p -Adic Number Fields Acting On W^* -Probability Spaces

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Received September 30, 2016; Revised December 11, 2016; Accepted February 09, 2017

Abstract In this paper, we study how a p -adic number field $\mathbb{Q}_p$ acts on an arbitrarily fixed W^* -algebra, and how it affects the original free-probabilistic information on the W^* -algebra, for each prime p . In particular, by understanding the σ -algebra $\sigma(\mathbb{Q}_p)$ of $\mathbb{Q}_p$ as a semigroup equipped with the setintersection, we act $\sigma(\mathbb{Q}_p)$ on a unital tracial W^* -probability space (M, tr) , creating the corresponding semigroup W^* -dynamical system. From such a dynamical system, construct the crossed product W^* -algebra equipped with a suitable linear functional. We study free probability on such W^* -dynamical operator-algebraic structures determined by primes, and those on corresponding free products of such structures over primes. As application, we study cases where given W^* -probability spaces are generated by countable discrete groups.

Keywords: Free Probability, Free Probability Spaces, p -Adic Number Fields, Von Neumann Algebras, W^* -Dynamical Systems, Crossed Product W^* -Algebras.

Cite This Article: Ilwoo Cho, “ p -Adic Number Fields Acting On W^* -Probability Spaces.” *Turkish Journal of Analysis and Number Theory*, vol. 5, no. 2 (2017): 31-56. doi: 10.12691/tjant-5-2-2.

1. Introduction

The main purposes of this paper are (i) to consider how a prime p acts on a fixed unital tracial W^* -probability space, (ii) to study operator-algebraic, and operator-theoretic information of crossed product W^* -algebras obtained from W^* -dynamical systems obtained from (i), and (iii) to investigate free probability on free product of crossed product W^* -algebras of (ii). Also, we apply our main results to the cases where a fixed unital tracial W^* -probability space is generated by a countable discrete group.

1.1. Preview and Motivation

The relations between *primes* and *operator algebras* have been studied in various different approaches. For instance, we studied how primes act on certain types of von Neumann algebras, especially, generated by p -adic, and *Adelic measure spaces* (e.g., [3,6]). Meanwhile, in [4] and [5], primes are regarded as *linear functionals* acting on *arithmetic functions*. In such a case, one can understand arithmetic functions as *Krein-space operators* (for fixed primes), under certain *Krein-space representations* (e.g., [8]). Also, in [7], we considered free-probabilistic structures on a *Hecke algebra* $\mathcal{H}(G_p)$ for a fixed prime p . For more about number-theoretic motivation, see e.g., [9,10,13].

In [1], we considered free-distributional data of certain operators in a C^* -algebra M_p , induced by a (pure-algebraic)

* -algebra $\mathcal{M}_p$, consisting of all measurable functions over $\mathbb{Q}_p$, under *Hilbert-space representation*, for a prime p . For all $j \in \mathbb{Z}$, we defined C^* -probability spaces (M_p, φ_j^p) , where φ_j^p are kind of “filtered” *linear functionals* implying the *number-theoretic* data on $\mathcal{M}_p$, in terms of the usual p -adic integration on $\mathbb{Q}_p$, for a prime p . From the system

$$\{(M_p, \varphi_j^p) : p \in \mathcal{P}, j \in \mathbb{Z}\},$$

of C^* -probability spaces, the free product C^* -probability space

$$(M_{\mathcal{P}}(\mathbb{Z}), \varphi) = \bigstar_{p \in \mathcal{P}, j \in \mathbb{Z}} (M_p, \varphi_j^p)$$

is constructed, and the free probability on this C^* -probability space, called the *Adelic probability space*, is studied.

In [2], we established *weighted-semicircular elements*, and corresponding *semicircular elements* in certain Banach * -probability spaces induced by a C^* -probability space

$$(M_p, \varphi_j^p), \text{ for } p \in \mathcal{P}, j \in \mathbb{Z}.$$

Especially, the weighted-semicircularity of [2] is dictated by free-distributional data on M_p obtained in [1].

Motivated by the main results of [1] and [2], we here study how our p -adic-analytic structures affect the original

free-probabilistic data on arbitrarily fixed *unital tracial* W^* -probability spaces. To do that, we establish suitable W^* -dynamical systems induced by the σ -algebra $\sigma(\mathbb{Q}_p)$ of $\mathbb{Q}_p$. From such dynamical systems, we construct corresponding *crossed product* W^* -algebras, and investigate W^* -probability on them. In particular, we focus on how p -adic-analytic data (containing number-theoretic information) affect the original free-distributional data.

Remark and Emphasis Even though our proceeding works are highly motivated by those of [1] and [2], the aims, approaches, tools, theories and results are totally independent from each other.

1.2. Overview

In Sections 2, we briefly introduce backgrounds and a motivation of our works. In particular, we introduce p -adic number fields $\mathbb{Q}_p$, for primes p , and consider p -adic calculus on $\mathbb{Q}_p$, based on p -adic calculus, we construct free-probabilistic models on the $*$ -algebras $\mathcal{M}_p$ consisting of all measurable functions on $\mathbb{Q}_p$, for primes $\mathbb{Q}_p$, *Free products of $*$ -probability spaces* is observed, too.

In Section 3, we understand the σ -algebra $\sigma(\mathbb{Q}_p)$ as a semigroup equipped with the *set-intersection*, for a fixed prime p , and then act this semigroup on an arbitrarily taken unital tracial W^* -probability space (M, tr) . A W^* -dynamical system is well-established by the semigroup acting on (M, tr) , and the corresponding *crossed product* W^* -algebra is considered. By determining a suitable linear functional on such a crossed product W^* -algebra, free-probabilistic informations are investigated.

In Section 4, certain “sub”-free-probabilistic structures of the W^* -probability spaces of Section 3 are established and studied.

In Section 5, from the W^* -probability spaces of Section 4, we construct *free product W^* -probability spaces*, and study operator-algebraic, and operator-theoretic (especially, spectral-theoretic) properties of such free product W^* -probability spaces.

As an application of our main results of Sections 4 and 5, we restrict our interests to the cases where a fixed unital tracial W^* -probability space is induced by a *countable discrete group*, in Section 6. For example, we concentrate on the case where a fixed group is a *free group* with multi generators.

2. Preliminaries

In this section, we briefly mention about backgrounds of our works.

2.1. Free Probability

Readers can check fundamental analytic-and-combinatorial *free probability* from [12] and [14] (and the cited papers

therein). *Free probability* is understood as the noncommutative operator-algebraic version of classical probability theory (covering commutative cases). The classical *independence* is replaced by the *freeness*, by replacing measures to linear functionals. It has various applications not only in pure mathematics (e.g., [11]), but also in related scientific topics (for example, see [1,2,3,4,8]). In particular, we will use combinatorial approach of *Speicher* (e.g., [12]).

Especially, in the text, without introducing detailed definitions and combinatorial backgrounds, *free moments* and *free cumulants* of operators will be computed. Also, we use *free product of C^* -probability spaces in the sense of [12] and [14]*, without detailed introduction. However, rough introduction would be given whenever they are needed in text.

2.2. Calculus on $\mathbb{Q}_p$

Let $\mathbb{Q}_p$ be the p -adic number fields for $p \in \mathcal{P}$, equipped with the non-Archimedean p -norms $|\cdot|_p$ (on $\mathbb{Q}$), where $\mathcal{P}$ is the set of all primes in the *natural numbers* (or the positive integers) $\mathbb{N}$. This *Banach space* $\mathbb{Q}_p$ is also understood as a *measure space*

$$\mathbb{Q}_p = (\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p),$$

equipped with the left-and-right additive invariant *Haar measure* μ_p on the σ -algebra $\sigma(\mathbb{Q}_p)$. Recall also that, $\mathbb{Q}_p$ is a well-defined *ring algebraically*.

As a *topological space*, the p -adic number field $\mathbb{Q}_p$ contains its *basis elements*

$$U_k = p^k \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}, \quad (2.2.1)$$

satisfying the basis property,

$$\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} U_k,$$

and the chain property,

$$\cdots \subset U_2 \subset U_1 \subset U_0 = \mathbb{Z}_p \subset U_{-1} \subset U_{-2} \subset \cdots,$$

and the measure-theoretic data,

$$\mu_p(U_k) = \frac{1}{p^k} = \mu_p(x + U_k), \text{ for all } k \in \mathbb{Z},$$

for all $x \in \mathbb{Q}_p$, where

$$U_0 = \mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p = 1 \right\} = \bigcup_{l=0}^{\infty} U_l$$

is the *unit disk* of $\mathbb{Q}_p$, consisting of all p -adic integers.

By understanding $\mathbb{Q}_p$ as a measure space, one can establish a $*$ -algebra $\mathcal{M}_p$ over $\mathbb{C}$ as a $*$ -algebra consisting of all μ_p -measurable functions f ,

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \text{ with } t_S \in \mathbb{C},$$

where the sum Σ means a finite sum, and χ_S are the usual characteristic functions of S . Of course, the adjoint f^* of f is defined to be

$$f^* = \sum_{S \in \sigma(\mathbb{Q}_p)} \overline{t_S} \chi_S \text{ in } \mathcal{M}_p,$$

where $\overline{z}$ mean the conjugates of z , for all $z \in \mathbb{C}$.

On $\mathcal{M}_p$, one can naturally define a linear functional φ_p ,

$$\varphi_p(f) = \int_{\mathbb{Q}_p} f d\mu_p, \text{ for all } f \in \mathcal{M}_p,$$

and hence, the pair $(\mathcal{M}_p, \varphi_p)$ orms a well-determined $*$ -probability space. Remark that it is a “commutative” $*$ -probability space (and hence, it is well-covered by (noncommutative) free probability theory).

Define now subsets ∂_k of $\mathbb{Q}_p$ by

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}. \quad (2.2.3)$$

We call such μ_p -measurable subsets ∂_k , the k -th boundaries of the basis elements U_k of (2.2.1), which are also μ_p -measurable subsets, for all $k \in \mathbb{Z}$. By the basis property in (2.2.1), one obtains that

$$\mathbb{Q}_p = \sqcup_{k \in \mathbb{Z}} \partial_k, \quad (2.2.4)$$

where $\sqcup$ means the disjoint union, and

$$\mu_p(\partial_k) = \mu_p(U_k) - \mu_p(U_{k+1}) = \frac{1}{p^k} - \frac{1}{p^{k+1}},$$

by the measure-theoretic information in (2.2.1), for all $k \in \mathbb{Z}$.

Now, let $\mathcal{M}_p$ be the vector space of all μ_p -measurable functions on $\mathbb{Q}_p$, i.e.,

$$\mathcal{M}_p = \{f : \mathbb{Q}_p \rightarrow \mathbb{C} : f \text{ is } \mu_p\text{-measurable}\}. \quad (2.2.5)$$

So, $f \in \mathcal{M}_p$, if and only if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \text{ with } t_S \in \mathbb{C},$$

where Σ means the finite sum, and χ_S are the usual characteristic functions of $S \in \sigma(\mathbb{Q}_p)$.

Then it forms a $*$ -algebra over $\mathbb{C}$. Indeed, the vector space $\mathcal{M}_p$ of (2.2.5) is an algebra under the usual functional addition, and multiplication. Also, this algebra $\mathcal{M}_p$ has the adjoint,

$$\left(\sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \right)^* \stackrel{\text{def}}{=} \sum_{S \in \sigma(\mathbb{Q}_p)} \overline{t_S} \chi_S,$$

where $t_S \in \mathbb{C}$, having their conjugates $\overline{t_S}$ in $\mathbb{C}$.

Let $f \in \mathcal{M}_p$. Then one can define the p -adic integral of f by

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \mu_p(S). \quad (2.2.6)$$

Note that, by (2.2.4), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a subset Λ_S of $\mathbb{Z}$, such that

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \emptyset\},$$

satisfying the following result.

Proposition 2.1. (See [1]) Let $S \in \sigma(\mathbb{Q}_p)$, and let $\chi_S \in \mathcal{M}_p$. Then there exist $r_j \in \mathbb{R}$, such that

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_S,$$

and

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right),$$

where Λ_S is in the sense of (2.2.7).

By (2.2.8), one obtains that if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in \mathcal{M}_p,$$

then

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \left(\sum_{j \in \Lambda_S} r_j^S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right),$$

where r_j^S are in the sense of (2.2.8), for all $j \in \Lambda_S$, for all $S \in \sigma(\mathbb{Q}_p)$, whenever Λ_S of (2.2.7) is nonempty in $\mathbb{Z}$.

2.3. Free Probability on $\mathcal{M}_p$

Throughout this section, fix a prime $p \in \mathcal{P}$, and let $\mathbb{Q}_p$ be the corresponding p -adic number field, and let $\mathcal{M}_p$ be the $*$ -algebra consisting of all μ_p -measurable functions on $\mathbb{Q}_p$. In this section, let's establish a suitable free probabilistic model on the $*$ -algebra $\mathcal{M}_p$. Remark that free probability provides a universal tool to study free distributions on “noncommutative” algebras, and hence, it covers the cases where given algebras are “commutative.” Remark that $\mathcal{M}_p$ is a commutative $*$ -algebra, but, for our later purposes, we construct free-probabilistic settings on $\mathcal{M}_p$.

Let U_k be the basis elements (2.2.1) of the topology for $\mathbb{Q}_p$, i.e.,

$$U_k = p^k \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}, \quad (2.3.1)$$

with their boundaries

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}.$$

Define a linear functional $\varphi_p : \mathcal{M}_p \rightarrow \mathbb{C}$ by

$$\varphi_p(f) = \int_{\mathbb{Q}_p} f d\mu_p, \text{ for all } f \in \mathcal{M}_p. \quad (2.3.2)$$

Then, by (2.3.2), one obtains that

$$\varphi_p(\chi_{U_j}) = \frac{1}{p^j}, \text{ and } \varphi_p(\chi_{\partial_j}) = \frac{1}{p^j} - \frac{1}{p^{j+1}},$$

since

$$\Lambda_{U_j} = \{k \in \mathbb{Z} : k \geq j\}, \text{ and } \Lambda_{\partial_j} = \{j\},$$

with help of (2.2.8), for all $j \in \mathbb{Z}$.

Moreover, by the commutativity on $\mathcal{M}_p$,

$$\varphi_p(f_1 f_2) = \varphi_p(f_2 f_1), \text{ for all } f_1, f_2 \in \mathcal{M}_p,$$

and hence, this linear functional φ_p of (2.3.2) is a trace on $\mathcal{M}_p$.

Definition 2.1. The free probability space $(\mathcal{M}_p, \varphi_p)$ is called the p -adic free probability space, for $p \in \mathcal{P}$, where φ_p is the linear functional (2.3.2) on $\mathcal{M}_p$.

Let U_k be in the sense of (2.3.1) in $\mathbb{Q}_p$, and $\chi_{U_k} \in \mathcal{M}_p$, for all $k \in \mathbb{Z}$. Then

$$\chi_{U_{k_1}} \chi_{U_{k_2}} = \chi_{U_{k_1} \cap U_{k_2}} = \chi_{U_{\max\{k_1, k_2\}}},$$

by (2.2.3), where $\max\{k_1, k_2\}$ means the maximum in $\{k_1, k_2\}$.

Say $k_1 \leq k_2$ in $\mathbb{Z}$. Then $U_{k_1} \supseteq U_{k_2}$ in $\mathbb{Q}_p$, by (2.2.3). Therefore, $U_{k_1} \cap U_{k_2} = U_{k_2}$ in $\mathbb{Q}_p$. So, if $k_1 \leq k_2$ in $\mathbb{Z}$, then

$$\chi_{U_{k_1}} \chi_{U_{k_2}} = \chi_{U_{k_1} \cap U_{k_2}} = \chi_{U_{k_2}} \text{ in } \mathcal{M}_p.$$

Thus, one can verify that

$$\varphi_p(\chi_{U_{k_1}} \chi_{U_{k_2}}) = \frac{1}{p^{\max\{k_1, k_2\}}}. \quad (2.3.3)$$

Inductive to (2.3.3), we obtain the following result.

Proposition 2.2. (See [1]) Let $(j_1, \dots, j_N) \in \mathbb{Z}^N$, for $N \in \mathbb{N}$. Then

$$\prod_{l=1}^N \chi_{U_{j_l}} = \chi_{U_{\max\{j_1, \dots, j_N\}}} \text{ in } \mathcal{M}_p, \quad (2.3.4)$$

and hence,

$$\varphi_p\left(\prod_{l=1}^N \chi_{U_{j_l}}\right) = \frac{1}{p^{\max\{j_1, \dots, j_N\}}}.$$

Now, let ∂_k be the k -th boundary $U_k \setminus U_{k+1}$ of U_k in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then, for $k_1, k_2 \in \mathbb{Z}$, one obtains that

$$\chi_{\partial_{k_1}} \chi_{\partial_{k_2}} = \chi_{\partial_{k_1} \cap \partial_{k_2}} = \delta_{k_1, k_2} \chi_{\partial_{k_1}}, \quad (2.3.5)$$

where δ means the Kronecker delta, and hence,

$$\begin{aligned} \varphi_p(\chi_{\partial_{k_1}} \chi_{\partial_{k_2}}) &= \delta_{k_1, k_2} \varphi_p(\chi_{\partial_{k_1}}) \\ &= \delta_{k_1, k_2} \left(\frac{1}{p^{k_1}} - \frac{1}{p^{k_1+1}} \right). \end{aligned}$$

So, we obtain the following computations.

Proposition 2.3. Let $(j_1, \dots, j_N) \in \mathbb{Z}^N$, for $N \in \mathbb{N}$. Then

$$\prod_{l=1}^N \chi_{\partial_{j_l}} = \delta_{(j_1, \dots, j_N)} \chi_{\partial_{j_1}} \text{ in } \mathcal{M}_p, \quad (2.3.6)$$

and hence,

$$\varphi_p\left(\prod_{l=1}^N \chi_{\partial_{j_l}}\right) = \delta_{(j_1, \dots, j_N)} \left(\frac{1}{p^{j_1}} - \frac{1}{p^{j_1+1}} \right),$$

where

$$\delta_{(j_1, \dots, j_N)} = \left(\prod_{l=1}^{N-1} \delta_{j_l, j_{l+1}} \right) (\delta_{j_N, j_1}).$$

Proof. The proof of (2.3.6) is done by (2.3.5).

Thus, one can get that, for any $S \in \sigma(\mathbb{Q}_p)$,

$$\varphi_p(\chi_S) = \varphi_p\left(\sum_{j \in \Lambda_S} \chi_{S \cap \partial_j}\right)$$

where Λ_S is in the sense of (2.2.8)

$$\begin{aligned} &= \sum_{j \in \Lambda_S} \varphi_p(\chi_{S \cap \partial_j}) = \sum_{j \in \Lambda_S} \mu_p(S \cap \partial_j) \\ &= \sum_{j \in \Lambda_S} r^j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \end{aligned} \quad (2.3.7)$$

where $0 \leq r_j \leq 1$ are in the sense of (2.2.8), for all $j \in \mathbb{Z}$.

