1. Introduction

Fibonacci and Lucas numbers have been studied by many researchers for a long time to get intrinsic theory and applications of these numbers in many research areas as Physics, Engineering, Architecture, Nature and Art. For example, the ratio of two consecutive numbers converges to the Golden ratio \( \alpha = \frac{1 + \sqrt{5}}{2} \) which was thoroughly interested in [13]. We should recall that, for \( k \in \mathbb{R}^+ \), \( k \)-Fibonacci \( \{F_k, n\}_{n \in \mathbb{N}} \) and \( k \)-Lucas \( \{L_k, n\}_{n \in \mathbb{N}} \) sequences have been defined by the recursive equations [9,10]:

\[
F_{k,n+2} = k F_{k,n+1} + F_{k,n}, \quad L_{k,n+2} = k L_{k,n+1} + L_{k,n},
\]

with initial conditions \( F_{k,0} = 1, \ F_{k,1} = k \) and \( L_{k,0} = 2, \ L_{k,1} = k \), respectively. For the special case \( k = 1 \), it is clear that these two sequences are simplified to the well-known Fibonacci and Lucas sequences, respectively. In this contribution, we shall define a new useful operator denoted by \( \delta_{p_1 p_2}^k \) for which we can formulate, extend and prove new results based on our previous ones [4,5,6]. In order to determine generating functions for \( k \)-Fibonacci numbers, \( k \) -Lucas numbers and Lucas polynomials, we combine between our indicated past techniques and these presented polishing approaches.

Let \( k \) and \( n \) be two positive integers and \( \{x_1, x_2, \ldots, x_n\} \) are set of given variables, recall [8] that the \( k \)-th elementary symmetric function \( e_k(x_1, x_2, \ldots, x_n) \) and the \( k \)-th complete homogeneous symmetric function \( h_k(x_1, x_2, \ldots, x_n) \) are defined respectively by

\[
e_k (x_1, x_2, \ldots, x_n) = \sum_{i_1 + i_2 + \cdots + i_n = k} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, 0 \leq k \leq n.
\]

With \( i_1, i_2, \ldots, i_n = 0 \) or 1,

\[
h_k (x_1, x_2, \ldots, x_n) = \sum_{i_1 + i_2 + \cdots + i_n = k} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, 0 \leq k \leq n.
\]

With \( i_1, i_2, \ldots, i_n \geq 0 \), First, we set \( e_0(x_1, x_2, \ldots, x_n) = 1 \) and \( h_0 (x_1, x_2, \ldots, x_n) = 1 \) (by convention).

For \( k > n \) or \( k < 0 \), we set

\[
e_k (x_1, x_2, \ldots, x_n) = 0 \quad \text{and} \quad h_k (x_1, x_2, \ldots, x_n) = 0.
\]

**Definition 1.** [1] Let \( B \) and \( P \) be any two alphabets, then we give \( S_n(B - P) \) by the following form:

\[
\prod_{p \in P} (1 - pt) = \prod_{b \in B} (1 - bt) = \sum_{n=0}^{\infty} S_n(B - P)t^n,
\]

with the condition \( S_n(B - P) = 0 \) for \( n < 0 \).

**Definition 2.** [2] Let \( g \) be any function on \( \mathbb{R}^n \), then we consider the divided difference operator as the following form:

\[
\delta_{x_1, x_{i+1}}(g) = \frac{g(x_1, \ldots, x_{i+1}, \ldots, x_n) - g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}{x_i - x_{i+1}}.
\]

**Definition 3.** [7] The symmetrizing operator \( \delta_{p_1 p_2}^k \) is defined by

\[
\delta_{p_1 p_2}^k g(p_1) = \frac{p_k^k g(p_1) - p_k^k g(p_2)}{p_1 - p_2} \quad \text{for all} \ k \in \mathbb{N}.
\]

**Remark 1.** Let \( P = \{p_1, p_2\} \) an alphabet, we have
2. The $k$-Lucas Numbers and Properties

The $k$-Lucas numbers have been defined in [11] for any number $k$ as follows.

**Definition 4.** [11] For any positive real number, the $k$-Lucas numbers, say $\{L_{k,n}\}_{n \in \mathbb{N}}$, are defined recurrently by

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1} \quad \text{for } n \geq 1,$$

with initial conditions $L_{k,0} = 2, L_{k,1} = k$.

