} \), then we have the weighted left rectangle rule
\[
\begin{align*}
\int_{a}^{b} f(t)w(t)dt &= \sum_{i=0}^{n-1} \frac{x_{i+1}}{\xi_{i}} f(x_{i}) + R_{wL}(I_{n}, f, w). \\
&= \frac{1}{2} \nu(L) \int_{a}^{b} (f).
\end{align*}
\]

The remainder \( R_{wL}(I_{n}, f, w) \) satisfies
\[
|R_{wL}(I_{n}, f, w)| \leq \frac{1}{2} \nu(L) \int_{a}^{b} (f).
\]

2) Similarly, choosing \( \xi_{i} = x_{i} \), we have the weighted right rectangle rule
\[
\begin{align*}
\int_{a}^{b} f(t)w(t)dt &= \sum_{i=0}^{n-1} \frac{x_{i+1}}{\xi_{i}} f(x_{i}) + R_{wR}(I_{n}, f, w). \\
&= \frac{1}{2} \nu(L) \int_{a}^{b} (f).
\end{align*}
\]

And, the remainder term \( R_{wR}(I_{n}, f, w) \) satisfies
\[
|R_{wR}(I_{n}, f, w)| \leq \frac{1}{2} \nu(L) \int_{a}^{b} (f).
\]

References