Also, if $S_1, S_2 \in \sigma(\mathbb{Q}_p)$, then

$$\begin{aligned} \chi_{S_1} \chi_{S_2} &= \left(\sum_{k \in \Lambda_{S_1}} \chi_{S_1 \cap \partial_k} \right) \left(\sum_{j \in \Lambda_{S_2}} \chi_{S_2 \cap \partial_j} \right) \\ &= \sum_{(k, j) \in \Lambda_{S_1} \times \Lambda_{S_2}} (\chi_{S_1 \cap \partial_k} \chi_{S_2 \cap \partial_j}) \\ &= \sum_{(k, j) \in \Lambda_{S_1} \times \Lambda_{S_2}} \delta_{k, j} \chi_{(S_1 \cap S_2) \cap \partial_j} \\ &= \sum_{j \in \Lambda_{S_1, S_2}} \chi_{(S_1 \cap S_2) \cap \partial_j}, \end{aligned} \quad (2.3.8)$$

where

$$\Lambda_{S_1, S_2} = \Lambda_{S_1} \cap \Lambda_{S_2},$$

because $\partial_k \cap \partial_j = \delta_{k, j} \partial_j$.

In (2.3.8), it is clear that, if Λ_{S_1, S_2} is empty, then

$$\chi_{S_1} \chi_{S_2} = 0_{\mathcal{M}_p} = \chi_{\emptyset}, \text{ the zero element of } \mathcal{M}_p,$$

where $\emptyset$ is the empty set in $\sigma(\mathbb{Q}_p)$.

Thus, one can get that there exist $w_j \in \mathbb{R}$, such that

$$0 \leq w_j \leq 1, \text{ for all } j \in \Lambda_{S_1, S_2}, \quad (2.3.9)$$

where

$$\varphi_p(\chi_{S_1} \chi_{S_2}) = \sum_{j \in \Lambda_{S_1, S_2}} w_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right),$$

by (2.3.8) and (2.2.10), for all $S_1, S_2 \in \sigma(\mathbb{Q}_p)$.

By (2.3.9), we obtain the following general result under induction.

Theorem 2.4. Let $S_l \in \sigma(\mathbb{Q}_p)$, and let $\chi_{S_l} \in (\mathcal{M}_p, \varphi_p)$ for $l = 1, \dots, N$, for $N \in \mathbb{N}$. Let

$$\Lambda_{S_1, \dots, S_N} = \bigcap_{l=1}^N \Lambda_{S_l} \text{ in } \mathbb{Z},$$

where Λ_{S_l} are in the sense of (2.2.7), for $l = 1, \dots, N$.

Then there exist $r_j \in \mathbb{R}$, such that

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \quad (2.3.10)$$

and

$$\varphi_p \left(\prod_{l=1}^N \chi_{S_l} \right) = \sum_{j \in \Lambda_{S_1, \dots, S_N}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right).$$

Proof. The proof of (2.3.10) is done by induction on (2.3.9). For more details, see [1] and [2].

Of course, if $\Lambda_{S_1, \dots, S_N}$ is empty in $\mathbb{Z}$, then the formula (2.3.10) vanishes. By (2.3.10), we obtain that, if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in (\mathcal{M}_p, \varphi_p), \text{ with } t_S \in \mathbb{C},$$

then

$$\begin{aligned} \varphi_p(f) &= \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \varphi_p(\chi_S) \\ &= \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \left(\sum_{j \in \Lambda_S} r_j^S \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right), \end{aligned} \quad (2.3.11)$$

where r_j^S are in the sense of (2.2.10), for all $j \in \Lambda_S$.

The above joint free-moment formula (2.3.11) provides a universal tool to compute the free-distributional data of free random variables in our p -adic free probability space $(\mathcal{M}_p, \varphi_p)$.

2.4. Free Product W^* -Probability Spaces

Let (A_k, φ_k) be arbitrary W^* -probability spaces, consisting of C^* -algebras A_k , and corresponding linear

functional φ_k , for $k \in \Delta$ where Δ is an arbitrary countable (finite or infinite) index set. The free product C^* -algebra A ,

$$A = \bigstar_{l \in \Delta} A_l$$

is the C^* -algebra generated by the noncommutative reduced words in $\bigcup_{l=1}^N A_l$, having a new linear functional

$$\varphi = \bigstar_{l \in \Delta} \varphi_l.$$

The C^* -algebra A is understood as a Banach space,

$$\mathbb{C} \oplus \left(\bigoplus_{n=1}^{\infty} \left(\bigoplus_{(i_1, \dots, i_n) \in \text{alt}(\Delta^n)} \left(\bigotimes_{k=1}^n A_{i_k}^o \right) \right) \right) \quad (2.4.1)$$

with

$$A_i^o = A_{i_k} \ominus \mathbb{C}, \text{ for all } k = 1, \dots, n,$$

where

$$\text{alt}(\Delta^n) = \left\{ (i_1, \dots, i_n) \left| \begin{array}{l} (i_1, \dots, i_n) \in \Delta^n \\ i_1 \neq i_2, i_2 \neq i_3 \\ \dots, i_{n-1} \neq i_n \end{array} \right. \right\},$$

for all $n \in \mathbb{N}$, and where the direct product $\oplus$, and the tensor product $\otimes$ are topological on Banach spaces.

In particular, if an element $a \in A$ is of the form of free reduced word,

$$\prod_{l=1}^n a_{i_l} \text{ in } A,$$

then one can understand a as an equivalent Banach-space vector

$$\bigotimes_{l=1}^n a_{i_l} \text{ in the Banach space } A \text{ of (2.4.1),}$$

contained in a direct summand, $\bigotimes_{k=1}^n A_{i_k}^o$ of (2.4.1).

We denote this relation by

$$a \stackrel{\text{equi}}{=} \bigotimes_{l=1}^n a_{i_l} \text{ in } A. \quad (2.4.2)$$

Of course, the left-hand side a means the C^* -algebra element of A , while, the right-hand side $\bigotimes_{l=1}^n a_{i_l}$ means the Banach-space vector of A in the sense of (2.4.1).

Remark that, if a is a free reduced word in A , then

$$a^k \stackrel{\text{equi}}{=} \left(\bigotimes_{l=1}^n a_{i_l} \right)^k = \bigotimes_{l=1}^n a_{i_l}^k \stackrel{\text{equi}}{=} \prod_{l=1}^n a_{i_l}^k \text{ in } A, \quad (2.4.3)$$

for all $k \in \mathbb{N}$.

Notation and Remark Let $a = \prod_{l=1}^n a_{i_l}$ be a free reduced word in A , as above. The power a^k in (2.4.3)

means the k -th power of a as a new element of A , which is regarded as a vector in

$$\bigotimes_{l=1}^n A_{i_k}^o \subset A \text{ in the sense of (2.4.1).}$$

To avoid the confusion, we may use the notation $a^{(k)}$, as a construction of new free “non-reduced” word,

$$a^{(k)} = \underbrace{a \cdot a \cdots a}_{k\text{-times}}$$

in A . Note that only if $i_n \neq i_k$ in the index set Δ , then the above free non-reduced word $a^{(k)}$ forms a free reduced word in A .

For example, let $a = a_{i_1} a_{i_2} a_{i_1}$ be a free reduced word with

$$(i_1, i_2, i_1) \in \text{alt}(\Delta^3),$$

as a vector,

$$a_{i_1} \otimes a_{i_2} \otimes a_{i_1} \text{ in } A \text{ of (2.4.1).}$$

Then

$$a^3 \stackrel{\text{equi}}{=} (a_{i_1} \otimes a_{i_2} \otimes a_{i_3})^3 \stackrel{\text{equi}}{=} a_{i_1}^3 a_{i_2}^3 a_{i_1}^3,$$

in A , but

$$\begin{aligned} a^{(3)} &= (a_{i_1} a_{i_2} a_{i_1})^{(3)} \\ &= a_{i_1} a_{i_2} a_{i_1} a_{i_1} a_{i_2} a_{i_1} a_{i_1} a_{i_2} a_{i_1} \text{ (as a non-reduced word)} \\ &= a_{i_1} a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1} \text{ (as a reduced word)} \end{aligned}$$

i.e.,

$$a^{(3)} = a_{i_1} a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1},$$

as free reduced words in A .

So, in the text below, if we use the term “ a^k ” for a fixed free reduced word a , then it is in the sense of (2.4.3). In the following text, we will not use the concept “ $a^{(k)}$.” However, we want to emphasize the difference between a^k and $a^{(k)}$ in the free product algebra A , at this moment.

Similar to a^k and $a^{(k)}$, one can understand the adjoints a^* and $a^{(*)}$ of a fixed free reduced word a in A . i.e.,

$$a^* \stackrel{\text{equi}}{=} \left(\bigotimes_{l=1}^n a_{i_l} \right)^* = \bigotimes_{l=1}^n a_{i_l}^* \stackrel{\text{equi}}{=} \prod_{l=1}^n a_{i_l}^* \text{ in } A,$$

and

$$\begin{aligned} a^{(*)} &= \left(\prod_{l=1}^n a_{i_l} \right)^{(*)} = (a_{i_1} a_{i_2} \cdots a_{i_n})^{(*)} \\ &= a_{i_n}^* a_{i_{n-1}}^* \cdots a_{i_2}^* a_{i_1}^* = \prod_{l=1}^n a_{i_{n-l+1}}^*, \end{aligned}$$

in A .

So, the free product linear functional φ on A satisfies that, whenever a is a reduced free word in A satisfying (2.4.2), then

$$\varphi(a^k) = \varphi \left(\prod_{l=1}^n a_{i_l}^k \right) = \prod_{l=1}^n \left(\varphi(a_{i_l}^k) \right), \quad (2.4.4)$$

by (2.4.3), for all $k \in \mathbb{N}$. Sometimes, by abusing (2.4.3), one can / may write

$$\varphi(a^k) \stackrel{\text{equi}}{=} \varphi \left(\bigotimes_{l=1}^n a_{i_l}^k \right) = \prod_{l=1}^n \varphi(a_{i_l}^k),$$

whenever $a = \prod_{l=1}^n a_{i_l}$ is a “free reduced word” in A , for all $k \in \mathbb{N}$.

Now, let

$$b = \sum_{l=1}^n b_{i_l} \in (A, \varphi).$$

We say that a is a *free sum* in A , if all summands $b_{i_1}, \dots, b_{i_n}$ of b are contained in “mutually-distinct” direct summands of a Banach space A of (2.4.1), as free reduced words. Then, similar to the above observation, one can realize that

$$b \stackrel{\text{equi}}{=} \bigotimes_{l=1}^n b_{i_l} \text{ in the Banach space } A \text{ of (2.4.1), (2.4.5)}$$

satisfying

$$\begin{aligned} \varphi(b^k) &\stackrel{\text{equi}}{=} \varphi \left(\left(\bigotimes_{l=1}^n b_{i_l} \right)^k \right) = \varphi \left(\bigotimes_{l=1}^n b_{i_l}^k \right) \\ &\stackrel{\text{equi}}{=} \varphi \left(\sum_{l=1}^n b_{i_l}^k \right) = \sum_{l=1}^n \varphi(b_{i_l}^k), \end{aligned}$$

for all $k \in \mathbb{N}$.

Here, remark that each summand $\varphi(b_{i_l}^k)$ of (2.4.5) satisfies (2.4.4).

Notation and Remark Similar to the free-reduced-word case, if b is a free sum in the sense of (2.4.5), then one can consider

$$\begin{aligned} b^{(k)} &= \left(\sum_{l=1}^n b_{i_l} \right)^{(k)} \\ &= \sum_{(i_1, \dots, i_k) \in [1, \dots, n]^k} (b_{i_1} b_{i_2} \cdots b_{i_k}), \end{aligned}$$

where the summands $b_{i_1} \cdots b_{i_k}$ are forms of free “non-reduced” words in A . In the following text, we will not use the concept “ $b^{(k)}$ ” in A . But, as before, we emphasize the difference between b^k and $b^{(k)}$, for a fixed free sum b of A .

For more about (free-probabilistic) free product algebras, and corresponding free probability spaces, see [11,12,14] and cited papers therein.

3. Act $\mathbb{Q}_p$ on W^* -Probability Spaces

In this section, we regard the σ -algebra $\sigma(\mathbb{Q}_p)$ of a p -adic number field $\mathbb{Q}_p$ as a monoid equipped with its binary operation, the set-intersection $\cap$. And we act this monoid $(\sigma(\mathbb{Q}_p), \cap)$ on arbitrarily given W^* -probability spaces. In other words, we construct a monoidal W^* -dynamical systems. And then, corresponding crossed product W^* -algebras are established. Free probability on such W^* -algebras is considered.

3.1. Act $\mathbb{Q}_p$ on W^* -Probability Spaces

Let M be a *von Neumann algebra*, i.e., it is a unital W^* -subalgebra of the operator algebra $B(H)$ consisting of all bounded operators on a Hilbert space H . Assume further that we fix a *linear functional* tr on M satisfying

$$tr(m_1 m_2) = tr(m_2 m_1),$$

for all $m_1, m_2 \in M$, and

$$tr(1_M) = 1,$$

where 1_M means the *identity operator* of M , which is the identity operator of $B(H)$. i.e., the linear functional tr is a *unital trace* on M . So, the pair (M, tr) forms a *unital tracial W^* -probability space* in the sense of [11] and [14].

For a prime p , let H_p be the L^2 -Hilbert space $L^2(\mathbb{Q}_p)$ over the p -adic number field $\mathbb{Q}_p$, i.e.,

$$H_p = L^2(\mathbb{Q}_p) = L^2(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p). \quad (3.1.1)$$

It has its inner product $\langle \cdot, \cdot \rangle_2$,

$$\langle h_1, h_2 \rangle_2 = \int_{\mathbb{Q}_p} h_1 h_2^* d\mu_p, \text{ for all } h_1, h_2 \in H_p,$$

and the corresponding L^2 -norm $\| \cdot \|_2$,

$$\|h\|_2 = \sqrt{\langle h, h \rangle_2}, \text{ for all } h \in H_p.$$

Remark that, for any $h \in H_p$, this vector h is expressed by

$$h = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \text{ with } t_S \in \mathbb{C},$$

where Σ is a finite or an infinite (limit of finite) sum(s) under L^2 -topology.

Let H be the Hilbert space where a fixed von Neumann algebra M acts, and let H_p be the L^2 -Hilbert

space (3.1.1). Then one has the *tensor product Hilbert space* $H_{M,p}$,

$$H_{M,p} = H \otimes H_p, \quad (3.1.2)$$

for $p \in \mathcal{P}$.

Understand now the σ -algebra $\sigma(\mathbb{Q}_p)$ of $\mathbb{Q}_p$ as a semigroup equipped with its binary operation, the *set-intersection* $\cap$. i.e.,

$$\sigma(\mathbb{Q}_p) = (\sigma(\mathbb{Q}_p), \cap).$$

Each element Y of $\sigma(\mathbb{Q}_p)$ is acting on H_p by

$$\begin{aligned} Y \bullet h &\rightarrow \chi_Y \left(\sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \right) \\ &= \sum_{S \in \sigma(\mathbb{Q}_p)} t_S t_Y \chi_S = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_{Y \cap S}, \end{aligned} \quad (3.1.3)$$

$$h = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in H_p.$$

for all

i.e., there exists a well-defined *semigroup-action* χ of the semigroup $\sigma(\mathbb{Q}_p)$, such that

$$\chi(K)(h) = \chi_K h, \text{ for all } h \in H_p, \quad (3.1.4)$$

acting on H_p , by (3.1.3).

This action χ of the semigroup $\sigma(\mathbb{Q}_p)$ acting on H_p , is called the *characteristic(-functional) action*. By the very definition (3.1.4) of χ , we denote the images $\chi(K)$ of χ simply by χ_K , for all $K \in \sigma(\mathbb{Q}_p)$.

With help of the action χ of (3.1.4), one can construct a *semigroup-action* π of $\sigma(\mathbb{Q}_p)$, acting on the fixed von Neumann algebra M as follows

$$\pi(Y)(m) = \chi_Y m \chi_Y^* = \chi_Y m \chi_Y, \text{ on } H_{M,p}, \quad (3.1.5)$$

where $H_{M,p}$ is the tensor product Hilbert space (3.1.2).

Indeed, one can understand χ_Y as $1_M \otimes \chi_Y$ on $H_{M,p}$, and m as $m \otimes \chi_{\mathbb{Q}_p}$ on $H_{M,p}$. Then the image $\pi(Y)(m)$ of (3.1.5) is regarded as a process sending

$$m \in M \otimes (\mathbb{C} \cdot \chi_{\mathbb{Q}_p}) \text{ to } \pi(Y)(m) \in M \otimes (\mathbb{C} \cdot \chi_Y), \quad (3.1.6)$$

for $Y \in \sigma(\mathbb{Q}_p)$, for all $m \in M$.

Theorem 3.1. *Let π be in the sense of (3.1.5). Then the triple $(M, \sigma(\mathbb{Q}_p), \pi)$ forms a well-defined semigroup W^* -dynamical system in $B(H_{M,p})$.*

Proof. It suffices to show that π is indeed a well-defined semigroup action of $\sigma(\mathbb{Q}_p)$ acting on M . By the definition (3.1.5), satisfying (3.1.6), one has that

$$\begin{aligned}\pi(K_1 \cap K_2)(m) &= \chi_{K_1 \cap K_2} m \chi_{K_1 \cap K_2} \\ &= \chi_{K_1 \cap K_2} m \chi_{K_2 \cap K_1} = \chi_{K_1} \chi_{K_2} m \chi_{K_2} \chi_{K_1}\end{aligned}$$

since $K_1 \cap K_2 = K_2 \cap K_1$ in $\sigma(G_p)$

$$\begin{aligned}&= \chi_{K_1} (\chi_{K_2} m \chi_{K_2}) \chi_{K_1} \\ &= \chi_{K_1} (\pi(K_2)(m)) \chi_{K_1} \\ &= \pi(K_1)(\pi(K_2)(m)) \\ &= (\pi(K_1) \circ \pi(K_2))(m),\end{aligned}$$

for all $m \in M$, for all $K_1, K_2 \in \sigma(G_p)$, i.e.,

$$\pi(K_1 \cap K_2) = \pi(K_1) \circ \pi(K_2),$$

for all $K_1, K_2 \in \sigma(G_p)$.

Therefore, the morphism π is a well-defined semigroup-action of $\sigma(\mathbb{Q}_p)$ acting on M in $B(H_{M,p})$.

The above theorem shows that there is a well-defined semigroup W^* -dynamical system $(M, \sigma(\mathbb{Q}_p), \pi)$ in $B(H_{M,p})$.

Definition 3.1. The semigroup W^* -dynamical system $(M, \sigma(\mathbb{Q}_p), \pi)$ is called the p -adic W^* -dynamical system in $B(H_{M,p})$.