Note that if $k$ is a real variable $x$ then $L_{k,n} = L_n(x)$ and they correspond to the Lucas polynomials defined by

$$L_{n+1}(x) = \begin{cases} 2 & \text{if } n = 0, \\ x & \text{if } n = 1, \\ xL_n(x) + L_{n-1}(x) & \text{if } n > 1. \end{cases}$$

Particular cases of the $k$-Lucas numbers are

- If $k = 1$, the classical Lucas numbers are obtained:
  
  \begin{align*}
  L_0 &= 2, L_1 = 1, \\
  L_{n+1} &= L_n + L_{n-1} \quad \text{for } n \geq 1, \\
  \{L_n\}_{n \in \mathbb{N}} &= \{2,1,3,4,7,11,24,\ldots\}.
  \end{align*}

- If $k = 2$, the Pell-Lucas numbers appear:
  
  \begin{align*}
  Q_0 &= 2, Q_1 = 2, \\
  Q_{n+1} &= 2Q_n + Q_{n-1} \quad \text{for } n \geq 1, \\
  \{Q_n\}_{n \in \mathbb{N}} &= \{2,2,6,14,34,\ldots\}.
  \end{align*}

The well-known Binet’s formula in the Lucas numbers theory allows us to express the $k$-Lucas number in function of the roots $r_1$ and $r_2$ of the characteristic equation, associated to the recurrence relation (2.1):

$$r^2 = kr + 1.$$  \hfill (2.2)

**Proposition 1.** (Binet’s formula) The $n$th $k$-Lucas number is given by

$$L_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

where $r_1, r_2$ are the roots of the characteristic equation (2.2) and $r_1 > r_2$.

**Proof.** The roots of the characteristic equation (2.2) are

$$\eta_1 = \frac{k + \sqrt{k^2 + 4}}{2} \quad \text{and} \quad \eta_2 = \frac{k - \sqrt{k^2 + 4}}{2}.$$

Note that, since $k > 0$, the $r_2 < 0 < r_1$ and $|r_2| < |r_1|$, $r_1 + r_2 = k$ and $r_1 - r_2 = \sqrt{k^2 + 4}$.

If $\sigma$ denotes the positive root of the characteristic equation, the general term may be written in the form [10]

$$L_{k,n} = \frac{\sigma^n - \sigma^{-n}}{\sigma + \sigma^{-1}}$$

and the limit of the quotient of two terms is

$$\lim_{n \to \infty} \frac{L_{k,n+r}}{L_{k,n}} = \sigma^r.$$  

In addition, the general term of the $k$-Lucas numbers may be obtained by the formula [10]:

$$L_{k,n} = kF_{k,n-1} + F_{k,n+1}.$$  

3. On the Symmetric Functions of Some Numbers and Polynomials

**Theorem 1.** [4] Let $P$ and $B$ be two alphabets, respectively, $\{p_1, p_2\}$ and $\{b_1, b_2, b_3\}$, then we have

$$\sum_{n=0}^{\infty} h_n(b_1, b_2, b_3)h_n(p_1, p_2) t^n$$

$$= \left(1 - p_1 p_2^2 (b_1, b_2, b_3) t^2 \right) - p_1 p_2^2 (b_1, b_2, b_3) h_1 (p_1, p_2) t^3$$

$$+ \left(\sum_{n=0}^{\infty} c_n(b_1, b_2, b_3) t^n\right) \left(\sum_{n=0}^{\infty} c_n(b_1, b_2, b_3) t^n\right)^2.$$  \hfill (3.1)

In the case $B = \{1\}$ based on Theorem 1, we deduce the following Lemmas.

**Lemma 1.** Given an alphabet $P = \{p_1, -p_2\}$, we have

$$\sum_{n=0}^{\infty} h_n(p_1, [-p_2]) t^n = \frac{1}{1 - (p_1 - p_2)^{t} + p_1 p_2 t^2}.$$  \hfill (3.2)

