First Zagreb Index, F-index and F-coindex of the Line Subdivision Graphs

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Abstract In this paper we investigate first Zagreb index, F-index and F-coindex of the line graph of some chemical graphs using the subdivision concept.

Keywords: chemical graphs, Zagreb index, F-index, F-coindex


1. Introduction

The line graph of a simple graph $G$, denoted by $L(G)$, is the graph whose vertices correspond to the edges of $G$ such that two vertices of $L(G)$ being adjacent if and only if the corresponding edges of $G$ share a common vertex [see [2,12]]. The subdivision graph $S(G)$ of a graph $G$ is obtained from $G$ by deleting every edge $uv$ of $G$ and replacing it by a vertex $w$ of degree 2 that is joined to $u$ and $v$ [see p.151 of [3]]. If $S(G)$ is the subdivision graph of a graph $G$, then the line subdivision of $G$ is $L(S(G))$.

Following [16], we can construct the Line Subdivision of a graph $G$, as follows:

(i) Replace each vertex $u \in V(G)$ by $K(u)$, the complete graph on $d_G(u)$ vertices;
(ii) There is an edge joining a vertex of $K(u_1)$ and a vertex of $K(u_2)$ in $L(S(G))$ if and only if there is an edge joining $u_1$ and $u_2$ in $G$;
(iii) For each vertex $v$ of $K(u)$, $d_{L(S(G))}(v) = d_G(u)$.

![Figure 1. Hydrocarbon graph $G$ and its Line subdivision](image)

A Hydrocarbon graph $G$ and its line subdivision is shown in Figure 1.

Topological indices are numbers associated with molecular graphs for the purpose of allowing quantitative structure-activity/property relationships. Topological indices correlate certain Physico-Chemical properties like boiling point, stability, strain energy etc of chemical compounds. One of the oldest most popular and extremely studied topological indices are well-known Zagreb indices first introduced in 1972 by Gutman and Trinajestic [8].

Let $G$ be a simple graph and let $V(G)$ and $E(G)$ be its vertex and edge sets, respectively. The edge connecting the vertices $u$ and $v$ will be denoted by $uv$. The complement $\overline{G}$ of the graph $G$ is the graph with vertex set $V(G)$, in which two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in $G$. The degree of the vertex $v$, denoted by $d_G(v)$, is the number of first neighbors of $v$ in the underlying graph $G$. Then the first and second Zagreb index are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G^2(v)$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v)$$

respectively.

There is another expression for the first Zagreb index namely

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

In 2008, bearing in mind Eq. (1.2), Doslic in [6] put forward the first Zagreb coindex, defined as

$$\overline{M}_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$
Recently, Furtula and Gutman [7] introduced a new topological index and named this index as forgotten topological index. They showed that the predictive ability of this index is almost similar to that of first Zagreb index. Throughout the present paper we name this index as F-index and denote it by $F(G)$, so

$$F(G) = \sum_{v \in V(G)} d_G^2(v). \quad (1.3)$$

There is another expression for the F-index namely

$$F(G) = \sum_{uv \in E(G)} \left[ d_G^2(u) + d_G^2(v) \right].$$

Similar to the first Zagreb coindex, the F-coindex of a graph $G$ is defined as

$$F(G) = \sum_{uv \in E(G)} \left[ d_G^2(u) + d_G^2(v) \right].$$

For more details on the topological indices and coindices we refer to the articles [1,4,9,10,11,13,18].

In 2011, Ranjini et al. calculated the explicit expressions for the Schultz indices of the subdivision graphs of the Tadpole, Wheel, Helm and Ladder graphs [15]. They also studied the Zagreb indices of the line graphs of the Tadpole, Wheel and Ladder graphs with subdivision in [14]. In 2015, Su and Xu calculated the general Sum-Tadpole, Wheel and Ladder graphs with subdivision in connectivity indices and coindices of the line graphs of the Tadpole, Wheel and Ladder graphs with subdivision [17]. In [12], Nadeem et al. computed $ABC_4$ and $GA_5$ indices of the line graphs of the Tadpole, Wheel and Ladder graphs by using the notion of subdivision.