Let $(M, \sigma(\mathbb{Q}_p), \pi)$ be the p -adic W^* -dynamical system in $B(H_{M,p})$. Then one can construct the corresponding crossed product W^* -algebra,

$$\mathcal{M}_{p,\pi} = M \rtimes_{\pi} \sigma(\mathbb{Q}_p) \quad (3.1.7)$$

by the W^* -subalgebra of $B(H_{M,p})$ generated by M and $\pi(\sigma(\mathbb{Q}_p))$ satisfying the π -relation:

$$\begin{aligned}(m_1 \chi_{S_1})(m_2 \chi_{S_2}) &= m_1 \chi_{S_1} m_2 \chi_{S_2} \chi_{S_1} \\ &= m_1 \chi_{S_1} m_2 \chi_{S_1}^2 \chi_{S_2} = m_1 (\chi_{S_2} m \chi_{S_1}) \chi_{S_1} \chi_{S_2} \\ &= m_1 (\pi(S_1)(m)) \chi_{S_1} \chi_{S_2} \\ &= m_1 (\pi(S_1)(m)) \chi_{S_1 \cap S_2},\end{aligned}$$

for all $S_1, S_2 \in \sigma(G_p)$, for all $m \in M$, and

$$\begin{aligned}(m \chi_S)^* &= \chi_S^* m^* = \chi_S m^* \chi_S \\ &= \chi_S m^* \chi_S^2 = (\chi_S m^* \chi_S) \chi_S \\ &= (\pi(S)(m^*)) \chi_S = (\pi(S)(m^*))^* \chi_S^*,\end{aligned}$$

i.e., the π -relation is

$$(m_1 \chi_{S_1})(m_2 \chi_{S_2}) = m_1 (\pi(S_1)(m_2)) \chi_{S_1 \cap S_2}, \quad (3.1.8)$$

and

$$(m \chi_S)^* = (\pi(S)(m^*)) \chi_S,$$

for all $m, m_1, m_2 \in M$, and for all $S, S_1, S_2 \in \sigma(G_p)$.

Definition 3.2. The crossed product W^* -algebra $\mathcal{M}_{p,\pi} = M \rtimes_{\pi} \sigma(\mathbb{Q}_p)$ of (3.1.7), with the W^* -relation (3.1.8), is called the p -adic dynamical π -algebra induced by the p -adic W^* -dynamical system $(M, \sigma(\mathbb{Q}_p), \pi)$.

Now, let $\mathcal{N}_p$ be the von Neumann algebra $L^\infty(\mathbb{Q}_p)$,

$$\mathcal{N}_p = L^\infty(\mathbb{Q}_p) = L^\infty(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p), \quad (3.1.9)$$

which is the L^∞ -algebra acting on H_p . i.e., all elements of $\mathcal{N}_p$ are understood as the multiplication operators on H_p , i.e., if f is an arbitrary element of $\mathcal{N}_p$ of (3.1.9), then it is acting on H_p by

$$f \in \mathcal{N}_p \mapsto T_f \in B(H_p),$$

where T_f is the multiplication operator on H_p ,

$$T_f(h) = fh, \text{ for all } h \in H_p.$$

As usual, we denote multiplication operators $T_f \in B(H_p)$ of f , simply by $f \in \mathcal{N}_p$.

Note that all elements f of $\mathcal{N}_p$ are expressed by the finite or infinite (limits of finite) sums

$$\sum_{Y \in \sigma(\mathbb{Q}_p)} t_Y \chi_Y \text{ with } t_Y \in \mathbb{C},$$

under L^∞ -topology.

Let's define the tensor product W^* -algebra $M \otimes_{\mathbb{C}} \mathcal{N}_p$ of a given von Neumann algebra M , and the von Neumann algebra $\mathcal{N}_p$, where $\otimes_{\mathbb{C}}$ means the tensor product of von Neumann algebras.

Define now a W^* -subalgebra $\mathcal{M}_{p,\pi}$ by the conditional tensor product W^* -algebra of M and $\mathcal{N}_p$.

$$\mathcal{M}_{p,\pi} = M \otimes_{\pi} \mathcal{N}_p \text{ in } M \otimes_{\mathbb{C}} \mathcal{N}_p, \quad (3.1.10)$$

generated by $M \cup \mathcal{N}_p$, satisfying the π -relation:

$$(m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2}) = (m_1 \pi(S_1)(m_2)) \otimes \chi_{S_1 \cap S_2}, \quad (3.1.11)$$

and

$$(m \otimes \chi_S)^* = (\pi(S)(m^*)) \otimes \chi_S^* = (\pi(S)(m^*)) \otimes \chi_S, \quad (3.1.12)$$

for all $m, m_1, m_2 \in M$, and $S, S_1, S_2 \in \sigma(\mathbb{Q}_p)$, under linearity and product topology.

Then we obtain the following isomorphism theorem.

Theorem 3.2. Let $\mathcal{M}_{p,\pi} = M \times_{\pi} \sigma(\mathbb{Q}_p)$ be the p -adic dynamical W^* -algebra (3.1.7) induced by the p -adic W^* -dynamical system $(M, \sigma(\mathbb{Q}_p), \pi)$, and let $\mathcal{M}_{p,\pi} = M \otimes_{\pi} \mathcal{N}_p$ be the conditional tensor product W^* -algebra (3.1.10) of M and $\mathcal{N}_p$, satisfying (3.1.11). Then

$$\mathcal{M}_{p,\pi}^{*-iso} = \mathcal{M}_{p,\pi}, \quad (3.1.12)$$

where “ $=$ ” means “being $*$ -isomorphic.”

Proof. Let $\mathcal{M}_{p,\pi}$ be the conditional tensor product W^* -algebra (3.1.10), satisfying the π -relation (3.1.11), and let $\mathcal{M}_{p,\pi}$ be the p -adic dynamical W^* -algebra (3.1.7) induced by the p -adic dynamical system $(M, \sigma(\mathbb{Q}_p), \pi)$, satisfying the π -relation (3.1.8). Define a morphism

$$\Phi: \mathcal{M}_{p,\pi} \rightarrow \mathcal{M}_{p,\pi}$$

by the linear transformation satisfying

$$\Phi(m \otimes \chi_S) = m \chi_S \text{ in } \mathcal{M}_{p,\pi}, \quad (3.1.13)$$

for all $m \otimes \chi_S \in \mathcal{M}_{p,\pi}$, for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

By the very definition (3.1.13) of the linear morphism Φ , one can verify it is bounded and injective. Moreover, for any generating elements $m \chi_S$ of $\mathcal{M}_{p,\pi}$, we can always find the elements $m \otimes \chi_S$ in $\mathcal{M}_{p,\pi}$. So, under topology, the morphism Φ is surjective, too. i.e., Φ is a bijective bounded linear transformation.

Observe that

$$\begin{aligned} & \Phi((m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2})) \\ &= \Phi((m_1(\pi(S_1)(m_2))) \otimes \chi_{S_1 \cap S_2}) \end{aligned}$$

by (3.1.11)

$$\begin{aligned} &= (m_1(\pi(S_1)(m_2))) \chi_{S_1 \cap S_2} \\ &= (m_1 \chi_{S_1})(m_2 \chi_{S_2}) \end{aligned}$$

by (3.1.8)

$$= (\Phi(m_1 \chi_{S_1}))(\Phi(m_2 \chi_{S_2})),$$

for all $m_1, m_2 \in M$, and $S_1, S_2 \in \sigma(\mathbb{Q}_p)$.

And hence, for any $T_1, T_2 \in \mathcal{M}_{p,\pi}$, one has

$$\Phi(T_1 T_2) = \Phi(T_1) \Phi(T_2) \text{ in } \mathcal{M}_{p,\pi}.$$

Therefore, this bijective linear transformation Φ is multiplicative, equivalently, it is a bounded algebra-isomorphism.

Also, it satisfies that

$$\Phi((m \otimes \chi_S)^*) = \Phi((\pi(S)(m^*)) \otimes \chi_S)$$

by (3.1.11)

$$\begin{aligned} &= (\pi(S)(m^*)) \chi_S \\ &= (m \chi_S)^* = (\Phi(x \otimes \chi_S))^*, \end{aligned}$$

in $\mathcal{M}_{p,\pi}$, for all $m \otimes \chi_S \in \mathcal{M}_{p,\pi}$, for $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

And hence, for any $T \in \mathcal{M}_{p,\pi}$,

$$\Phi(T^*) = (\Phi(T))^* \text{ in } \mathcal{M}_{p,\pi}.$$

Thus, the bounded algebra-isomorphism Φ is a $*$ -isomorphism.

So, two von Neumann algebras $\mathcal{M}_{p,\pi}$ and $\mathcal{M}_{p,\pi}$ are $*$ -isomorphic from each other since there exists a $*$ -isomorphism Φ of (3.1.13).

By the structure theorem (3.1.12), one can understand the conditional tensor product W^* -algebra

$$\mathcal{M}_{p,\pi} = M \otimes_{\pi} \mathcal{N}_p$$

and the p -adic dynamical W^* -algebra

$$\mathcal{M}_{p,\pi} = M \times_{\pi} \sigma(\mathbb{Q}_p),$$

as the same von Neumann algebra. In the rest of this paper, we use $\mathcal{M}_{p,\pi}$ and $\mathcal{M}_{p,\pi}$ alternatively, and we call these identified von Neumann algebras, a p -adic dynamical W^* -algebra.

3.2. Free Probability on p -Adic Dynamical W^* -Algebra $\mathcal{M}_{p,\pi}$

In this section, we fix the p -adic dynamical W^* -algebra

$$\mathcal{M}_{p,\pi} = M \times_{\pi} \sigma(\mathbb{Q}_p)$$

induced by a p -dynamical W^* -dynamical system $(M, \sigma(\mathbb{Q}_p), \pi)$ which is also understood as the conditional tensor product W^* -algebra

$$\mathcal{M}_{p,\pi} = M \otimes_{\pi} \mathcal{N}_p,$$

where $\mathcal{N}_p$ is the L^∞ -von Neumann algebra $L^\infty(\mathbb{Q}_p)$.

As we assumed at the beginning of Section 9, let tr be a fixed unital trace on M . i.e., the pair (M, tr) forms a unital tracial W^* -probability space. Define now a linear functional ψ_p on the L^∞ -Neumann algebra $\mathcal{N}_p$ by

$$\psi_p(f) = \int_{\mathbb{Q}_p} f d\mu_p, \text{ for all } f \in \mathcal{N}_p. \quad (3.2.1)$$

So, for any $S \in \sigma(\mathbb{Q}_p)$, the linear functional ψ_p of (3.2.1) satisfies that

$$\psi_p(\chi_S) = \mu_p(S) = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \quad (3.2.2)$$

for some $0 \leq r_j \leq 1$, for all

$$j \in \Lambda_S = \{k \in \mathbb{Z} : S \cap \partial_k \neq \emptyset\},$$

by (2.2.8).

Since the morphism ψ_p of (3.2.1) is a well-defined linear functional, the pair $(\mathcal{N}_p, \psi_p)$ is a W^* -probability space.

Remark 3.1. Check the similarity-but-difference between our (pure-algebraic) $*$ -probability spaces $(\mathcal{M}_p, \varphi_p)$, and the W^* -probability spaces $(\mathcal{N}_p, \psi_p)$.

By (3.2.2), one obtains that

$$\psi_p(\chi_{U_k}) = \frac{1}{p^k}, \text{ and } \psi_p(\chi_{\partial_k}) = \frac{1}{p^k} - \frac{1}{p^{k+1}},$$

for all $p \in \mathcal{P}, k \in \mathbb{Z}$.

Define now a bounded linear transformation from our p -adic dynamical W^* -algebra $\mathcal{M}_{p,\pi} = \mathcal{M}_{p,\pi}$ to the L^∞ -von Neumann algebra $\mathcal{N}_p$

$$E^p : \mathcal{M}_{p,\pi} \rightarrow \mathcal{N}_p$$

by

$$E^p \left(\sum_{S \in \sigma(\mathbb{Q}_p)} m_S \chi_S \right) = \sum_{S \in \sigma(\mathbb{Q}_p)} \text{tr}(m_S) \chi_S. \quad (3.2.3)$$

Proposition 3.3. The linear morphisms E_j^p of (3.2.3) are conditional expectations from $\mathcal{M}_{p,\pi}$ onto $\mathcal{N}_p$, for all $j \in \mathbb{Z}$.

Proof. Since $\mathcal{M}_{p,\pi}^{*-iso} = \mathcal{M}_{p,\pi} = M \otimes_\pi \mathcal{N}_p$ by (3.1.12), the linear transformation E^p is bounded by the very definition (3.2.3).

Now, let $T = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in \mathcal{N}_p$ with $t_S \in \mathbb{C}$. Then

$$E^p(T) = \sum_{S \in \sigma(\mathbb{Q}_p)} \text{tr}(t_S) \chi_S = \sum_{S \in \sigma(G_p) \cap \{B_{k_0}^\alpha\}} t_S \chi_S = T, \quad (3.2.4)$$

in $\mathcal{N}_p$.

Now, let

$$T_j = \sum_{S \in \sigma(\mathbb{Q}_p)} t_{S_j} \chi_{S_j} \in \mathcal{N}_p, \text{ with } t_{S_j} \in \mathbb{C}, \text{ for } j=1,2,$$

and

$$R = \sum_{S \in \sigma(\mathbb{Q}_p)} m_S \chi_S \in \mathcal{M}_{p,\pi}, \text{ with } m_S \in M.$$

Then

$$E^p(T_1 R T_2) = E^p \left(\sum_{(S_1, S, S_2) \in \sigma(\mathbb{Q}_p)^3} t_{S_1} m_S \chi_{S_2} \chi_{S_1} \cap S \cap S_2 \right)$$

where X^k mean the iterated Cartesian product $X \times \dots \times X$ (k -times), for all $k \in \mathbb{N}$, for all sets X

$$\begin{aligned} &= \sum_{(S_1, S, S_2) \in \sigma(\mathbb{Q}_p)} \text{tr}(t_{S_1} m_S \chi_{S_2}) \chi_{S_1} \cap S \cap S_2 \\ &= \sum_{(S_1, S, S_2) \in \sigma(\mathbb{Q}_p)^3} t_{S_1} (\text{tr}(m_S)) t_{S_2} \chi_{S_1} \cap (S) \cap S_2 \\ &= \sum_{(S_1, S, S_2) \in \sigma(\mathbb{Q}_p)^3} t_{S_1} (\text{tr}(m_S)) t_{S_2} \chi_{S_1} (\chi_S) \chi_{S_2} \\ &= \left(\sum_{S_1 \in \sigma(\mathbb{Q}_p)} t_{S_1} \chi_{S_1} \right) \cdot \left(\sum_{S \in \sigma(\mathbb{Q}_p)} \text{tr}(m_S) \chi_S \right) \\ &\quad \cdot \left(\sum_{S_2 \in \sigma(\mathbb{Q}_p)} t_{S_2} \chi_{S_2} \right) \\ &= T_1 (E^p(R)) T_2, \end{aligned}$$

i.e.,

$$E^p(T_1 R T_2) = T_1 (E^p(R)) T_2 \text{ in } \mathcal{N}_p, \quad (3.2.5)$$

for all $T_1, T_2 \in \mathcal{N}_p$, and $R \in \mathcal{M}_{p,\pi}$.

Also, E^p satisfies that, for any R (as above),

$$\begin{aligned} E^p(R^*) &= E^p \left(\left(\sum_{S_1 \in \sigma(\mathbb{Q}_p)} m_S \chi_S \right)^* \right) \\ &= E^p \left(\sum_{S_1 \in \sigma(\mathbb{Q}_p)} \pi(S) (m_S^*) \chi_S \right) \end{aligned}$$

by (3.1.8) or (3.1.11)

$$\begin{aligned} &= \sum_{S \in \sigma(\mathbb{Q}_p)} \text{tr}(\pi(S) (m_S^*)) \chi_S \\ &= \sum_{S \in \sigma(\mathbb{Q}_p)} \text{tr}((\pi(S) (m_S))^*) \chi_S \\ &= \sum_{S \in \sigma(\mathbb{Q}_p)} \overline{\text{tr}(\pi(S) (m_S))} \chi_S = (E^p(R))^*, \end{aligned}$$

i.e.,

$$E^p(R^*) = (E^p(R))^* \text{ in } \mathcal{N}_p, \text{ for all } R \in \mathcal{M}_{p,\pi}. \quad (3.2.6)$$

Therefore, this surjective bounded linear transformation E^p of (3.2.3) is a conditional expectation from $\mathcal{M}_{p,\pi} = \mathcal{M}_{p,\pi}$ onto $\mathcal{N}_p$, by (3.2.4), (3.2.5) and (3.2.6).

So, the pair $(\mathcal{M}_{p,\pi}, E^p)$ of our p -adic dynamical W^* -algebra $\mathcal{M}_{p,\pi} = \mathcal{M}_{p,\pi}$, and the conditional expectation E^p of (3.2.3) forms an amalgamated $\mathcal{N}_p$ -valued W^* -probability space with amalgamation over $M_{B_{k_0}^o}$ (e.g., [12]).

Remember that the von Neumann algebra $\mathcal{N}_p$ forms a well-determined W^* -probability space $(\mathcal{N}_p, \psi_p)$, where ψ_p is in the sense of (3.2.1). One can define a linear functional $\psi_{p,\pi}$ on $\mathcal{M}_{p,\pi}$ by

$$\psi_{p,\pi} = \psi_p \circ E^p : \mathcal{M}_{p,\pi} \rightarrow \mathbb{C}, \quad (3.2.9)$$

where E^p is the conditional expectation (3.2.3) from $\mathcal{M}_{p,\pi}$ onto $\mathcal{N}_p$, and ψ_p is the linear functional (3.2.1) on $\mathcal{N}_p$, for $p \in \mathcal{P}$.

Clearly, by the linearity of ψ_p and that of E^p , the morphism $\psi_{p,\pi}$ of (3.2.9) is a well-defined linear functional on $\mathcal{M}_{p,\pi}$, inducing a corresponding W^* -probability space $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$.

Definition 3.3. The W^* -probability spaces $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$ of a p -adic dynamical W^* -algebra $\mathcal{M}_{p,\pi}$, and the linear functional $\psi_{p,\pi}$ of (3.2.9) is called the p -adic dynamical W^* -probability spaces.

3.3. On p -Adic Dynamical W^* -Probability Spaces $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$

Let $p \in \mathcal{P}$, and let

$$\mathcal{M}_{p,\pi} = M \rtimes_{\pi} \sigma(\mathbb{Q}_p) = M \otimes_{\pi} \mathcal{N}_p = \mathcal{M}_{p,\pi}$$

be a p -adic dynamical W^* -algebra, and let $\psi_{p,\pi} = \psi_p \circ E^p$ be the linear functional (3.2.9) on $\mathcal{M}_{p,\pi}$, where ψ_p is the linear functional (3.2.1) on $\mathcal{N}_p$, and $E^p : \mathcal{M}_{p,\pi} \rightarrow \mathcal{N}_p$ is the conditional expectation (3.2.3). Let $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$ be the corresponding p -adic dynamical W^* -probability space.