**Proof.** Let $\sum_{n=0}^{\infty} p_1^n t^n$ and $1 - p_1 t$ be two sequences such that $\sum_{n=0}^{\infty} p_1^n t^n$ and $(1 - p_1 t)$ be two sequences. Following Lemma 1, the left-hand side of the formula (3.2) can be written as:

$$\delta_{p_1[-p_2]} \sum_{n=0}^{\infty} p_1^n t^n = p_1 \sum_{n=0}^{\infty} p_1^n t^n - p_2 \sum_{n=0}^{\infty} [-p_2]^n t^n$$

$$= \sum_{n=0}^{\infty} \frac{p_1^{n+1} [-p_2]^{n+1}}{p_1 - [-p_2]} t^n$$

$$= \sum_{n=0}^{\infty} h_n(p_1, [-p_2]) t^n.$$  

While the right-hand side can be expressed as

$$\delta_{p_1[-p_2]} \frac{1}{1 - p_1 t} = p_1 \frac{1}{1 - p_1 t} - \frac{[-p_2]}{1 - [-p_2] t}$$

$$= p_1 \frac{1}{1 - p_1 t} - \frac{[-p_2]}{1 + [p_2] t}$$

$$= \frac{p_1^{n+1} [-p_2]^{n+1}}{p_1 [-p_2]}$$

$$= \frac{p_1 (1 + p_2 t + p_2 (1 - p_2 t))}{(p_1 + p_2) (1 - p_1 t) (1 + p_2 t)}$$

$$= \frac{p_1 + p_2}{(p_1 + p_2) (1 - p_1 t - p_1 p_2 t^2)}$$

$$= \frac{1}{1 - (p_1 - p_2) t - p_1 p_2 t^2}.$$
This completes the proof.

**Lemma 2.** Given an alphabet \( P = \{p_1, -p_2\} \), we have
\[
\sum_{n=0}^{\infty} h_{n+1}(p_1, -p_2) t^n = \frac{p_1 - p_2 + p_1 p_2 t}{1 - (p_1 - p_2) t - p_1 p_2 t^2}.
\]
(3.3)

**Proof.** Let \( \sum_{n=0}^{\infty} p_1^n t^n \) and \((1 - p_1 t)\) be two sequences such that \( \sum_{n=0}^{\infty} p_1^n t^n = \frac{1}{1 - p_1 t} \), the left-hand side of the formula (3.3) can be written as:
\[
\delta^2 p_1[-p_2] \sum_{n=0}^{\infty} p_1^n t^n = \frac{p_1^2 \sum_{n=0}^{\infty} p_1^n t^n}{p_1} \left( \frac{-p_2}{1 - p_2} \right) - \frac{p_1}{1 - p_2} t^n
\]
\[= \sum_{n=0}^{\infty} p_1^{n+2} \left( -p_2 \right)^n t^{n+2}
= \sum_{n=0}^{\infty} p_1^{n+2} \left( -p_2 \right)^n t^n.
\]
(3.4)

From which we have the following theorem.

**Theorem 3.** For \( n \in \mathbb{N} \), the generating function of the \( k \)-Lucas numbers is given by
\[
\sum_{n=0}^{\infty} L_{k,n} t^n = \frac{2 - k t}{1 - k t - t^2}.
\]
(3.6)

• Put \( k = 2 \) in the relationship (3.6) we have
\[
\sum_{n=0}^{\infty} Q_{2,n} t^n = \frac{2 - 2 t}{1 - 2 t - t^2},
\]
which represents a generating function for Pell-Lucas numbers [5].

Replacing \( t \) by \((-t)\) in (3.4) and (3.6), we have the following theorems.

**Theorem 4.** We have the following a new generating function of the \( k \)-Fibonacci numbers at negative indices as
\[
\sum_{n=0}^{\infty} F_{k,-n} t^n = \frac{1}{t^2 - k t - 1}.
\]

**Proof.** The ordinary generating function associated is defined by
\[
G(F_{k,n} t) = \sum_{n=0}^{\infty} F_{k,n} t^n.
\]
Using the initial conditions, we get
\[
\sum_{n=0}^{\infty} F_{k,n} t^n = F_{k,0} t^0 + F_{k,1} t + \sum_{n=2}^{\infty} F_{k,n} t^n
= 1 + k t + \sum_{n=2}^{\infty} k F_{k,n-1} t^n + \sum_{n=2}^{\infty} F_{k,n-2} t^n.
\]
Consider that \( j = n - 2 \) and \( p = n - 1 \). Then can be written by
\[
= 1 + k t + \sum_{n=2}^{\infty} k F_{k,n-1} t^n + t^2 \sum_{j=0}^{\infty} F_{k,n} t^n
= 1 + k t + \sum_{p=0}^{\infty} F_{k,n-2} t^n - k t + t^2 \sum_{j=0}^{\infty} F_{k,n} t^n.
\]
which is equivalent to
\[
\left(1 - kt - t^2\right) \sum_{n=0}^{\infty} F_{k,n} t^n = 1
\]
\[
\Rightarrow \sum_{n=0}^{\infty} F_{k,n} t^n = \frac{1}{1 - kt - t^2}.
\]
Replacing \(t\) by \((-t)\), we have
\[
\sum_{n=0}^{\infty} (-1)^n F_{k,n} t^n = \frac{1}{1 + kt - t^2},
\]
therefore
\[
\sum_{n=0}^{\infty} (-1)^{n+1} F_{k,n} t^n = \frac{1}{t^2 - kt - 1}
\]
\[
\Rightarrow \sum_{n=0}^{\infty} F_{k,-n} t^n = \frac{1}{t^2 - kt - 1}.
\]
This completes the proof.