In this paper we compute first Zagreb index, F-index and F-coindex of Dandelion graph $D(n, l)$, Comet graph $C(n, l)$, Fence graph $P_n [P_2]$, Closed fence graph $C_n [P_2]$, Friendship graph $F_m$, t-fold bristled of $C_n$ and t-fold bristled of $P_n$, Tadpole graph $T_{n,k}$, Wheel $W_{n+1}$ and Ladder graph $L_n$.

2. Main Results

We begin with a lemma used in the proof of our results.

**Lemma 2.1.** (55) Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then

$$\overline{F}(G) = (n-1)M_1(G) - F(G).$$

Let $D(n, l)$ be a Dandelion graph with $n$ vertices consisted of a copy of the star $S_{n-1}$ and a copy of the path $P_l$ with vertices $P_0, P_1, \ldots, P_{l-1}$, where $P_0$ is identified with a star center. (See Figure 2)

**Theorem 2.2.** Let $G$ be the line graph of the subdivision graph of a Dandelion graph. Then

$$M_1(G) = (n-l+1)^3 + (n-l+1) + 8(l-2)$$

$$F(G) = (n-l+1)^4 + (n-l+1) + 16(l-2)$$

$$\overline{F}(G) = (2n-2) \left[ (n-l+1)^3 + (n-l+1) + 8(l-2) \right]$$

$$- \left[ (n-l+1)^4 + (n-l+1) + 16(l-2) \right].$$

**Proof.** The number of vertices in $G$ are $2n-2$ among which $n-l+1$ vertices are of degree $n-l+1, n-l+1$ vertices are of degree 1 and the remaining $2l-4$ vertices are of degree 2. Using Eqs.(1.1) and (1.3) and lemma 2.1 we have,

$$M_1(G) = (n-l+1)(n-l+1)^2$$

$$+ (n-l+1)(l-1)^2 + (2l-4)(2)^2$$

$$= (n-l+1)^3 + (n-l+1) + 8(l-2)$$

$$F(G) = (n-l+1)(n-l+1)^3$$

$$+ (n-l+1)(l-1)^3 + (2l-4)(2)^3$$

$$= (n-l+1)^4 + (n-l+1) + 16(l-2).$$

$$\overline{F}(G) = (2n-2) \left[ (n-l+1)^3 + (n-l+1) + 8(l-2) \right]$$

$$- \left[ (n-l+1)^4 + (n-l+1) + 16(l-2) \right].$$

For a positive integer $l \leq n$ let $C(n, l)$ be a Comet graph with $n$ vertices consisted of a copy of the Complete graph $K_{n-l+1}$ and a copy of the Path $P_l$ with vertices $P_0, P_1, \ldots, P_{l-1}$, where $P_0$ is identified with a vertex from $K_{n-l+1}$. (See Figure 3)

**Theorem 2.3.** Let $G$ be the line graph of the subdivision graph of a Comet graph, then

$$M_1(G) = (n-l+1)^3 + (n-l+1)^4 + 4l - 3$$

$$F(G) = (n-l+1)^4 + (n-l+1)^5 + 8l - 7$$

$$\overline{F}(G) = \left[ (n-l+1)^2 + n + l - 3 \right]$$

$$\times \left[ (n-l+1)^3 + (n-l+1)^4 + 4l - 3 \right]$$

$$- \left[ (n-l+1)^4 + (n-l+1)^5 + 8l - 7 \right].$$

[Figure 2. Dandelion graph $D(17,8)$]

[Figure 3. Comet graph $C(7,3)$]
Proof. For \( l = 2 \) the proof is easy, so we consider the case \( l > 2 \). The subdivision graph \( S(G(n,l)) \) contains \( (n-l)^2 + n + l - 2 \) edges, so \( G \) contain \( (n-l)^2 + n + l - 2 \) vertices among which, \( n-l+1 \) vertices are of degree \( n-l+1 \), \((n-l)^2 \) vertices are of degree \( n-l \), \( l-1 \) vertices are of degree 2 and one vertex is of degree 1. Now, using Eqs. (1.1) and (1.3) and lemma 2.1 we have the proof.

Theorem 2.4. Let \( G \) be the line graph of the subdivision graph of a Fence graph \( P_n[2] \) (See Figure 4), then
\[
M_1(G) = 250n^2 - 392
\]
\[
F(G) = 1250n - 2176
\]
\[
\overline{F}(G) = 2500n^2 - 7420n + 5703.
\]

Proof. The number of vertices in \( G \) are \( 12 + 10(n-2) \) among which \( 10(n-2) \) vertices are of degree 5 and 12 vertices are of degree 3. Thus, using Eqs. (1.1) and (1.3) and lemma 2.1 we have the proof.