In this section, we study free-distributional data on $\mathcal{M}_{p,\pi}$ in terms of $\psi_{p,\pi}$, by computing free moments, and free cumulants of arbitrarily fixed free random variables of $(\mathcal{M}_{p,\pi}, \psi_{p,\pi}^o)$. In particular, we concentrate on

studying free distributions of generating elements $m\chi_S$ of $\mathcal{M}_{p,\pi}$, for $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

Lemma 3.4. Let $m\chi_S \in (\mathcal{M}_{p,\pi}, \psi_{p,\pi})$, for $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Then there exist $r_j \in \mathbb{R}$, such that

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_S, \quad (3.3.1)$$

and

$$\psi_{p,\pi}(m\chi_S) = \text{tr}(m) \left(\sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right).$$

Proof. The proof of (3.3.1) is done by straightforward computations by the definition (3.2.9) of $\psi_{p,\pi}$. Indeed,

$$\begin{aligned} \psi_{p,\pi}(m\chi_S) &= \psi_p(E^p(m\chi_S)) = \psi_p(\text{tr}(m)\chi_S) \\ &= \text{tr}(m)\psi_p(\chi_S) = \text{tr}(m) \left(\sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right), \end{aligned}$$

by (2.2.8).

From now on, for convenience, we will use the following notations;

Notation Denote $\pi(S)(m)$ simply by m^S in $\mathcal{M}_{p,\pi}$, for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

Now, observe that, if $T_j = m_j\chi_{S_j} \in (\mathcal{M}_{p,\pi}, \psi_{p,\pi})$, for $j = 1, \dots, n$, for $n \in \mathbb{N}$, then

$$\begin{aligned} &\psi_{p,\pi} \left(\prod_{j=1}^n T_j \right) \\ &= \psi_{p,\pi} (m_1 m_2^{S_1} m_3^{S_1 \cap S_2} \dots m_n^{S_1 \cap \dots \cap S_{n-1}} \chi_{S_1 \cap S_2 \cap \dots \cap S_n}) \\ &= \text{tr} \left(m_1 \prod_{j=2}^n m_j^{S_1 \cap \dots \cap S_{j-1}} \right) \mu_p \left(\left(\bigcap_{j=1}^n S_j \right) \right) \\ &= \text{tr} \left(m_1 \prod_{j=2}^n m_j^{S_1 \cap \dots \cap S_{j-1}} \right) \left(\sum_{j \in \Lambda_{S_1, \dots, S_n}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right), \end{aligned} \quad (3.3.2)$$

by (3.3.1), where

$$\Lambda_{S_1, \dots, S_n} = \bigcap_{l=1}^n \Lambda_{S_l},$$

where Λ_{S_l} are in the sense of (2.2.8).

Proposition 3.5. Let $m_j \in M$, and $S_j \in \sigma(\mathbb{Q}_p)$, for $j = 1, \dots, n$, for $n \in \mathbb{N}$. Then

$$\begin{aligned} &\psi_{p,\pi} \left(\prod_{j=1}^n m_j \chi_{S_j} \right) \\ &= \text{tr} \left(m_1 \prod_{j=2}^n m_j^{S_1 \cap \dots \cap S_{j-1}} \right) \left(\sum_{j \in \Lambda_{S_1, \dots, S_n}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) \right), \end{aligned} \quad (3.3.3)$$

where

$$\Lambda_{S_1, \dots, S_n} = \bigcap_{l=1}^n \Lambda_{S_l},$$

where Λ_{S_l} are in the sense of (2.2.8).

Proof. The free-moment formula (3.3.3) is proven by (3.3.2), under the π -relation (3.1.8) on $\mathcal{M}_{p,\pi}$.

The above joint free-moment formula (3.3.3) provides the free-distributional data of generating elements $m\chi_S$ of our p -adic dynamical W^* -probability space $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$.

Based on (3.3.3), one can obtain that, for any

$$T_j = m_j \chi_S \in (\mathcal{M}_{p,\pi}, \psi_{p,\pi}),$$

for $j = 1, \dots, n$, for $n \in \mathbb{N}$.

$$\begin{aligned} k_n^{p,\pi}(T_1, \dots, T_n) \\ = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \psi_{p,\pi} \left(\prod_{l \in V} T_l \right) \right) \mu(\pi, 1_n) \end{aligned}$$

by the *Möbius inversion* of [12]

$$\begin{aligned} = \sum_{\pi \in NC(n)} \left(\prod_{V = (i_1, \dots, i_k) \in \pi} \text{tr} \left(m_{i_1} \prod_{l=2}^k m_{i_l}^{S_1 \cap \dots \cap S_{i_l-1}} \right) \right. \\ \left. \cdot \mu \left(\left(\bigcap_{l=1}^k S_{i_l} \right) \cap B_k^o \right) \right) \mu(\pi, 1_n), \end{aligned} \quad (3.3.4)$$

by (3.3.2) and (3.3.3), where $k_n^{p,\pi}(\dots)$ is the *free cumulant with respect to $\psi_{p,\pi}$* (e.g., [12]).

Now, let $T_j = m_j \chi_{S_j}$ be free random variables in $(\mathcal{M}_{p,\pi}, \psi_k^o)$, with $m_j \in M$, and $S_j \in \sigma(\mathbb{Q}_p)$, for $j = 1, 2$, and let $(i_1, \dots, i_n)$ be the n -tuples in $\{1, 2\}^n$, for $n \in \mathbb{N}$. Then

$$\begin{aligned} k_n^{p,\pi}(T_{i_1}, \dots, T_{i_n}) \\ = \sum_{\pi \in NC(\{i_1, \dots, i_n\})} \left(\prod_{V \in \pi} \psi_{p,\pi}(V) \right) \mu(\pi, 1_n), \end{aligned} \quad (3.3.5)$$

with

$$\psi_{p,\pi}(V) = \text{tr} \left(m_{i_{j_1}} \prod_{l=2}^{|V|} m_{i_{j_l}}^{S_1 \cap \dots \cap S_{i_{j_l-1}}} \right) \mu_p([S]),$$

where

$$[S] = \begin{cases} S_1 \cap S_2 & \text{if } V \text{ contains both 1 and 2} \\ S_1 & \text{if } V = \left(\underbrace{1, \dots, 1}_{|V| \text{-times}} \right) \\ S_2 & \text{if } V = \left(\underbrace{2, \dots, 2}_{|V| \text{-times}} \right), \end{cases}$$

by (3.3.4).

In particular, by (3.3.5), we obtain the following freeness condition on the p -adic dynamical W^* -algebra $\mathcal{M}_{p,\pi}$.

Theorem 3.6. Let $T_j = m_j \chi_{S_j}$ be free random variables in the p -adic dynamical W^* -probability space $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$, with $m_j \in M$, and $S_j \in \sigma(\mathbb{Q}_p)$, for $j = 1, 2$. If

$$\mu_p(S_j) = \mu(S_1 \cap S_2), \quad \forall j = 1, 2, \text{ and} \quad (3.3.6)$$

$$m_j^{S_1} = m_j = m_j^{S_2} \text{ in } M, \quad \forall j = 1, 2, \quad (3.3.7)$$

and if m_1 and m_2 are free in (M, tr) , then T_1 and T_2 are free in $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$.

Proof. Let $(i_1, \dots, i_n)$ be a “mixed” n -tuple in $\{1, 2\}^n$, for $n \in \mathbb{N} \setminus \{1\}$. Then

$$\begin{aligned} k_n^{p,\pi}(T_{i_1}, \dots, T_{i_n}) \\ = \sum_{\pi \in NC((i_1, \dots, i_n))} \left(\prod_{V \in \pi} \psi_k^o(V) \right) \mu(\pi, 1_n), \end{aligned} \quad (3.3.8)$$

with

$$\psi_k^o(V) = \text{tr} \left(m_{i_{j_1}} \prod_{l=2}^{|V|} m_{i_{j_l}}^{S_{i_{j_l-1}}} \right) \mu_p([S]),$$

for all $V \in \pi$, for all $\pi \in NC((i_1, \dots, i_n))$, where

$$[S] = \begin{cases} S_1 \cap S_2 & \text{if } V \text{ contains both 1 and 2} \\ S_1 & \text{if } V = \left(\underbrace{1, \dots, 1}_{|V| \text{-times}} \right) \\ S_2 & \text{if } V = \left(\underbrace{2, \dots, 2}_{|V| \text{-times}} \right), \end{cases}$$

by (3.3.5).

Under the conditions (3.3.6) and (3.3.7), one has that

$$\psi_{p,\pi}(V) = \text{tr} \left(\prod_{l=1}^{|V|} m_{i_{j_l}} \right) \mu_p(S_1),$$

for all $V \in \pi$, for all $\pi \in NC((i_1, \dots, i_n))$.

So, the formula (3.3.8) satisfies that

$$\begin{aligned} k_n^{p,\pi}(T_{i_1}, \dots, T_{i_n}) \\ = \sum_{\pi \in NC((i_1, \dots, i_n))} \left(\prod_{V \in \pi} \text{tr} \left(\prod_{l=1}^{|V|} m_{i_{j_l}} \right) \mu_p(S_1) \right) \mu(\pi, 1_n) \\ = (\mu_p(S_1)) \left(\sum_{\pi \in NC((i_1, \dots, i_n))} \prod_{V \in \pi} \text{tr} \left(\prod_{l=1}^{|V|} m_{i_{j_l}} \right) \mu(\pi, 1_n) \right) \\ = (\mu_p(S_1)) (k_n^{tr}(m_{i_1}, \dots, m_{i_n})) \end{aligned}$$

where $k_n^{tr}(\dots)$ is the free cumulant in terms of tr on the W^* -probability space (M, tr)

$$= (\mu_p(S_1))(0) = 0,$$

by the assumption that m_1 and m_2 are free in (M, tr) .

Indeed, since all mixed free cumulants of m_1 and m_2 vanish, under conditions (3.3.6) and (3.3.7), all mixed free cumulants of T_1 and T_2 vanish, equivalently, T_1 and T_2 are free on $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$.

The above theorem shows that, under the assumptions (3.3.6) and (3.3.7), the freeness on (M, tr) implies the freeness on $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$.

4. Certain p -Adic Dynamical Subalgebras of $\mathcal{M}_{p,\pi}$

Let p be a fixed prime in $\mathcal{P}$, and let (M, tr) be a unital tracial W^* -probability space. And let

$$\mathcal{M}_{p,\pi} = M \rtimes_{\sigma} (\mathbb{Q}_p) = M \otimes_{\pi} \mathcal{N}_p = \mathfrak{M}_{p,\pi}$$

be a p -adic dynamical W^* -algebra, inducing the p -adic dynamical W^* -probability space $(\mathcal{M}_{p,\pi}, \psi_{p,\pi})$.

In this section, we are interested in certain sub-structures of the p -adic dynamical W^* -algebra $\mathcal{M}_{p,\pi} = \mathfrak{M}_{p,\pi}$. Define a W^* -subalgebra $\mathfrak{G}_p^w$ of $\mathcal{N}_p$ by the W^* -algebra generated by

$$(\chi_{\partial_j} \in \mathcal{N}_p : j \in \mathbb{Z}),$$

i.e.,

$$\mathfrak{G}_p^w = W^*\left(\left\{\chi_{\partial_j}\right\}_{j \in \mathbb{Z}}\right) \text{ in } \mathcal{N}_p. \quad (4.1)$$

Since χ_{∂_j} are the mutually-orthogonal projections in $\mathcal{N}_p$, satisfying

$$\chi_{\partial_j}^* = \chi_{\partial_j} = \chi_{\partial_j}^2, \text{ for all } j \in \mathbb{Z}, \quad (4.2)$$

and

$$\chi_{\partial_{j_1}} \chi_{\partial_{j_2}} = \delta_{j_1, j_2} \chi_{\partial_{j_1}}, \text{ for all } j_1, j_2 \in \mathbb{Z},$$

it is not difficult to obtain the following structure theorem of the W^* -subalgebra $\mathfrak{G}_p^w$ of $\mathcal{N}_p$.

Proposition 4.1. *Let $\mathfrak{G}_p^w$ be the W^* -subalgebra (4.1) of $\mathcal{N}_p$. Then*

$$\mathfrak{G}_p^w \stackrel{*iso}{=} \bigoplus_{j \in \mathbb{Z}} (\mathbb{C} \cdot \chi_{\partial_j}) \stackrel{*iso}{=} \mathbb{C}^{\oplus |\mathbb{Z}|}, \quad (4.3)$$

in $\mathcal{N}_p$, where “ $\oplus$ ” means the direct product of W^* -algebras.

Proof. The structure theorem (4.3) is obtained by the relation (4.2) of the generators $\{\chi_{\partial_j}\}_{j \in \mathbb{Z}}$ of $\mathfrak{G}_p^w$.

Let $\mathfrak{G}_p^w$ be the W^* -subalgebra (4.1) of $\mathcal{N}_p$. Then one can construct the W^* -subalgebra $\mathfrak{G}_{p,\pi}^w$ of our p -adic dynamical W^* -algebra $\mathfrak{M}_{p,\pi}$ by

$$\mathfrak{G}_{p,\pi}^w = M \otimes_{\pi} \mathfrak{G}_p^w \text{ in } \mathfrak{M}_{p,\pi} = M \otimes_{\pi} \mathcal{N}_p, \quad (4.4)$$

where $\otimes_{\pi}$ is the conditional tensor product under the π -relation.

Since $\mathfrak{M}_{p,\pi} = \mathcal{M}_{p,\pi}$, the W^* -algebra $\mathfrak{G}_{p,\pi}^w$ of (4.4) is understood as the W^* -subalgebra of $\mathcal{M}_{p,\pi}$, too.

Theorem 4.2. *Let $\mathfrak{G}_{p,\pi}^w$ be the W^* -subalgebra (4.4) of our p -adic dynamical W^* -algebra $\mathcal{M}_{p,\pi} = \mathfrak{M}_{p,\pi}$. Then*

$$\mathfrak{G}_{p,\pi}^w \stackrel{*iso}{=} \bigoplus_{j \in \mathbb{Z}} \left(M \otimes_{\pi} (\mathbb{C} \cdot \chi_{\partial_j}) \right) \stackrel{*iso}{=} M^{\otimes |\mathbb{Z}|}. \quad (4.5)$$

Proof. Observe that

$$\mathfrak{G}_{p,\pi}^w = M \otimes_{\pi} \mathfrak{G}_p^w$$

where $\mathfrak{G}_p^w$ is in the sense of (4.1) in $\mathcal{N}_p$

$$\stackrel{*iso}{=} M \otimes_{\pi} \left(\bigoplus_{j \in \mathbb{Z}} (\mathbb{C} \cdot \chi_{\partial_j}) \right)$$

by (4.3)

$$\stackrel{*iso}{=} \bigoplus_{j \in \mathbb{Z}} \left(M \otimes_{\pi} (\mathbb{C} \cdot \chi_{\partial_j}) \right)$$

$$\stackrel{*iso}{=} \bigoplus_{j \in \mathbb{Z}} (M \otimes_{\mathbb{C}} \mathbb{C})$$

$$\stackrel{*iso}{=} M^{\otimes |\mathbb{Z}|}.$$

By (4.5), one can understand the W^* -subalgebra $\mathfrak{G}_{p,\pi}^w$ is a certain diagonal-like subalgebra of $\mathcal{M}_{p,\pi}$.

Definition 4.1. *Let $p \in \mathcal{P}$, and let $\mathcal{M}_{p,\pi}$ be the p -adic dynamical W^* -algebra induced by the p -adic W^* -dynamical system $(M, \sigma(\mathbb{Q}_p), \pi)$ and let $\mathfrak{G}_{p,\pi}^w$ be the W^* -subalgebra (4.4) of $\mathcal{M}_{p,\pi}$. This W^* -algebra $\mathfrak{G}_{p,\pi}^w$ is called the p -(adic)-dynamical diagonal W^* -subalgebra of $\mathcal{M}_{p,\pi}$.*

Recall that $\mathcal{M}_{p,\pi} = \mathfrak{M}_{p,\pi}$ has its well-defined linear functional $\psi_{p,\pi}$ of (3.2.9). So, one can naturally restrict this linear functional on $\mathfrak{G}_{p,\pi}^w$. For convenience, we

denote such a restricted linear functional again by $\psi_{p,\pi}$. Thus, a W^* -probability space $(\mathfrak{G}_{p,\pi}^w, \psi_{p,\pi})$ is well-determined.

Definition 4.2. Let $\mathfrak{G}_{p,\pi}^w$ be the p -dynamical diagonal W^* -subalgebra of a p -adic dynamical W^* -algebra $\mathcal{M}_{p,\pi} = \mathcal{M}_{p,\pi}$, for a prime p . Then the corresponding W^* -probability space $(\mathfrak{G}_{p,\pi}^w, \psi_{p,\pi})$ is said to be the p -dynamical diagonal W^* -probability space.

Let $(\mathfrak{G}_{p,\pi}^w, \psi_{p,\pi})$ be a p -dynamical diagonal W^* -probability space, and let

$$T_{m,j} \stackrel{\text{def}}{=} m \otimes \chi_{\partial j} = m \chi_{\partial j} \in (\mathfrak{G}_{p,\pi}^w, \psi_{p,\pi}), \quad (4.6)$$

for all $m \in M$, $j \in \mathbb{Z}$.

Let $T_{m,j} \in (\mathfrak{G}_{p,\pi}^w, \psi_{p,\pi})$ be in the sense of (4.6). Then, for any $n \in \mathbb{N}$,

$$\begin{aligned} T_{m,j}^n &= \left(m m^{\partial j} m^{\partial j \cap \partial j} \dots m^{\partial j \cap \dots \cap \partial j} \right) \chi_{\partial j} \\ &= \left(m \left(m^{\partial j} \right)^{n-1} \right) \chi_{\partial j}, \end{aligned} \quad (4.7)$$

by the π -relation on $\mathfrak{G}_{p,\pi} = M \otimes_{\pi} \mathfrak{G}_p^w$.

Therefore, one can get the following free-distributional data of free random variables $T_{m,j}$ of (4.6), for all $m \in M$, $j \in \mathbb{Z}$.

Proposition 4.3. Let $T_{m,j}$ be a free random variable in a p -dynamical diagonal W^* -probability space $(\mathfrak{G}_{p,\pi}^w, \psi_{p,\pi})$ for $m \in M$, $j \in \mathbb{Z}$. Then

$$\psi_{p,\pi}(T_{m,j}^n) = \text{tr} \left(m \left(m^{\partial j} \right)^{n-1} \right) \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \quad (4.8)$$

for all $n \in \mathbb{N}$. Moreover, if $(r_1, \dots, r_n) \in \{1, *\}^n$, for $n \in \mathbb{N}$, then

$$\psi_{p,\pi}(T_{m,j}^{r_1} T_{m,j}^{r_2} \dots T_{m,j}^{r_n}) = t_n \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \quad (4.9)$$

with

$$t_n = \text{tr} \left(m_1^{r_1} \left(m_2^{r_2} \right)^{\partial j} \left(m_3^{r_3} \right)^{\partial j} \dots \left(m_n^{r_n} \right)^{\partial j} \right).$$

Proof. The free-moment formula (4.8) is obtained by (4.7) and (3.3.3). So, it suffices to prove the joint free-moment formula (4.9).