**Theorem 5.** We have the following a new generating function of the \(k\)-Lucas numbers at negative indices as
\[
\sum_{n=0}^{\infty} L_{k,-n} t^n = \frac{2 + kt}{1 + kt - t^2}.
\]

*Put \(k = 2\) in the relationship (3.7) we have
\[
\sum_{n=0}^{\infty} Q n t^n = \frac{2 + 2t}{1 + 2t - t^2},
\]
which represents a generating function for Pell-Lucas numbers at negative indices [3].

Choosing \(p_1\) and \(p_2\) such that \(p_1 p_2 = 1\) \((p_1 - p_2 = x)\ and substituting in (3.2) and (3.3) we end up with
\[
\sum_{n=0}^{\infty} F_n(x) t^n = \frac{1}{1 - xt - t^2},
\]
which represents a generating function of the Fibonacci polynomials
\[
\sum_{n=0}^{\infty} h_{n+1}(p_1,[-p_2]) t^n = \frac{x + t}{1 - xt - kt^2},
\]
which represents a new generating functions.

* Multiplying the equation (3.8) by \((2 + x^2)\) and subtract it from (3.9) by \((x)\), we obtain
\[
\sum_{n=0}^{\infty} \left((x^2 + 2)h_n(p_1,[-p_2]) - xh_{n+1}(p_1,[-p_2])\right) t^n
\]
\[
= \frac{2 - xt}{1 - xt - t^2}.
\]
Thus we get the following theorem.

**Theorem 6.** We have the following a generating function of the Lucas polynomials as
\[
\sum_{n=0}^{\infty} L_n(x) t^n = \frac{2 - xt}{1 - xt - t^2}.
\]

**Proof.** The ordinary generating function associated is defined by
\[
G(L_n(x), t) = \sum_{n=0}^{\infty} L_n(x) t^n.
\]

Using the initial conditions, we get
\[
\sum_{n=0}^{\infty} L_n(x) t^n = L_0(x) t^0 + L_1(x) t + \sum_{n=2}^{\infty} L_n(x) t^n
\]
\[
= 2 + xt + \sum_{n=2}^{\infty} (xL_{n-1}(x) + L_{n-2}(x)) t^n
\]
\[
= 2 + xt + \sum_{n=2}^{\infty} L_{n-1}(x) t^n + \sum_{n=2}^{\infty} L_{n-2}(x) t^n.
\]

Consider that \(j = n - 2\) and \(p = n - 1\). Then can be written by
\[
= 2 + xt + x \sum_{n=2}^{\infty} L_{n-1}(x) t^n + \sum_{n=2}^{\infty} L_{n-2}(x) t^n.
\]

which is equivalent to
\[
(1 - xt - t^2) \sum_{n=0}^{\infty} L_n(x) t^n = 2 - xt
\]
\[
\Rightarrow \sum_{n=0}^{\infty} L_n(x) t^n = \frac{2 - xt}{1 - xt - t^2}.
\]
This completes the proof.

Replacing \(t\) by \((-t)\) in (3.8) and (3.10), we have the following theorems.

**Theorem 7.** We have the following a new generating function of the Lucas polynomials at negative coefficient as
\[
\sum_{n=0}^{\infty} F_n(-x) t^n = \frac{1}{1 + xt - t^2}.
\]

**Theorem 8.** We have the following a new generating function of the Lucas polynomials at negative coefficient as
\[
\sum_{n=0}^{\infty} L_n(-x) t^n = \frac{2 + xt}{1 + xt - t^2}.
\]

**4. Conclusion**

In this paper, a new theorem has been proposed in order to determine the generating functions. The proposed theorem is based on the symmetric functions. The obtained results agree with the results obtained in some previous works.
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References