A Friendship graph (or Dutch windmill graph) \( F_m \) is a graph with \( 2m+1 \) vertices and \( 3m \) edges constructed by joining \( m \) copies of the cycle graph \( C_3 \) with a common vertex. (See Figure 5)

\[ \begin{align*}
M_1(G) &= 8m\left(n^2 + 2\right) \\
F(G) &= 16m\left(n^3 + 2\right) \\
\overline{F}(G) &= 8m\left(4m^3 - m^2 + 12m - 6\right).
\end{align*} \]

Proof. The number of vertices in \( G \) are \( 6m \) among which \( 2m \) vertices are of degrees 2m, and \( 4m \) vertices are of degree 2. Hence, using Eqs. (1.1) and (1.3) and lemma 2.1 we can get the proof.

For a given graph \( G \), its t-fold bristled graph \( Brst(G) \) is obtained by attaching \( t \) vertices of degree one to each vertex of \( G \). (See Figure 6)

Theorem 2.6. Let \( G \) be the line graph of the subdivision of a t-fold bristled graph of \( C_n \), then
\[
M_1(G) = t(n + (2n + tn)(t + 2)^2)
\]
\[
F(G) = t(n + (2n + tn)(t + 2)^3)
\]
\[
\overline{F}(G) = (2n + 2nt - 1)\left[tn + (2n + tn)(t + 2)^2\right] - [tn + (2n + tn)(t + 2)^3].
\]

Proof. The number of vertices of \( G \) are \( 2n(1 + t) \), among which \( tn \) vertices are of degree one and \( 2n + tn \) vertices are of degree \( t + 2 \). Thus, using Eqs. (1.1) and (1.3) and lemma 2.1 we have the proof.

With reference to the above theorems, the proof of next theorems are easy, so we omit the proofs.

Theorem 2.7. Let \( G \) be the line graph of the subdivision of a Closed fence graph \( C_n[2] \) (See Figure 4), then
\[
M_1(G) = 250(n-1)
\]
\[
F(G) = 1250(n-1)
\]
\[
\overline{F}(G) = 2500n^2 - 6500n + 400.
\]

Theorem 2.8. Let \( G \) be the line graph of the subdivision of a t-fold bristled graph of \( P_n \) (See Figure 6), then
\[
M_1(G) = t(n - 4) + n(5t + 4) - 6
\]
\[
F(G) = nt\left(t^2 + 6t + 13\right) - 6t(t + 3) + 8n - 14
\]
\[
\overline{F}(G) = nt\left(n^2 - 11t + 6tn - t^2 + 9n - 28\right)
+ 4n^2 - 18n + 6t^2 + 18t + 20.
\]

Theorem 2.9. Let \( G \) be the line graph of the subdivision graph of Wheel \( W_{n+1} \), then
\[
M_1(G) = n^3 + 27n
\]
\[
F(G) = n(n^3 + 81)
\]
\[
\overline{F}(G) = 2n(3n^3 - n^2 + 108n - 108).
\]
A Tadpole graph $T_{n,k}$ is a special type of graph consisting of a Cycle graph with $n$ (at least 3) vertices and a Path graph with $k$ vertices, connected with a bridge. (See Figure 7)

**Theorem 2.10.** Let $G$ be the line graph of the subdivision graph of a Tadpole graph $T_{n,k}$, then

$$M_1(G) = 4(2n + 2k + 3)$$

$$F(G) = 16n + 16k + 50$$

$$\overline{F}(G) = 4(2n + 2k - 1)(2n + 2k + 3) - 16(n + k) - 50.$$ 

A Ladder graph $L_n$ is a graph obtained as the Cartesian Product of two path graphs, one of which has only one edge (See Figure 8).

**Theorem 2.11.** Let $G$ be the line graph of the subdivision graph of a Ladder graph $L_n$, then

$$M_1(G) = 54n - 76$$

$$F(G) = 162n - 260$$

$$\overline{F}(G) = 324n^2 - 888n + 640.$$ 

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**References**