Let $n \in \mathbb{N}$, and $(r_1, \dots, r_n) \in \{1, *\}^n$. Then

$$\begin{aligned} T_{m,j}^{r_1} T_{m,j}^{r_2} \dots T_{m,j}^{r_n} &= \left(m \otimes \chi_{\partial j} \right)^{r_1} \dots \left(m \otimes \chi_{\partial j} \right)^{r_n} \\ &= \left(m^{r_1} \otimes \chi_{\partial j}^{r_1} \right) \dots \left(m^{r_n} \otimes \chi_{\partial j}^{r_n} \right) \end{aligned}$$

$$\begin{aligned} &= \left(m^{r_1} \left(m^{r_2} \right)^{\partial j} \left(m^{r_3} \right)^{\partial j \cap \partial j} \dots \left(m^{r_n} \right)^{\partial j \cap \dots \cap \partial j} \right) \\ &\quad \otimes \chi_{\partial j \cap \dots \cap \partial j \cap \partial j} \end{aligned}$$

by the π -relation on $\mathfrak{G}_{p,\pi}^w = M \otimes_{\pi} \mathfrak{G}_p^w$

$$= \left(m^{r_1} \left(m^{r_2} \right)^{\partial j} \left(m^{r_3} \right)^{\partial j} \dots \left(m^{r_n} \right)^{\partial j} \right) \otimes \chi_{\partial j}$$

i.e.,

$$\begin{aligned} &T_{m,j}^{r_1} T_{m,j}^{r_2} \dots T_{m,j}^{r_n} \\ &= \left(m^{r_1} \left(m^{r_2} \right)^{\partial j} \left(m^{r_3} \right)^{\partial j} \dots \left(m^{r_n} \right)^{\partial j} \right) \chi_{\partial j}, \end{aligned} \quad (4.10)$$

in $\mathfrak{G}_{p,\pi}^w \subset \mathcal{M}_{p,\pi}$.

Therefore, by (4.10), one has

$$\begin{aligned} &\psi_{p,\pi}(T_{m,j}^{r_1} T_{m,j}^{r_2} \dots T_{m,j}^{r_n}) \\ &= \text{tr} \left(m^{r_1} \left(m^{r_2} \right)^{\partial j} \left(m^{r_3} \right)^{\partial j} \dots \left(m^{r_n} \right)^{\partial j} \right) \psi(\chi_{\partial j}) \\ &= \text{tr} \left(m^{r_1} \left(m^{r_2} \right)^{\partial j} \left(m^{r_3} \right)^{\partial j} \dots \left(m^{r_n} \right)^{\partial j} \right) \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right). \end{aligned}$$

Therefore, the joint free-moment formula (4.9) holds.

The above free-moment formulas (4.8) and (4.9) characterize the free-moment of operators $T_{m,j}$ in the sense of (4.6) in a p -dynamical diagonal W^* -probability space $(\mathfrak{G}_{p,\pi}^w, \psi_{p,\pi})$. So, we obtain the following generalized free-distributional data.

Theorem 4.4. Let $T = \sum_{k=1}^N T_{m_k, j_k}$ be a free random variable in a p -dynamical diagonal W^* -probability space $(\mathfrak{G}_{p,\pi}^w, \psi_{p,\pi})$, where T_{m_k, j_k} are in the sense of (4.6), for $m_k \in M$, $j_k \in \mathbb{Z}$, for all $k=1, \dots, N$, for $N \in \mathbb{N}$. Then

$$\begin{aligned} &\psi_{p,\pi} \left(\prod_{l=1}^n T^{r_l} \right) \\ &= \sum_{l=1}^N \text{tr} \left(m_l^{r_l} \left(\left(m_l^{r_l} \right)^{\partial j_l} \right)^{n-1} \right) \left(\frac{1}{p^{j_l}} - \frac{1}{p^{j_l+1}} \right), \end{aligned} \quad (4.11)$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. Let T be given as above in $(\mathfrak{G}_{p,\pi}^w, \psi_{p,\pi})$. Then,

for any $(r_1, \dots, r_n) \in \{1, *\}^n$, for $n \in \mathbb{N}$, one can get that

$$\begin{aligned} T^{r_1} T^{r_2} T^{r_n} &= \sum_{(l_1, \dots, l_n) \in \{1, \dots, N\}^n} T_{m_{l_1}, j_{l_1}}^{r_1} T_{m_{l_2}, j_{l_2}}^{r_2} \dots T_{m_{l_n}, j_{l_n}}^{r_n} \\ &= \sum_{(l_1, \dots, l_n) \in \{1, \dots, N\}^n} \left(m_{l_1}^{r_1} \left(m_{l_2}^{r_2} \right)^{\partial j_{l_1}} \left(m_{l_3}^{r_3} \right)^{\partial j_{l_1} \cap \partial j_{l_2}} \dots \left(m_{l_n}^{r_n} \right)^{\partial j_{l_1} \cap \dots \cap \partial j_{l_{n-1}}} \right) \chi_{\bigcap_{t=1}^n \partial j_{l_t}} \end{aligned}$$

$$= \sum_{\substack{(l_1, \dots, l_n) \in \{1, \dots, N\}^n \\ \text{n-times}}} \begin{pmatrix} m_{l_1}^\eta \left(m_{l_1}^\eta\right)^{\partial_{j_1}} \\ \left(m_{l_1}^\eta\right)^{\partial_{j_1} \cap \partial_{j_2}} \\ \dots \left(m_{l_n}^\eta\right)^{\partial_{j_1} \cap \dots \cap \partial_{j_n}} \end{pmatrix} \chi_{\partial_{j_1} \cap \dots \cap \partial_{j_n}}$$

since $\partial_{k_1} \cap \partial_{k_2} = \delta_{k_1, k_2} \partial_{k_1}$, for all $k_1, k_2 \in \mathbb{Z}$, or by (4.5)

$$= \sum_{l=1}^N \left(\underbrace{m_l^\eta \left(m_l^\eta\right)^{\partial_{j_1}} \left(m_l^\eta\right)^{\partial_{j_2}} \dots \left(m_l^\eta\right)^{\partial_{j_n}}}_{n\text{-many}} \right) \chi_{\partial_{j_l}}$$

$$= \sum_{l=1}^N \left(m_l^\eta \left(\left(m_l^\eta\right)^{\partial_{j_l}} \right)^{n-1} \right) \chi_{\partial_{j_l}},$$

and hence, we obtain that

$$\psi_{p,\pi} \left(T^\eta T^{\eta_2} T^{\eta_n} \right)$$

$$= \sum_{l=1}^N \text{tr} \left(m_l^\eta \left(\left(m_l^\eta\right)^{\partial_{j_l}} \right)^{n-1} \right) \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right).$$

Therefore, the free-moment formula (4.11) holds.

The above joint free-moment formula (4.11) characterizes the free-distributional data of elements of a p -dynamical diagonal W^* -probability space $(\mathfrak{S}_{p,\pi}^w, \psi_{p,\pi})$, under linearity.

Recall that the Euler totient function ϕ is an arithmetic function,

$$\phi: \mathbb{N} \rightarrow \mathbb{C}$$

defined by

$$\phi(n) = \left| \left\{ k \in \mathbb{N} \mid 1 \leq k \leq n, \text{ and } \gcd(k, n) = 1 \right\} \right|, \quad (4.12)$$

for all $n \in \mathbb{N}$.

It is well-known that if ϕ is the Euler totient function (4.12), then

$$\phi(n) = n \left(\prod_{p \in P, p|n} \left(1 - \frac{1}{p} \right) \right), \text{ for all } n \in \mathbb{N}, \quad (4.13)$$

where “ $p|n$ ” means “ p divides n ,” or “ p is a divisor of n .”

For instance,

$$\phi(p) = p - 1 = p \left(1 - \frac{1}{p} \right), \quad (4.14)$$

by (4.12) and (4.13), for all $p \in \mathcal{P}$.

Now, observe that, if $T_{m,j}$ is in the sense of (4.6), and if

$$(\eta_1, \dots, \eta_n) \in \{1, *\}^n, \text{ for } n \in \mathbb{N},$$

then

$$\psi_{p,\pi} \left(\prod_{l=1}^n T_{m,j}^\eta \right)$$

$$= \text{tr} \left(m^\eta \left(m^{\eta_2} \right)^{\partial_j} \dots \left(m^{\eta_n} \right)^{\partial_j} \right) \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right)$$

by (4.9) and (4.11)

$$= \text{tr} \left(m^\eta \left(m^{\eta_2} \right)^{\partial_j} \dots \left(m^{\eta_n} \right)^{\partial_j} \right) \frac{1}{p^j} \left(1 - \frac{1}{p} \right)$$

$$= \text{tr} \left(m^\eta \left(m^{\eta_2} \right)^{\partial_j} \dots \left(m^{\eta_n} \right)^{\partial_j} \right) \frac{p}{p^{j+1}} \left(1 - \frac{1}{p} \right) \quad (4.15)$$

$$= \text{tr} \left(m^\eta \left(m^{\eta_2} \right)^{\partial_j} \dots \left(m^{\eta_n} \right)^{\partial_j} \right) \frac{1}{p^{j+1}} \left(p \left(1 - \frac{1}{p} \right) \right)$$

$$= \frac{\phi(p)}{p^{j+1}} \text{tr} \left(m^\eta \left(m^{\eta_2} \right)^{\partial_j} \dots \left(m^{\eta_n} \right)^{\partial_j} \right),$$

for all $m \in M$, $j \in \mathbb{Z}$. Similar to (4.15), one can get that

$$\psi_{p,\pi} \left(T_{m,j}^n \right) = \frac{\phi(p)}{p^{j+1}} \text{tr} \left(m \left(m^{\partial_j} \right)^{n-1} \right), \quad (4.15)'$$

for all $n \in \mathbb{N}$, by (4.8).

Motivated by (4.15) and (4.15)', we define a new linear functional $\tau_{p,\pi}$ on $\mathfrak{S}_{p,\pi}^w$ by

$$\tau_{p,\pi} = \frac{1}{\phi(p)} \psi_{p,\pi} \text{ on } \mathfrak{S}_{p,\pi}^w. \quad (4.16)$$

Then the pair $(\mathfrak{S}_{p,\pi}^w, \tau_{p,\pi})$ is a well-determined W^* -probability space, where $\tau_{p,\pi}$ is in the sense of (4.16).

Definition 4.3. The W^* -probability space $(\mathfrak{S}_{p,\pi}^w, \tau_{p,\pi})$ of a p -dynamical diagonal W^* -algebra $\mathfrak{S}_{p,\pi}^2$, and the linear functional $\tau_{p,\pi}^w = \frac{1}{\phi(p)} \psi_{p,\pi}$ of (4.16) is called the p -dynamical W^* -probability space of $\mathfrak{S}_{p,\pi}^w$.

By the very construction, our p -dynamical W^* -probability space $(\mathfrak{S}_{p,\pi}^w, \tau_{p,\pi})$ has the following free-distributional information.

Theorem 4.5. Let $T = \sum_{k=1}^N T_{m_k, j_k}$ be a free random variable in a p -dynamical W^* -probability space $(\mathfrak{S}_{p,\pi}^w, \tau_{p,\pi})$, where T_{m_k, j_k} are in the sense of (4.6) for $m_k \in M$, $j_k \in \mathbb{Z}$, for all $k = 1, \dots, N$, for $N \in \mathbb{N}$, and $\tau_{p,\pi}$ is in the sense of (4.16). Then

$$\tau_{p,\pi} \left(\prod_{l=1}^n T^\eta \right) = \sum_{l=1}^N \frac{1}{p^{j_l+1}} \text{tr} \left(m_l^\eta \left(\left(m_l^\eta\right)^{\partial_{j_l}} \right)^{n-1} \right), \quad (4.17)$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. Let T be given as above in the p -dynamical W^* -probability space $(\mathfrak{S}_{p,\pi}^w, \tau_{p,\pi})$. Then, for any

$(r_1, \dots, r_n) \in \{1, *\}^n$, for $n \in \mathbb{N}$, one can get that

$$\begin{aligned} & T^{\eta_1} T^{\eta_2} \dots T^{\eta_n} \\ &= \sum_{(l_1, \dots, l_n) \in \{1, \dots, N\}^n} \left(T_{m_{l_1}, j_{l_1}}^{\eta_1} T_{m_{l_2}, j_{l_2}}^{\eta_2} \dots T_{m_{l_n}, j_{l_n}}^{\eta_n} \right) \\ &= \sum_{(l_1, \dots, l_n) \in \{1, \dots, N\}^n} \left(\begin{matrix} m_{l_1}^{\eta_1} \left(m_{l_2}^{\eta_2} \right)^{\partial j_{l_1}} \\ \left(m_{l_3}^{\eta_3} \right)^{\partial j_{l_1} \cap \partial j_{l_2}} \dots \\ \left(m_{l_n}^{\eta_n} \right)^{\partial j_{l_1} \cap \dots \cap \partial j_{l_{n-1}}} \end{matrix} \right) \chi_{\bigcap_{t=1}^n \partial j_{l_t}} \\ &= \sum_{\left(\underbrace{l_1, \dots, l_n}_{n\text{-times}} \right) \in \{1, \dots, N\}^n} \left(\begin{matrix} m_l^{\eta_1} \left(m_l^{\eta_1} \right)^{\partial j_l} \\ \left(m_l^{\eta_1} \right)^{\partial j_l \cap \partial j_l} \dots \\ \left(m_l^{\eta_1} \right)^{\partial j_l \cap \dots \cap \partial j_l} \end{matrix} \right) \chi_{\partial j_l \cap \dots \cap \partial j_l} \end{aligned}$$

since $\partial_{k_1} \cap \partial_{k_2} = \delta_{k_1, k_2} \partial_{k_1}$, for all $k_1, k_2 \in \mathbb{Z}$, or by (4.5)

$$\begin{aligned} &= \sum_{l=1}^N \left(\underbrace{m_l^{\eta_1} \left(m_l^{\eta_1} \right)^{\partial j_l} \left(m_l^{\eta_1} \right)^{\partial j_l} \dots \left(m_l^{\eta_1} \right)^{\partial j_l}}_{n\text{-many}} \right) \chi_{\partial j_l} \\ &= \sum_{l=1}^N \left(m_l^{\eta_1} \left(\left(m_l^{\eta_1} \right)^{\partial j_l} \right)^{n-1} \right) \chi_{\partial j_l}, \end{aligned}$$

and hence, we obtain that

$$\begin{aligned} & \tau_{p,\pi} \left(T^{\eta_1} T^{\eta_2} \dots T^{\eta_n} \right) \\ &= \sum_{l=1}^N \frac{1}{\phi(p)} \text{tr} \left(m_l^{\eta_1} \left(\left(m_l^{\eta_1} \right)^{\partial j_l} \right)^{n-1} \right) \left(\frac{1}{p^{j_l}} - \frac{1}{p^{j_l+1}} \right) \\ &= \sum_{l=1}^N \frac{1}{\phi(p)} \text{tr} \left(m_l^{\eta_1} \left(\left(m_l^{\eta_1} \right)^{\partial j_l} \right)^{n-1} \right) \frac{p}{p^{j_l+1}} \left(1 - \frac{1}{p} \right) \\ &= \sum_{l=1}^N \frac{1}{\phi(p)} \text{tr} \left(m_l^{\eta_1} \left(\left(m_l^{\eta_1} \right)^{\partial j_l} \right)^{n-1} \right) \frac{\phi(p)}{p^{j_l+1}} \\ &= \sum_{l=1}^N \frac{1}{p^{j_l+1}} \text{tr} \left(m_l^{\eta_1} \left(\left(m_l^{\eta_1} \right)^{\partial j_l} \right)^{n-1} \right). \end{aligned}$$

Therefore, the free-moment formula (4.17) holds.

By the above free-moment formula (4.17), we obtain the following corollary.

Corollary 4.6. Let $T_{m,j}$ be in the sense of (4.6) in the p -dynamical W^* -probability $(\mathfrak{S}_{p,\pi}^w, \tau_{p,\pi})$. Then

$$\tau_{p,\pi} \left(T_{m,j}^n \right) = \frac{1}{p^{j+1}} \text{tr} \left(m \left(m^{\partial j} \right)^{n-1} \right), \quad (4.18)$$

and

$$\tau_{p,\pi} \left(\prod_{l=1}^n T_{m,j}^{\eta_l} \right) = \frac{1}{p^{j+1}} \text{tr} \left(m^{\eta_1} \left(m^{\eta_2} \right)^{\partial j} \dots \left(m^{\eta_n} \right)^{\partial j} \right),$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. The proof of (4.18) is done by (4.17). Also, the free-moment formulas of (4.18) can be proven by (4.15) and (4.15)'; under the definition (4.16).

5. Free Product W^* -Probability Spaces of

$$\left\{ \left(\mathfrak{S}_{p,\pi_p}^w, \tau_{p,\pi_p} \right) \right\}_{p \in \mathcal{P}}$$

For a prime $p \in \mathcal{P}$, one can have a corresponding p -dynamical diagonal W^* -algebra $\mathfrak{S}_{p,\pi_p}^w$, from a W^* -dynamical system

$$\left((M, \text{tr}), \sigma(\mathbb{Q}_p), \pi_p \right),$$

where (M, tr) is a fixed unital tracial W^* -probability space. By defining a linear functional τ_{p,π_p} on $\mathfrak{S}_{p,\pi_p}^w$, one obtains the p -dynamical W^* -probability space

$$\mathfrak{S}_{p,\pi_p}^w \stackrel{\text{denote}}{=} \left(\mathfrak{S}_{p,\pi_p}^w, \tau_{p,\pi_p} \right).$$

For a fixed W^* -probability space (M, tr) , assume we have family $\mathcal{S}_{\mathcal{P}}$ of W^* -probability spaces $\mathfrak{S}_{q,\pi_q}^w$, for all $q \in \mathcal{P}$. i.e., in the similar manner with Section 3, suppose we have W^* -probability spaces

$$\mathcal{S}_{\mathcal{P}} = \left\{ \mathfrak{S}_{p,\pi_p}^w = \left(\mathfrak{S}_{p,\pi_p}^w, \tau_{p,\pi_p} \right) : p \in \mathcal{P} \right\}, \quad (5.1)$$

where π_p are in the sense of (3.1.5) for every prime p in $\mathcal{P}$.

Let $\mathcal{S}_{\mathcal{P}}$ be the family (5.1) of prime-dynamical W^* -probability spaces $\mathfrak{S}_{p,\pi_p}$'s. Now, we consider the free product W^* -probability space $\mathfrak{S}$,

$$\mathfrak{S} \stackrel{\text{denote}}{=} (\mathfrak{S}, \tau) \stackrel{\text{def}}{=} \bigstar_{X \in \mathcal{S}_{\mathcal{P}}} X.$$

i.e.,

$$\begin{aligned} \mathfrak{S} = (\mathfrak{S}, \tau) &= \bigstar_{p \in \mathcal{P}} \mathfrak{S}_{p,\pi_p}^w = \bigstar_{p \in \mathcal{P}} \left(\mathfrak{S}_{p,\pi_p}^w, \tau_{p,\pi_p} \right) \\ &= \left(\bigstar_{p \in \mathcal{P}} \mathfrak{S}_{p,\pi_p}^w, \bigstar_{p \in \mathcal{P}} \tau_{p,\pi_p} \right). \end{aligned} \quad (5.2)$$

Definition 5.1. The free product W^* -probability space $\mathfrak{G}$ of (5.2) is called the Adelic-dynamical W^* -probability space over (M, tr) .

By the very definition (5.2) of the Adelic-dynamical W^* -probability space $\mathfrak{G}$ over (M, tr) , we obtain the following structure theorem.

Theorem 5.1. Let $\mathfrak{G} = (\mathfrak{G}, \tau)$ be our Adelic-dynamical W^* -probability space over (M, tr) . Then $\mathfrak{G}$ is $*$ -isomorphic to the conditional tensor product W^* -algebra,

$$\mathfrak{G} \stackrel{*}{=} M \otimes_{\pi} \left(\bigstar_{p \in \mathcal{P}} \mathfrak{G}_p^w \right), \quad (5.3)$$

where $\mathfrak{G}_p^w$ are the W^* -subalgebras (4.1) of $\mathcal{N}_p = L^\infty(\mathbb{Q}_p)$, for all $p \in \mathcal{P}$, and

$$\pi = \bigstar_{p \in \mathcal{P}} \pi_p,$$

satisfying that

$$M \otimes_{\pi} \mathfrak{G}_q^w = M \otimes_{\pi_q} \mathfrak{G}_q^w = \mathfrak{G}_{q, \pi_q}^w, \text{ for all } q \in \mathcal{P}.$$

Proof. Let S be the Adelic-dynamical W^* -probability space (5.2) over a fixed W^* -probability space (M, tr) . Then, by the very construction (5.2), one obtains that

$$\begin{aligned} \mathfrak{G} &= \bigstar_{p \in \mathcal{P}} \mathfrak{G}_{p, \pi_p}^w = \bigstar_{p \in \mathcal{P}} \left(M \otimes_{\pi_p} \mathfrak{G}_p^w \right) \\ &\stackrel{*}{=} M \otimes_{\pi} \left(\bigstar_{p \in \mathcal{P}} \mathfrak{G}_p^w \right), \end{aligned}$$

in terms of the linear functional $\tau = \bigstar_{p \in \mathcal{P}} \tau_{p, \pi_p}$, where

$$\pi = \bigstar_{p \in \mathcal{P}} \pi_p, \text{ satisfying,}$$

$$M \otimes_{\pi} \mathfrak{G}_p^w = M \otimes_{\pi_q} \mathfrak{G}_q^w = \mathfrak{G}_{q, \pi_q}^w,$$

for all $q \in \mathcal{P}$.

By the structure theorem (5.3), we understand the free product W^* -algebra $\mathfrak{G}$ and its $*$ -isomorphic W^* -algebra,

$$\mathfrak{G}_M \stackrel{def}{=} M \otimes_{\pi} \left(\bigstar_{p \in \mathcal{P}} \mathfrak{G}_p^w \right) \quad (5.4)$$

as the same W^* -algebra.

Notation and Assumption For convenience, we denote the conditional-tensor factor, which is a free product W^* -algebra,

$$\bigstar_{p \in \mathcal{P}} \mathfrak{G}_p^w \text{ by } \mathfrak{G}^w. \quad (5.5)$$

i.e.,

$$\mathfrak{G} = \mathfrak{G}_M = M \otimes_{\pi} \mathfrak{G}^w,$$

where $\mathfrak{G}_M$ is in the sense of (5.4), and $\mathfrak{G}^w$ is in the sense of (5.5).

By the structure theorem (5.3) of the Adelic-dynamical W^* -probability space $\mathfrak{G}$ over (M, tr) , we understand $\mathfrak{G}$ and $\mathfrak{G}_M$ as the same W^* -algebras, and denote them by $\mathfrak{G}$.

Let $\mathfrak{G} = \mathfrak{G}_M$ be the Adelic-dynamical W^* -probability space over (M, tr) , and let

$$T = \prod_{n=1}^N \left(m_n \chi_{\partial_{j_n}} \right) \in \mathfrak{G}, \quad (5.6)$$

where

$$m_n \chi_{\partial_{j_n}} \in \left(\mathfrak{G}_{p_n, \pi_{p_n}}^w, \tau_{p_n, \pi_{p_n}} \right) \text{ in } \mathfrak{G}, \quad (5.6)'$$

with $m_n \in M$, $j_n \in \mathbb{Z}$, for $N \in \mathbb{N}$.

We are interested in the cases where a free random variable T of (5.6) is a free reduced word in $\mathfrak{G}$, equivalently, we are interested in the case where the prime-sequence

$$(p_1, \dots, p_N) \in \text{alt}(\mathcal{P}^N), \text{ for } N \in \mathbb{N},$$

from T of (5.6) is alternating (See (5.6)', and Section 2.4).

By Section 2.4, for a fixed free reduced word T of (5.6), we have

$$\begin{aligned} T^k &\stackrel{equi}{=} \left(\bigotimes_{n=1}^N \left(m_n \chi_{\partial_{j_n}} \right) \right)^k = \bigotimes_{n=1}^N \left(m_n \chi_{\partial_{j_n}} \right)^k \\ &= \bigotimes_{n=1}^N \left(m_n \left(m_n^{\partial_{j_n}} \right)^{k-1} \right) \chi_{\partial_{j_n}} \\ &\stackrel{equi}{=} \prod_{n=1}^N \left(m \left(m_n^{\partial_{j_n}} \right)^{k-1} \right) \chi_{\partial_{j_n}}, \end{aligned} \quad (5.7)$$

for all $k \in \mathbb{N}$, meanwhile,

$$T^{(k)} = \underbrace{T \dots T}_{n\text{-times}}, \text{ for all } k \in \mathbb{N},$$

as a free “non-reduced” word in $\mathfrak{G}$. Remark that only if $j_n \neq j_1$ in $\mathbb{Z}$, then the above free non-reduced word $T^{(k)}$ of T forms a free reduced word in $\mathfrak{G}$.

Here, we are interested in the free-distributional data $\left\{ \tau(T^n) \right\}_{n=1}^\infty$ of the free reduced word T of (5.6) in the Adelic-dynamical W^* -probability space $\mathfrak{G}$ over (M, tr) .

Theorem 5.2. Let $T = \prod_{n=1}^N \left(m_n \chi_{\partial_{j_n}} \right)$ be a free reduced word (5.6) in the Adelicdynamical W^* -probability space $\mathfrak{G} = (\mathfrak{G}, \tau)$ over a fixed W^* -probability space (M, tr) , where

$$m_n \chi_{\partial_{j_n}} \in \left(\mathfrak{G}_{p_n, \pi_{p_n}}^w, \tau_{p_n, \pi_{p_n}} \right), \text{ as in (5.6)'}$$

for $n=1, \dots, N$, for $N \in \mathbb{N}$. Then there exist $r_0 \in \mathbb{Q}$, such that

$$r_0 = \prod_{n=1}^N \frac{1}{p_n^{j_n+1}} \text{ in } \mathbb{Q}, \quad (5.8)$$

and

$$\tau(T^k) = r_0 \left(\prod_{n=1}^N \text{tr} \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \right),$$

for all $k \in \mathbb{N}$.

Note that the quantity r_0 is free from the choice of the

powers k of T^k , for all $k \in \mathbb{N}$.

Proof. Let T be a free reduced word of $\mathfrak{G}$ in the sense of (5.6). Then, by (5.7), one obtains that

$$T^k = \prod_{n=1}^N \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \chi_{\partial j_n} \text{ in } \mathfrak{G},$$

for all $k \in \mathbb{N}$. So,

$$\begin{aligned} \tau(T^k) &= \tau \left(\prod_{n=1}^N \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \chi_{\partial j_n} \right) \\ &= \prod_{n=1}^N \tau_{p_n, \pi_{p_n}} \left(\left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \chi_{\partial j_n} \right) \\ &= \prod_{n=1}^N \frac{1}{p_n^{j_n+1}} \text{tr} \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \end{aligned}$$

by (4.17) and (4.18)

$$= \left(\prod_{n=1}^N \frac{1}{p_n^{j_n+1}} \right) \left(\prod_{n=1}^N \text{tr} \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \right),$$

for all $k \in \mathbb{N}$.

The above free-moment formula (5.8) shows that, under certain rational-scalar-multiples, the free-distributional data of T^k are determined by the free-distributional data on a fixed W^* -probability space (M, tr) , where T is free reduced word in the sense of (5.6) whose factors satisfy (5.6)'. Now, let

$$S = \sum_{n=1}^N \left(m_n \chi_{\partial j_n} \right) \in \mathfrak{G} = (\mathfrak{G}, \tau), \quad (5.9)$$

with

$$m_n \chi_{\partial j_n} \in \left(\mathfrak{G}_{p_n, \pi_{p_n}}, \tau_{p_n, \pi_{p_n}} \right), \quad (5.9)'$$

for $n=1, \dots, N$, for $N \in \mathbb{N}$.

Suppose S of (5.9) is a free sum in $\mathfrak{G}$. Equivalently, assume that the primes

$$\{p_1, \dots, p_N\} \in \mathcal{P}$$

from the operator S of (5.9), satisfying (5.9)', are mutually-distinct from each other in $\mathcal{P}$. Since the primes $p_1, \dots, p_N$ are mutually-distinct from each other,

all summands $m_n \chi_{\partial j_n}$ of S are contained in the mutually-distinct free blocks $\mathfrak{G}_{p_n, \pi_{p_n}}$ of (5.9)', for all $n=1, \dots, N$, and hence, they are free from each other in $\mathfrak{G}$.

Now, recall that

$$\begin{aligned} S^k &\stackrel{\text{equi}}{=} \left(\bigoplus_{n=1}^N m_n \chi_{\partial j_n} \right)^k = \bigoplus_{n=1}^N \left(m_n \chi_{\partial j_n} \right)^k \\ &= \bigoplus_{n=1}^N \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \chi_{\partial j_n} \\ &\stackrel{\text{equi}}{=} \sum_{n=1}^N \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \chi_{\partial j_n}, \end{aligned} \quad (5.10)$$

for all $k \in \mathbb{N}$, by Section 2.4. Meanwhile,

$$S^{(k)} = \sum_{(i_1, \dots, i_k) \in \{1, \dots, N\}^k} \left(\left(m_{n_{i_1}} \chi_{\partial j_{i_1}} \right) \dots \left(m_{n_{i_k}} \chi_{\partial j_{i_k}} \right) \right),$$

where the summands

$$\left(\left(m_{n_{i_1}} \chi_{\partial j_{i_1}} \right) \dots \left(m_{n_{i_k}} \chi_{\partial j_{i_k}} \right) \right)$$

are free non-reduced words in $\mathfrak{G}$, for all

$$(i_1, \dots, i_k) \in \{1, \dots, N\}^k,$$

for all $k \in \mathbb{N}$. (See Section 2.4.)

We are interested in the free-distributional data of $\{S^k\}_{k=1}^\infty$ for a free sum S of (5.9) in $\mathfrak{G}$.

Theorem 5.3. Let S be a free sum (5.9) in the Adelic-dynamical W^* -probability space $\mathfrak{G}$, satisfying (5.9)'. Then

$$\tau(S^k) = \sum_{n=1}^N \left(\frac{1}{p_n^{j_n+1}} \text{tr} \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \right), \quad (5.11)$$

for all $k \in \mathbb{N}$.

Proof. By (5.10), if S of (5.9) is a free word in $\mathfrak{G}$, then one has that

$$S^k = \sum_{n=1}^N \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \chi_{\partial j_n},$$

in $\mathfrak{G}$, for all $k \in \mathbb{N}$. Thus,

$$\begin{aligned} \tau(S^k) &= \tau \left(\sum_{n=1}^N \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \chi_{\partial j_n} \right) \\ &= \sum_{n=1}^N \tau_{p_n, \pi_{p_n}} \left(\left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \chi_{\partial j_n} \right) \\ &= \sum_{n=1}^N \left(\frac{1}{p_n^{j_n+1}} \text{tr} \left(m_n \left(m_n^{\partial j_n} \right)^{k-1} \right) \right), \end{aligned}$$

for all $k \in \mathbb{N}$.

6. Adelic-Dynamical W^* -Probability Spaces over Group W^* -Probability Spaces

In this section, let's apply the main results of Sections 4 and 5 to certain special cases. In particular, we are interested in the cases where fixed unital tracial W^* -probability spaces (M, tr) are induced by countable discrete groups. In the following, all groups G are automatically assumed to be countable discrete groups.

Let G be a group. Then one can construct the corresponding group Hilbert space,

$$H_G = l^2(G), \text{ the } l^2\text{-space generated by } G, \quad (6.1)$$

equipped with its orthonormal basis B_G ,

$$B_G = \{\xi_g : g \in G\},$$

satisfying

$$\langle \xi_{g_1} : \xi_{g_2} \rangle_G = \delta_{g_1, g_2},$$

where $\langle \cdot, \cdot \rangle_G$ means the l^2 -inner product of the l^2 -space H_G of (6.1).

Then the group G acts on H_G via an action u , satisfying that

$$u(g)(\xi_{gg'}), \text{ for all } g, g' \in G. \quad (6.2)$$

For convenience, we denote $u(g)$ by u_g , for all $g \in G$. Then these images of the action u satisfy

$$u_{g_1} u_{g_2} = u_{g_1 g_2}, \text{ for all } g_1, g_2 \in G, \quad (6.3)$$

and

$$u_g^* = u_{g^{-1}}, \text{ for all } g \in G,$$

where g^{-1} mean the inverses of g .

So, one can realize that

$$u_g u_g^* = u_{g g^{-1}} = u_{e_G} = u_{g^{-1} g} = u_g^* u_g, \quad (6.4)$$

for all $g \in G$, by (6.2) and (6.3), where e_G is the group-identity of G , and hence u_{e_G} is the identity operator 1_{H_G} on the Hilbert space H_G of (6.1).

The above relation (6.4) guarantees that u_g are unitary operators on H_G , satisfying

$$u_g^* = u_g^{-1}, \text{ for all } g \in G,$$

where u_g^{-1} mean the inverses of u_g on H_G .

So, we call the action u of G acting on H_G , the *left-regular unitary representation*. Also, the Hilbert-space representation (H_G, u) of a group G is also called the *left-regular unitary representation* of G .

Define now a W^* -subalgebra M_G of the operator algebra $B(H_G)$ by

$$M_G \stackrel{\text{def}}{=} \overline{\mathbb{C}[u(G)]}^w = \overline{\mathbb{C}[u_g \in B(H_G) : g \in G]}^w, \quad (6.5)$$

where $\bar{Y}^w$ mean the W^* -topology closures of subsets Y of $B(H_G)$.

We call the W^* -subalgebra M_G of (6.5), the group W^* -algebra of G (in $B(H_G)$). By the very definition (6.5) of M_G , if $m \in M_G$, then

$$m = \sum_{g \in G} t_g u_g \text{ with } t_g \in \mathbb{C},$$

where Σ means a finite, or an infinite (limits of finite) sum(s under W^* -topology).

Thus, on the group W^* -algebra, we define the canonical trace tr_G by

$$tr_G \left(\sum_{g \in G} t_g u_g \right) = t_{e_G}. \quad (6.6)$$

So, one can understand the canonical tracial values mean the coefficients of the u_{e_G} -terms of elements of M_G .

Remark that

$$tr_G(1_{H_G}) = tr_G(u_{e_G}) = 1,$$

and hence, the linear functional tr_G of (6.6) is a unital trace.

Definition 6.1. Let M_G be the group W^* -algebra (6.5) of G in $B(H_G)$, and let tr_G be the canonical trace (6.6) on M_G . Then the unital tracial W^* -probability space (M_G, tr_G) is said to be the group W^* -probability space of G .

Let G be a given group, and let (M_G, tr_G) be the corresponding group W^* -probability space of G . Then, by our general cases in Sections 3, 4 and 5, one can construct the W^* -dynamical systems

$$((M_G, tr_G), \sigma(\mathbb{Q}_p), \pi_p),$$

and the corresponding crossed product W^* -algebras

$$\mathcal{M}_{p, \pi_p}^G = M_G \times_{\pi_p} \sigma(\mathbb{Q}_p)^{*-\text{iso}} = M_G \times_{\pi_p} \mathcal{N}_p = \mathfrak{M}_{p, \pi_p}^G,$$

for all $p \in \mathcal{P}$.

Thus, one obtains the p -dynamical diagonal subalgebras

$$\mathfrak{G}_{p, \pi_p}^G = M_G \otimes_{\pi_p} \mathfrak{G}_p^G,$$

where $\mathfrak{G}_p^G$ are the p -adic diagonal W^* -subalgebra of $\mathcal{N}_p$, for all $p \in \mathcal{P}$.

So, as in Section 5, we have the system

$$\left\{ \mathfrak{S}_{p,\pi_p}^G \stackrel{\text{denote}}{=} \left(\mathfrak{S}_{p,\pi_p}^G, \tau_{p,\pi_p}^G \right) : p \in \mathcal{P} \right\} \quad (6.7)$$

of W^* -probability spaces $\mathfrak{S}_{p,\pi_p}^G$'s.

From the family $\left\{ \mathfrak{S}_{p,\pi_p}^G \right\}_{p \in \mathcal{P}}$ of (6.7), we can have the

corresponding Adelic-dynamical W^* -probability space $\mathfrak{S}^G$,

$$\mathfrak{S}^G \stackrel{\text{denote}}{=} \left(\mathfrak{S}^G, \tau^G \right) \stackrel{\text{def}}{=} \star_{p \in \mathcal{P}} \left(\mathfrak{S}_{p,\pi_p}^G, \tau_{p,\pi_p}^G \right), \quad (6.8)$$

i.e.,

$$\mathfrak{S}^G = \star_{p \in \mathcal{P}} \mathfrak{S}_{p,\pi_p}^G, \text{ and } \tau^G = \star_{p \in \mathcal{P}} \tau_{p,\pi_p}^G.$$

Let $T = u_g \chi_{\partial_j} \in \mathfrak{S}_{p,\pi_p}^G$ be a generating element of the Adelic-dynamical W^* -probability space $\mathfrak{S}^G$, for $g \in G$, and $j \in \mathbb{Z}$, where u_g are in the sense of (6.2). Then one obtains that

$$\begin{aligned} \tau^G(u_g \chi_{\partial_j}) &= \tau_{p,\pi_p}^G(u_g \chi_{\partial_j}) \\ &= \frac{1}{p^{j+1}} \text{tr}_G(u_g) = \delta_{e_G, g} \left(\frac{1}{p^{j+1}} \right), \end{aligned}$$

i.e.,

$$\tau^G(u_g \chi_{\partial_j}) = \frac{\delta_{e_G, g}}{p^{j+1}}, \quad (6.9)$$

by (4.17) and (5.8), for all $g \in G$, and $j \in \mathbb{Z}$, for all $p \in \mathcal{P}$, where e_G is the group-identity of G .

Thus, one can obtain the following generalized result of (6.9).

Theorem 6.1. Let $u_{g_l} \chi_{\partial_{j_l}}$ be free random variables of the Adelic-dynamical W^* -probability space $\mathfrak{S}^G = (\mathfrak{S}^G, \tau^G)$ of (6.8), for a fixed group G , for $g_l \in G$, $j_l \in \mathbb{Z}$, for $l = 1, \dots, N$, for $N \in \mathbb{N}$. Assume that

$$u_{g_l} \chi_{\partial_{j_l}} \in \mathfrak{S}_{p_l, \pi_{p_l}}^G \text{ in } \mathfrak{S}^G, \text{ for } l = 1, \dots, N. \quad (6.10)$$

Let $(p_1, \dots, p_N)$ be the N -tuple in primes obtained from (6.10). If this N -tuple is an alternating sequence, equivalently, if

$$(p_1, \dots, p_N) \in \text{alt}(\mathcal{P}^N),$$

and if

$$u_{g_l}^{\partial_{j_l}} = u_{g_l^o} \text{ in } M_G, \text{ for some } g_l^o \text{ in } G,$$

for all $l = 1, \dots, N$, then we have that

$$\tau^G \left(\left(\prod_{l=1}^N u_{g_l} \chi_{\partial_{j_l}} \right)^n \right) = \prod_{l=1}^N \delta_{e_G, g_l} (g_l^o)^{n-1} p_l^{-(j_l+1)}, \quad (6.11)$$

for all $n \in \mathbb{N}$.

Proof. By (6.10), we have

$$T_l = u_{g_l} \chi_{\partial_{j_l}} \in \mathfrak{S}_{p_l, \pi_{p_l}}^G, \text{ for } l = 1, \dots, N,$$

in the Adelic-dynamical W^* -probability space $\mathfrak{S}^G$ of a group G , for $N \in \mathbb{N}$. Assume that the prime-sequence $(p_1, \dots, p_N)$ is an alternating N -tuple in $\mathcal{P}$, and suppose

$$u_{g_l}^{\partial_{j_l}} = u_{g_l^o} \text{ in } M_G, \text{ for some } g_l^o \in G, \quad (6.12)$$

for all $l = 1, \dots, N$.

By the alternating property of $(p_1, \dots, p_N)$, the operator

$$T = \prod_{l=1}^N T_l \quad (6.13)$$

forms a free reduced word in $\mathfrak{S}^G$, satisfying that

$$T^n \stackrel{\text{equi}}{=} \left(\bigotimes_{l=1}^N T_l \right)^n = \bigotimes_{l=1}^N T_l^n \stackrel{\text{equi}}{=} \prod_{l=1}^N T_l^n, \quad (6.14)$$

for all $n \in \mathbb{N}$.

Thus, by (6.14), one can get that

$$\tau^G(T^n) = \prod_{l=1}^N \tau_{p_l, \pi_{p_l}}^G(T_l^n)$$

by (6.14)

$$\begin{aligned} &= \prod_{l=1}^N \tau_{p_l, \pi_{p_l}}^G \left(u_{g_l} \left(u_{g_l}^{\partial_{j_l}} \right)^{n-1} \chi_{\partial_{j_l}} \right) \\ &= \prod_{l=1}^N \frac{1}{p_l^{j_l+1}} \text{tr}_G \left(u_{g_l} \left(u_{g_l^o} \right)^{n-1} \right) \end{aligned}$$

by (6.12)

$$= \prod_{l=1}^N \frac{1}{p_l^{j_l+1}} \delta_{e_G, g_l} (g_l^o)^{n-1}.$$

Remark here that the existence of $g_l^o \in G$ for $g_l \in G$, in the assumption (6.12), is guaranteed by the very definition of the semigroup-action π_{p_l} of the semigroup $\sigma(\mathbb{Q}_p)$, and the structure theorem (5.3), which can be re-expressed by (5.4). i.e.,

$$u_{g_l}^{\partial_{j_l}} = \pi_{p_l}(\partial_{j_l})(u_{g_l}) \in M_G \otimes_{\pi} \mathfrak{S}_{p_l, \pi_{p_l}}^G \text{ in } \mathfrak{S}^G,$$

as a $*$ -homomorphic image, and hence, there does exist

$$g_l^o \in \left(M_G \otimes_{\pi} \mathbb{C} \cdot \chi_{\partial_{j_l}} \right)^{*-\text{iso}} = M_G,$$

such that $u_{g_l^o} = u_{g_l}^{\partial j_l}$. Therefore, the condition (6.12) in the above theorem is meaningful.

Let T be a free reduced word in $\mathfrak{S}^G$ in the sense of (6.13) above. Notice the difference between T^n and $T^{(n)}$, as we discussed in Section 2.4, i.e.,

$$T^n = \left(\bigotimes_{l=1}^N T_l \right)^n = \bigotimes_{l=1}^N T_l^n = \prod_{l=1}^N T_l \text{ in } \mathfrak{S}^G,$$

meanwhile,

$$T^{(n)} = \underbrace{TT \dots T}_{n\text{-times}} \text{ in } \mathfrak{S}^G,$$

as a “non-reduced” word, for all $n \in \mathbb{N}$. Thus, one can verify that

$$\tau^G(T^n) \neq \tau^G(T^{(n)}), \text{ in general,}$$

whenever $n \neq 1$ in $\mathbb{N}$.

By the free-moment formula (6.11), one obtains the following corollary.

Corollary 6.2. Let $u_g \chi_{\partial j} \in \mathfrak{S}_{p, \pi_p}^G$ in the Adelic-dynamical

W^* -probability space $\mathfrak{S}^G = (\mathfrak{S}^G, \tau^G)$ of a group G ,

for $g \in G$, and $j \in \mathbb{Z}$. If $u_g^{\partial j} = u_{g_o}$ in M_G , for some $g_o \in G$, then

$$\tau^G \left(\left(u_g \chi_{\partial j} \right)^n \right) = \delta_{e_G, g g_o^{n-1} p^{-(j+1)}}, \quad (6.15)$$

for all $n \in \mathbb{N}$.

Note that, in the above corollary case, one has

$$\left(u_g \chi_{\partial j} \right)^n = \left(u_g \chi_{\partial j} \right)^{(n)}, \text{ for all } n \in \mathbb{N},$$

and hence, we automatically obtain that

$$\begin{aligned} \tau^G \left(\left(u_g \chi_{\partial j} \right)^{(n)} \right) &= \tau^G \left(\left(u_g \chi_{\partial j} \right)^n \right) \\ &= \delta_{e_G, g g_o^{n-1} p^{-(j+1)}}, \end{aligned} \quad (6.16)$$

by (6.15), for all $n \in \mathbb{N}$.

Indeed, the formula (6.16) holds, because $u_g \chi_{\partial j}$ is chosen in a single free block $\mathfrak{S}_{p, \pi_p}^G$ of $\mathfrak{S}^G$.

However, one may take

$$T_l = u_g \chi_{\partial j} \in \mathfrak{S}_{p, \pi_p}^G \text{ in } \mathfrak{S}^G,$$

for $l = 1, \dots, N$, for $N \in \mathbb{N} \setminus \{1\}$, where

$$(p_1, \dots, p_N) \in \text{alt}(\mathcal{P}^N).$$

Then, even though $T_1, \dots, T_N$ are identical to $u_g \chi_{\partial j}$, the free reduced word

$$T = T_1 T_2 \dots T_N \in \mathfrak{S}^G$$

satisfy that

$$\begin{aligned} \tau^G(T) &= \prod_{l=1}^N \tau_{p_l, \pi_{p_l}}^G(T_l) = \prod_{l=1}^N \frac{\delta_{e_G, g}}{p_l^{j+1}} \\ &= \delta_{e_G, g} \left(\prod_{l=1}^N p_l^{-(j+1)} \right), \end{aligned} \quad (6.17)$$

by (6.11).

Corollary 6.3. Let $T_l = u_g \chi_{\partial j} \in \mathfrak{S}_{p_l, \pi_{p_l}}^G$ in $\mathfrak{S}^G$, for all

$l = 1, \dots, N$, for $N \in \mathbb{N} \setminus \{1\}$, for a fixed $g \in G$, and

$j \in \mathbb{Z}$, and let $T = \prod_{l=1}^N T_l$ in $\mathfrak{S}^G$. If the corresponding prime-sequence satisfies

$$(p_1, \dots, p_N) \in \text{alt}(\mathcal{P}^N),$$

then

$$\tau^G(T) = \delta_{e_G, g} \left(\prod_{l=1}^N p_l^{-(j+1)} \right). \quad (6.18)$$

Proof. The proof of (6.18) is done by (6.11) and (6.17).

Now, let's concentrate on free blocks

$$\mathfrak{S}_{p, \pi_p}^G = M_G \otimes_{\pi_p} \mathfrak{S}_p^G, \text{ for } p \in \mathcal{P}$$

of our Adelic-dynamical W^* -probability space $\mathfrak{S}^G$ of a group G .

Definition 6.2. Let $\mathfrak{S}_{p, \pi_p}^G$ be a free block of the

Adelic-dynamical W^* -probability space $\mathfrak{S}^G$ of a group G , for a fixed $p \in \mathcal{P}$. We say that this fixed free block $\mathfrak{S}_{p, \pi_p}^G$ is π_p -independent in $S\mathfrak{S}^G$, if

$$u_g^{\partial j} = \pi_p(\partial_j)(u_g) = u_g, \quad (6.19)$$

for all $g \in G$, for all $j \in \mathbb{Z}$.

By the very definition (6.19), one can realize that if a free block $\mathfrak{S}_{p, \pi_p}^G$ of the Adelic-dynamical

W^* -probability space $\mathfrak{S}^G$ of a group G is π_p -independent, then the semigroup-action π_p not only induces a $*$ -isomorphism from M_G to $M_G \otimes_{\mathbb{C}}$

$(\mathbb{C} \cdot \chi_{\partial j})$, for all $j \in \mathbb{Z}$, but also forms the identity

morphism from M_G to $M_G \otimes_{\mathbb{C}} (\mathbb{C} \cdot \chi_{\partial j})$, for all $j \in \mathbb{Z}$.

Therefore, one can characterize the π_p -independence

(6.19) of a free block $\mathfrak{S}_{p, \pi_p}^G$ of $\mathfrak{S}^G$ as follows.

Theorem 6.4. Let $\mathfrak{S}_{p, \pi_p}^G$ be a free block of the

Adelic-dynamical W^* -probability space $\mathfrak{S}^G$ of a group

G , for a fixed $p \in \mathcal{P}$. Then $\mathfrak{S}_{p,\pi_p}^G$ is π_p -independent in the sense of (6.19), if and only if

$$\mathfrak{S}_{p,\pi_p}^G \stackrel{\text{def}}{=} M_G \otimes_{\pi_p} \mathfrak{S}_p^{*-iso} = M_G \otimes_{\mathbb{C}} \mathfrak{S}_p^G, \quad (6.20)$$

in $\mathfrak{S}^G$.

Proof. ($\Rightarrow$) By the discussions in the very above paragraphs, if a free block $\mathfrak{S}_{p,\pi_p}^G$ is π_p -independent, then

$$\begin{aligned} \mathfrak{S}_{p,\pi_p}^G &\stackrel{\text{def}}{=} M_G \otimes_{\pi_p} \mathfrak{S}_p^G \\ &\stackrel{*iso}{=} M_G \otimes_{\pi_p} \left(\bigoplus_{j \in \mathbb{Z}} (\mathbb{C} \cdot \chi_{\partial_j}) \right) \\ &\stackrel{*iso}{=} \bigoplus_{j \in \mathbb{Z}} \left(M_G \otimes_{\pi_p} (\mathbb{C} \cdot \chi_{\partial_j}) \right) \\ &\stackrel{*iso}{=} \bigoplus_{j \in \mathbb{Z}} \left(M_G \otimes_{\mathbb{C}} (\mathbb{C} \cdot \chi_{\partial_j}) \right) \\ &\stackrel{*iso}{=} M_G \otimes_{\mathbb{C}} \left(\bigoplus_{j \in \mathbb{Z}} (\mathbb{C} \cdot \chi_{\partial_j}) \right) \\ &\stackrel{*iso}{=} M_G \otimes_{\mathbb{C}} \mathfrak{S}_p^G. \end{aligned}$$

($\Leftarrow$) Conversely, if a free block $\mathfrak{S}_{p,\pi_p}^G$ of $\mathfrak{S}^G$ is $*$ -isomorphic to $M_G \otimes_{\mathbb{C}} \mathfrak{S}_p^G$, then the semigroup-action π_p of the sub-semigroup $\langle \{\partial_j\}_{j \in \mathbb{Z}} \rangle$, generated by $\{\partial_j\}_{j \in \mathbb{Z}}$ of the semigroup $\sigma(\mathbb{Q}_p)$ satisfies (6.19). i.e., for all $j \in \mathbb{Z}$, the action π_p forms the identity $*$ -isomorphism from

$$M_G \text{ onto } M_G \otimes_{\mathbb{C}} (\mathbb{C} \cdot \chi_{\partial_j}).$$

Therefore, the above $*$ -isomorphic relation holds, then $\mathfrak{S}_{p,\pi_p}^G$ is π_p -independent in the Adelic-dynamical W^* -probability space $\mathfrak{S}^G$.

Then above characterization (6.20) shows that a free block $\mathfrak{S}_{p,\pi_p}^G$ of $\mathfrak{S}^G$ is π_p -independent, if and only if

$$\mathfrak{S}_{p,\pi_p}^G \stackrel{*iso}{=} M_G \otimes_{\mathbb{C}} \mathfrak{S}_p^G,$$

equivalently, the conditionality on tensor product can be ignored.

Theorem 6.5. Let $\mathfrak{S}^G = (\mathfrak{S}^G, \tau^G)$ be the Adelic-dynamical W^* -probability space of a group G , and let $\mathfrak{S}_{p_l,\pi_{p_l}}^G$ be some free blocks of $\mathfrak{S}^G$, for $l=1, \dots, N$, for $N \in \mathbb{N}$. Let

$$T_l = u_{g_l} \chi_{\partial_{j_l}} \in \mathfrak{S}_{p_l,\pi_{p_l}}^G, \text{ for } l=1, \dots, N,$$

where $g_l \in G$, and $j_l \in \mathbb{Z}$. If $\mathfrak{S}_{p_l,\pi_{p_l}}^G$ are π_{p_l} -independent in the sense of (6.19), for all $l=1, \dots, N$, in $\mathfrak{S}^G$, and if

$$(p_1, \dots, p_N) \in \text{alt}(\mathcal{P}^N),$$

then we obtain

$$\tau^G \left(\left(\prod_{l=1}^N T_l \right)^n \right) = \prod_{l=1}^N \delta_{e_G \cdot g_l^n p_l^{-(j_l+1)}}, \quad (6.21)$$

for all $n \in \mathbb{N}$.

Proof. By the assumption that the prime-sequence $(p_1, \dots, p_N)$ is an alternating sequence in $\mathcal{P}$, the operator T is a free reduced word in $\mathfrak{S}^G$, satisfying that

$$T^n = \prod_{l=1}^N T_l^n \text{ in } \mathfrak{S}^G, \text{ for all } n \in \mathbb{N}.$$

So, for any $n \in \mathbb{N}$, one has that

$$T^n = \prod_{l=1}^N \left(u_{g_l} \chi_{\partial_{j_l}} \right)^n = \prod_{l=1}^N u_{g_l} \left(u_{g_l}^{\partial_j} \right)^{n-1} \chi_{\partial_{j_l}}.$$

However, by the π_{p_l} -independence (6.19) of $\mathfrak{S}_{p_l,\pi_{p_l}}^G$, for all $l=1, \dots, N$, one can refine that

$$T^n = \prod_{l=1}^N u_{g_l}^n \chi_{\partial_{j_l}} = \prod_{l=1}^N u_{g_l^n} \chi_{\partial_{j_l}},$$

for all $n \in \mathbb{N}$, by (6.20).

Therefore, we have that

$$\begin{aligned} \tau^G(T^n) &= \tau^G \left(\prod_{l=1}^N u_{g_l^n} \chi_{\partial_{j_l}} \right) = \prod_{l=1}^N \tau_{p_l,\pi_{p_l}}^G \left(u_{g_l^n} \chi_{\partial_{j_l}} \right) \\ &= \prod_{l=1}^N \frac{\delta_{e_G \cdot g_l^n p_l^{-(j_l+1)}}}{p_l^{j_l+1}}, \end{aligned}$$

for all $n \in \mathbb{N}$.

So, by (6.15), (6.16) and (6.21), one can get the following corollary.

Corollary 6.6. Let $u_g \chi_{\partial_j} \in \mathfrak{S}_{p,\pi_p}^G$ in $\mathfrak{S}^G$, for $p \in \mathcal{P}$, where $g \in G$, and $j \in \mathbb{Z}$. Assume that the free block $\mathfrak{S}_{p,\pi_p}^G$ is π_p -independent in $\mathfrak{S}^G$. Then

$$\begin{aligned} \tau^G \left(\left(u_g \chi_{\partial_j} \right)^n \right) &= \tau^G \left(\left(u_g \chi_{\partial_j} \right)^{(n)} \right) \\ &= \delta_{e_G \cdot g^n p^{-(j+1)}}, \end{aligned} \quad (6.22)$$

for all $n \in \mathbb{N}$.

Proof. Let $T = u_g \chi_{\partial_j} \in \mathfrak{S}_{p,\pi_p}^G$, where $\mathfrak{S}_{p,\pi_p}^G$ is π_p -independent in $\mathfrak{S}^G$. Then

$$T^n = T^{(n)} = u_g \left(u_g^{\partial j} \right)^{n-1} \chi_{\partial j},$$

for all $n \in \mathbb{N}$, because T is contained in a single free block $\mathfrak{S}_{p, \pi_p}^G$ in $\mathfrak{S}^G$. Thus,

$$\tau^G(T^n) = \tau^G(T^{(n)}), \text{ for all } n \in \mathbb{N},$$

as we discussed in (6.16).

By the π_p -independence (6.19) of $\mathfrak{S}_{p, \pi_p}^G$,

$$u_g^{\partial j} = u_g, \text{ for all } g \in G,$$

by (6.20).

So, one obtains that

$$\tau^G(T^n) = \tau^G \left(u_g^n \chi_{\partial j} \right) = \tau_{p, \pi_p}^G \left(u_g^n \chi_{\partial j} \right) = \frac{\delta_{e_G \cdot g^n}}{p^{j+1}},$$

for all $n \in \mathbb{N}$.

Therefore, the free-moment formula (6.22) holds.

We finish this section with a special case where a given group G is the free group F_n with n -generators. Recall that the free group F_n is a non-abelian group generated by its n -many generators $\{x_i\}_{i=1}^n$, with no relators (or relations) on it, i.e.,

$$F_n = \langle \{x_1, x_2, \dots, x_n\} \rangle,$$

for all $n \in \mathbb{N}$. In particular, we are interested in the cases where $n > 1$. We cannot help emphasizing the importance of free groups not only in algebra, and combinatorics, but also in operator algebra theory (e.g., see [11] and [14]).

In the rest of this section, let F_n be the free group with n -generators $\{x_i\}_{i=1}^n$, for $n \in \mathbb{N} \setminus \{1\}$, and

$$\mathfrak{S}_{p, \pi_p}^{(n)} = \left(\mathfrak{S}_{p, \pi_p}^{(n)}, \tau_{p, \pi_p}^{(n)} \right),$$

the p -adic dynamical W^* -probability spaces, for all $p \in \mathcal{P}$, and let

$$\mathfrak{S}^{(n)} = \left(\mathfrak{S}^{(n)}, \tau^{(n)} \right)$$

be the corresponding Adelic-dynamical W^* -probability space.

We concentrate on the generating operators

$$T_{i,j}^p \stackrel{\text{def}}{=} u_{x_i} \chi_{\partial j} \in \mathfrak{S}_{p, \pi_p}^{(n)} \text{ in } \mathfrak{S}^{(n)}, \quad (6.23)$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, for all $i = 1, \dots, n$.

Lemma 6.7. Let $T_{i,j}^p$ be in the sense of (6.23) in the Adelic-dynamical W^* -probability space $\mathfrak{S}^{(n)}$ of the given free group F_n , for $p \in \mathcal{P}$, $j \in \mathbb{Z}$, for $i = 1, \dots, n$. If $\mathfrak{S}_{p, \pi_p}^{(n)}$ is π_p -independent in $\mathfrak{S}^{(n)}$, then

$$\tau^{(n)} \left(\left(T_{i,j}^p \right)^k \right) = 0, \text{ for all } k \in \mathbb{N}. \quad (6.24)$$

Proof. By (6.21), or (6.22), one can get that

$$\tau^{(n)} \left(\left(T_{i,j}^p \right)^k \right) = \tau_{p, \pi_p}^{(n)} \left(\left(T_{i,j}^p \right)^k \right) = \frac{\delta_{e_n \cdot x_i^k}}{p^{j+1}},$$

under the π_p -independence (6.19) of $\mathfrak{S}_{p, \pi_p}^{(n)}$ in $\mathfrak{S}^{(n)}$, for all $k \in \mathbb{N}$, where e_n means the group-identity of F_n .

By the very construction of the free group F_n , if x_i is a generator of F_n , then $x_i^k \neq e_n$, and hence, $\delta_{e_n \cdot x_i^k} = 0$, for all $k \in \mathbb{N}$. Therefore, the vanishing free-moment formula (6.24) holds.

More generally, one can get the following result.

Lemma 6.8. Let $T_{i_l, j_l}^{p_l}$ be in the sense of (6.23) in $\mathfrak{S}^{(n)}$, for $p_l \in \mathcal{P}$, $j_l \in \mathbb{Z}$, for $i_l = 1, \dots, n$, for all $l = 1, \dots, N$, for $N \in \mathbb{N}$. Suppose the free blocks $\mathfrak{S}_{p_l, \pi_{p_l}}^{(n)}$ are π_{p_l} -independent in $\mathfrak{S}^{(n)}$, for all $l = 1, \dots, N$. Let

$$T = \prod_{l=1}^N T_{i_l, j_l}^{p_l} \in \mathfrak{S}^{(n)}.$$

If

$$(p_1, \dots, p_N) \in \text{alt}(\mathcal{P}^n),$$

then we have

$$\tau^{(n)}(T^k) = 0, \text{ for all } k \in \mathbb{N}. \quad (6.25)$$

Proof. By assumption, the operator T is a free reduced word in $\mathfrak{S}^{(n)}$. So,

$$\begin{aligned} \tau^{(n)}(T^k) &= \tau^{(n)} \left(\prod_{l=1}^N \left(T_{i_l, j_l}^{p_l} \right)^k \right) \\ &= \prod_{l=1}^N \tau_{p_l, \pi_{p_l}}^{(n)} \left(\left(T_{i_l, j_l}^{p_l} \right)^k \right) = 0, \end{aligned}$$

by (6.24), under the π_l -independence, for all $l = 1, \dots, N$.

Observe that:

Lemma 6.9. Let $T = T_{i,j}^p = u_{x_i} \chi_j \in \mathfrak{S}_{p, \pi_p}^{(n)}$ be in the sense of (6.23) in the Adelic-dynamical W^* -probability space $\mathfrak{S}^{(n)}$ of the given free group F_n , for $p \in \mathcal{P}$. Assume that the free block $\mathfrak{S}_{p, \pi_p}^{(n)}$ is π_p -independent in $\mathfrak{S}^{(n)}$. Let W be an operator of $\mathfrak{S}^{(n)}$, which is either

$$W = (TT^*)^s, \text{ or } W = (T^*T)^s,$$

for some $s \in \mathbb{N}$. Then

$$\tau^{(n)}(W^k) = p^{-(j+1)}, \text{ for all } k \in \mathbb{N}, \quad (6.26)$$

for all $i = 1, \dots, n$.

Proof. Let $T = T_{i,j}^p = u_{x_i} \chi_{\partial_j}$ in a fixed free block $\mathfrak{S}_{p,\pi_p}^{(n)}$ of the Adelic-dynamical W^* -probability space $\mathfrak{S}^{(n)}$, for $p \in \mathcal{P}$. Then the adjoint T^* is also contained in $\mathfrak{S}_{p,\pi_p}^{(n)}$, and hence,

$$TT^*, T^*T \in \mathfrak{S}_{p,\pi}^{(n)} \text{ in } \mathfrak{S}^{(n)},$$

for all $i = 1, \dots, n$. Thus, for any $s \in \mathbb{N}$,

$$(TT^*)^s, (T^*T)^s \in \mathfrak{S}_{p,\pi}^{(n)} \text{ in } \mathfrak{S}^{(n)}, \text{ too,}$$

for all $i = 1, \dots, n$.

Observe that

$$\begin{aligned} TT^* &= \left(u_{x_i} \chi_{\partial_j} \right) \left(u_{x_i}^{-1} \chi_{\partial_j} \right) \\ &= u_{x_i} \left(u_{x_i}^{-1} \chi_{\partial_j} \right) = u_{x_i} u_{x_i}^{-1} \chi_{\partial_j} \\ &= u_{x_i x_i^{-1}} \chi_{\partial_j} = u_{e_n} \chi_{\partial_j}, \end{aligned}$$

in $\mathfrak{S}_{p,\pi_p}^{(n)}$ inside $\mathfrak{S}^{(n)}$, for all $i = 1, \dots, n$, by the π_p -independence of $\mathfrak{S}_{p,\pi_p}^{(n)}$ in $\mathfrak{S}^{(n)}$. Similarly, we have

$$T^*T = u_{e_n} \chi_{\partial_j} \in \mathfrak{S}_{p,\pi_p}^{(n)} \subset \mathfrak{S}^{(n)},$$

i.e., one obtains that

$$TT^* = u_{e_G} \chi_{\partial_j} = T^*T \text{ in } \mathfrak{S}_{p,\pi_p}^{(n)} \text{ inside } \mathfrak{S}^{(n)}, \quad (6.27)$$

By (6.27), let's concentrate on TT^* . For any $s \in \mathbb{N}$,

$$(TT^*)^s = \left(u_{e_G} \chi_{\partial_j} \right)^s = u_{e_G^s} \chi_{\partial_j} = u_{e_G} \chi_{\partial_j} = TT^*, \quad (6.28)$$

and similarly,

$$(T^*T)^s = T^*T, \text{ in } \mathfrak{S}^{(n)},$$

by (6.27), under the π_p -independence of $\mathfrak{S}_{p,\pi_p}^{(n)}$ in $\mathfrak{S}^{(n)}$.

So, by (6.28), we have that if $W \in \mathfrak{S}^{(n)}$ is either $W = (TT^*)^s$, or $W = (T^*T)^s$, for some $s \in \mathbb{N}$, then

$$W = TT^* = T^*T \text{ in } \mathfrak{S}_{p,\pi_p}^{(n)},$$

by (6.27).

Thus, if either $W = (TT^*)^s$, or $W = (T^*T)^s$, then

$$\begin{aligned} \tau^{(n)}(W^k) &= \tau^{(n)} \left((TT^*)^k \right) = \tau^{(n)} \left((T^*T)^k \right) \\ &= \tau_{p,\pi_p}^{(n)} \left(\left(u_{e_G} \chi_{\partial_j} \right)^k \right) = \tau_{p,\pi_p}^{(n)} \left(u_{e_G^k} \chi_{\partial_j} \right) \\ &= \tau_{p,\pi_p}^{(n)} \left(u_{e_G} \chi_{\partial_j} \right) = \frac{1}{p^{j+1}}, \end{aligned}$$

for all $k \in \mathbb{N}$, for all $i = 1, \dots, n$. Therefore, the free-distributional data (6.26) is obtained.

Motivated by (6.26), we obtain the following generalized result.

Proposition 6.10. Let $T = T_{i,j}^p = u_{x_i} \chi_j \in \mathfrak{S}_{p,\pi_p}^{(n)}$ be in the sense of (6.23) in the Adelic-dynamical W^* -probability space $\mathfrak{S}^{(n)}$ of the given free group F_n , for $p \in \mathcal{P}$, and $n \in \mathbb{N}$. Assume that the free block $\mathfrak{S}_{p,\pi_p}^{(n)}$ is π_p -independent in $\mathfrak{S}^{(n)}$. Let

$$\mathfrak{r} = (r_1, \dots, r_N) \in \{1, *\}^N, \text{ for } N \in \mathbb{N}, \setminus \{1\},$$

and let

$$\#_1(\mathfrak{r}) = \text{the number of } 1\text{'s in } \mathfrak{r},$$

and

$$\#_*(\mathfrak{r}) = \text{the number of } *\text{'s in } \mathfrak{r}.$$

If $\#_1(\mathfrak{r}) = \#_*(\mathfrak{r})$ in $\mathbb{N}$, then

$$\tau^{(n)} \left(\left(\prod_{l=1}^N (T^{\eta_l}) \right)^k \right) = p^{-(j+1)}, \text{ for all } k \in \mathbb{N}, \quad (6.29)$$

for all $i = 1, \dots, n$.

Proof. As we discussed in the proof of (6.26), if T is given as above, then

$$W = \prod_{l=1}^N T^{\eta_l} \in \mathfrak{S}_{p,\pi_p}^{(n)} \text{ in } \mathfrak{S}^{(n)},$$

and hence, W^k are contained in the same free block $\mathfrak{S}_{p,\pi_p}^{(n)}$ in $\mathfrak{S}^{(n)}$, for all $k \in \mathbb{N}$. Under the

π_p -independence of $\mathfrak{S}_{p,\pi_p}^{(n)}$, if

$$\#_1(\mathfrak{r}) = \#_*(\mathfrak{r}) \text{ in } \mathbb{N},$$

then

$$W = u_{e_G} \chi_{\partial_j} \text{ in } \mathfrak{S}^{(n)},$$

satisfying

$$W^k = W, \text{ for all } k \in \mathbb{N}.$$

Therefore, one has that

$$\tau^{(n)}(W^k) = \tau^{(n)}(W) = \tau_{p,\pi_p}^{(n)}(W) = \frac{1}{p^{j+1}},$$

for all $k \in \mathbb{N}$.

By the above three lemmas and proposition, we obtain the following free-distributional data on the Adelic-dynamical W^* -probability space $\mathfrak{G}^{(n)}$ of the given free group F_n .

Theorem 6.11. *Let $\mathfrak{G}^{(n)}$ be the Adelic-dynamical W^* -probability space of a given free group F_n , for $n \in \mathbb{N} \setminus \{1\}$, and let $T_{i,j}^p$ be in the sense of (6.23) in $\mathfrak{G}^{(n)}$, contained in the free block $\mathfrak{G}_{p,\pi_p}^{(n)}$, for $i=1, \dots, n$. Assume that this free block $\mathfrak{G}_{p,\pi_p}^{(n)}$ is π_p -independent in $\mathfrak{G}^{(n)}$. Let*

$$\mathfrak{r} = (r_1, r_2, \dots, r_N) \in \{1, *\}^N, \text{ for } N \in \mathbb{N}.$$

Also, let

$$\#_1(\mathfrak{r}) = \text{the number of } 1\text{'s in } \mathfrak{r},$$

and

$$\#_*(\mathfrak{r}) = \text{the number of } *\text{'s in } \mathfrak{r}.$$

Then we obtain that

$$\tau^{(n)} \left(\left(\prod_{l=1}^N \left(T_{i,j}^p \right)^\eta \right)^k \right) = \begin{cases} p^{-(j+1)} & \text{if } \#_1(\mathfrak{r}) = \#_*(\mathfrak{r}) \\ 0 & \text{otherwise,} \end{cases} \quad (6.30)$$

for all $k \in \mathbb{N}$, for all $i=1, \dots, n$.

Proof. Let $T_{i,j}^p = u_{x_i} \chi_{\partial_j}$ be in the sense of (6.23) in the π_p -independent free block $\mathfrak{G}_{p,\pi_p}^{(n)}$ of our Adelic-dynamical W^* -probability space $\mathfrak{G}^{(n)}$ of the free group F_n , for $i=1, \dots, n$. Let

$$\mathfrak{r} = (r_1, \dots, r_N) \in \{1, *\}^N, \text{ for } N \in \mathbb{N}.$$

and let

$$T_{\mathfrak{r}} = \prod_{l=1}^N \left(T_{i,j}^p \right)^\eta \in \mathfrak{G}_{p,\pi}^{(n)} \text{ in } \mathfrak{G}^{(n)}.$$

As we discussed in the proof of (6.26), indeed, the operator $T_{\mathfrak{r}}$ is contained in the free block $\mathfrak{G}_{p,\pi_p}^{(n)}$ of $\mathfrak{G}^{(n)}$.

Assume first $\#_1(\mathfrak{r}) \neq \#_*(\mathfrak{r})$, say $\#_1(\mathfrak{r}) > \#_*(\mathfrak{r})$. Then

$$T_{\mathfrak{r}} = u_{x_i} \#_1(\mathfrak{r}) - \#_*(\mathfrak{r}) \chi_{\partial_j},$$

and hence,

$$\tau^{(n)} \left(T_{\mathfrak{r}}^k \right) = 0, \text{ for all } k \in \mathbb{N},$$

by (6.24) and (6.25).

Similarly, one can verify that if $\#_1(\mathfrak{r}) = \#_*(\mathfrak{r})$, then

$$\tau^{(n)} \left(T_{\mathfrak{r}}^k \right) = 0, \text{ for all } k \in \mathbb{N}.$$

Suppose now that $\#_1(\mathfrak{r}) < \#_*(\mathfrak{r})$. Then, by (6.29), one obtains that

$$\tau^{(n)} \left(T_{\mathfrak{r}}^k \right) = \frac{1}{p^{j+1}}, \text{ for all } k \in \mathbb{N}.$$

The above free-moment formula (6.30) characterizes the free distributions of individual generating elements $u_{x_i} \chi_{\partial_j}$ of the Adelic-dynamical W^* -algebra $\mathfrak{G}^{(n)}$ of the free group F_n , for all $i=1, \dots, n$, for all $j \in \mathbb{Z}$, whenever a free block $\mathfrak{G}_{p,\pi_p}^{(n)}$ is π_p -independent in $S\mathfrak{G}^{(n)}$, for $p \in \mathcal{P}$.

By the above free-distributional data (6.30), we obtain the following result.

Theorem 6.12. *Let $T_l = T_{i_l, j_l}^{p_l} \in \mathfrak{G}_{p_l, \pi_{p_l}}^{(n)}$ be in the sense of (6.23) in the Adelic-dynamical W^* -probability space $\mathfrak{G}^{(n)}$ of a free group F_n , for $n \in \mathbb{N} \setminus \{1\}$, for $l=1, \dots, N$, for $N \in \mathbb{N}$. Suppose the corresponding prime-sequence*

$$(p_1, \dots, p_N) \in \text{alt}(\mathcal{P}^N)$$

is an alternating sequence in $\mathcal{P}$, and assume that the free blocks $\mathfrak{G}_{p_l, \pi_{p_l}}^{(n)}$ are π_{p_l} -independent in $\mathfrak{G}^{(n)}$, for all $l=1, \dots, N$. Let

$$T = \prod_{l=1}^N \left(\prod_{s=1}^{\eta_l} T_l^{\eta_l, s} \right) \in \mathfrak{G}^{(n)},$$

where

$$\mathfrak{r}_l = (r_{l,1}, r_{l,2}, \dots, r_{l,\eta_l}) \in \{1, *\}^{\eta_l},$$

for all $l=1, \dots, N$.

Then we obtain that

$$\tau^{(n)} \left(T^k \right) = \begin{cases} \prod_{l=1}^N p_l^{-(j_l+1)} & \text{if } \#_1(\mathfrak{r}) = \#_*(\mathfrak{r}), \forall l=1, \dots, N \\ 0 & \text{otherwise} \end{cases}, \quad (6.31)$$

for all $k \in \mathbb{N}$.

Proof. Under hypothesis, the factors $\prod_{s=1}^{\eta_l} T_l^{\eta_l, s}$ of T are contained in the free blocks $\mathfrak{G}_{p_l, \pi_{p_l}}^{(n)}$ of $\mathfrak{G}^{(n)}$, for all $l=1, \dots, N$, i.e.,

$$\prod_{s=1}^{\eta_l} T_l^{\eta_l, s} \in \mathfrak{G}_{p_l, \pi_{p_l}}^{(n)}, \text{ for all } l=1, \dots, N.$$

By the assumption that $(p_1, \dots, p_N)$ is an alternating sequence in $\mathcal{P}$, the operator T is a free reduced word in $\mathfrak{G}^{(n)}$, satisfying that

$$T^k = \prod_{l=1}^N \left(\prod_{s=1}^{t_l} T_l^{\eta_{l,s}} \right)^k, \text{ for all } k \in \mathbb{N}.$$

Thus, one has that

$$\tau_{p_l, \pi_{p_l}}^{(n)}(T^k) = \prod_{l=1}^N \tau_{p_l, \pi_{p_l}}^{(n)} \left(\left(\prod_{s=1}^{t_l} T_l^{\eta_{l,s}} \right)^k \right),$$

with

$$\tau_{p_l, \pi_{p_l}}^{(n)} \left(\left(\prod_{s=1}^{t_l} T_l^{\eta_{l,s}} \right)^k \right) = \begin{cases} p_l^{-(j_l+1)} & \text{if } \#_1(\mathbf{r}_l) = \#_*(\mathbf{r}_l), \\ 0 & \text{otherwise,} \end{cases}$$

for all $k \in \mathbb{N}$, by (6.30), under the π_{p_l} -independence

(6.19) of $\mathfrak{S}_{p_l, \pi_{p_l}}^{(n)}$ in $\mathfrak{S}^{(n)}$, for all $l = 1, \dots, N$,

Therefore, we have that

$$\tau^{(n)}(T^k) = \begin{cases} \prod_{l=1}^N p_l^{-(j_l+1)} & \text{if } \#_1(\mathbf{r}_l) = \#_*(\mathbf{r}_l), \forall l = 1, \dots, N \\ 0 & \text{otherwise,} \end{cases}$$

for all $k \in \mathbb{N}$.

Therefore, the free-moment formula (6.31) holds true.

The above theorem characterizes the free distributions of generating elements $T_{i,j}^p$'s of $\mathfrak{S}^{(n)}$ in terms of the joint free-moment formula (6.31).

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