

Generalization of Horadam's Sequence

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Abstract In this paper a new class of Fibonacci like sequence is introduced. Here we consider non-homogeneous recurrence relation to obtain generalization of Horadam's Sequence. Some identities concerning this new sequence are obtained and proved. Some examples are given in support of the results.

Keywords: pseudo fibonacci numbers, non-homogeneous recurrence relation

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1. Introduction

Fibonacci sequence is generalized in many ways. One of the most widely used extension is the one given by A.F. Horadam [2]. He defined the generalized sequence of numbers as follows. Let $\{W_n\}$ be a sequence defined by

$$W_n = W_n(a, b; p, q) = pW_{n-1} - qW_{n-2}, \quad (1.1)$$

for $n \geq 2$ where p, q, a and b are integers with $W_0 = a$ and $W_1 = b$.

One can see that $\{W_n\}$ reduces to generalized Fibonacci sequence $\{U_n\}$ when $a=0$ and $b=1$, and subsequently to the Fibonacci sequence $\{F_n\}$ when $p=1$, $q=-1$. That is $U_n = W_n(0, 1; p, q)$ and $F_n = W_n(0, 1; 1, -1)$. In a series of papers [1,3,4,5,8] various properties of the generalized sequence $\{W_n\}$ have been developed. Using Horadam Sequence and the concept of Pseudo Fibonacci sequence [A224508] defined earlier in [6] and studied further in [7], we now define a new extension of the sequence $\{W_n\}$.

Definition 1. The generalized Pseudo Fibonacci (GPF) Sequence $\{G_n\}$ is defined as the sequence satisfying the following non-homogeneous recurrence relation

$$G_{n+2} = pG_{n+1} - qG_n + At^n \quad (1.2)$$

for $n \geq 0$, $A \neq 0$ and $t \neq 0$, α, β with $G_0 = a$ and $G_1 = b$.

Here a, b, p, q are integers and α, β are distinct roots of characteristic equation $x^2 - px + q = 0$ of the corresponding homogeneous equation.

First few GPF numbers are given below:

$$\begin{aligned} G_0 &= a, G_1 = b, \\ G_2 &= (pb - qa) + A, \end{aligned}$$

$$G_3 = (p^2b - pqa - qb) + pA + At,$$

$$G_4 = (p^3b - p^2qa - 2pqb + q^2a) + (p^2 - q)A + pAt + At^2,$$

$$\begin{aligned} G_5 &= (p^4b - p^3qa - 3p^2qb + 2q^2a + q^2b) \\ &+ (p^3 - 2pq)A + (p^2 - q)pAt + pAt^2 + At^3. \end{aligned}$$

Observe that each GPF number G_n , $n \geq 2$ consist of two parts. The first part is an expression in p, q, a and b , while the second is a polynomial in t whose coefficients are A times terms in p and q . This is shown in the following tables:

Table 1. First part of G_n , $n \geq 2$

n	Expression in p, q, a, b
2	$pb - qa$
3	$p^2b - pqa - qb$
4	$p^3b - 2pqb - p^2qa + q^2a$
5	$p^4b - 3p^2qb + q^2b - p^3qa + 2q^2a$
6	$p^5b - 4p^3qb - p^4qa + 3q^2a + 2q^2$

Table 2. Second part of G_n , $n \geq 2$

n	A	At	At ²	At ³
2	1			
3	p	1		
4	p ² - q	p	1	
5	p ³ - 2pq	p ² - q	p	1

From the above tables, we have the following relation between GPF and Horadam numbers W_n .

Theorem 2. For $m \geq 2$, the term

$$G_m = W_m(a, b; p, q) + A \sum_{k=1}^{m-1} W_k(0, 1; p, q) t^{m-k-1}$$

of the sequence $\{G_m\}$ satisfy the non-homogeneous recurrence relation

$$G_{m+2} = pG_{m+1} - qG_m + At^m.$$

Proof. Consider

$$\begin{aligned} & \sum_{k=1}^{m+1} W_k(0,1;p,q)t^{m-k+1} \\ &= W_1t^m + W_2t^{m-1} + W_3t^{m-2} + \dots + W_{m+1} \\ &= W_1t^m + (pW_1 - qW_0)t^{m-1} + (pW_2 - qW_1)t^{m-2} \\ &+ \dots + (pW_m - qW_{m-1}) \\ &= W_1t^m + p(W_1t^{m-1} + W_2t^{m-2} + \dots + W_m) \\ &- q(W_0t^{m-1} + W_1t^{m-2} + \dots + W_{m-1}) \\ &= W_1t^m + p \sum_{k=1}^m W_k(0,1;p,q)t^{m-k} \\ &- q \sum_{k=0}^{m-1} W_k(0,1;p,q)t^{m-k-1}. \end{aligned}$$

That is,

$$\begin{aligned} & \sum_{k=1}^{m+1} W_k(0,1;p,q)t^{m-k+1} = W_1(0,1;p,q)t^m \\ & + p \sum_{k=1}^m W_k(0,1;p,q)t^{m-k} - q \sum_{k=0}^{m-1} W_k(0,1;p,q)t^{m-k-1} \end{aligned} \tag{1.3}$$

Now

$$G_{m+2} = W_{m+2}(a,b;p,q) + A \sum_{k=1}^{m+1} W_k(0,1;p,q)t^{m-k+1}.$$

Using equations (1.1) and (1.3), we write

$$\begin{aligned} G_{m+2} &= pW_{m+1}(a,b;p,q) - qW_m(a,b;p,q) \\ &+ Ap \sum_{k=1}^m W_k(0,1;p,q)t^{m-k} \\ &- qA \sum_{k=0}^{m-1} W_k(0,1;p,q)t^{m-k-1} + AW_1(0,1;p,q)t^m \\ &= p[W_{m+1}(a,b;p,q) + A \sum_{k=1}^m W_k t^{m-k}] \\ &- q[W(a,b;p,q) + A \sum_{k=0}^{m-1} W_k(0,1;p,q)t^{m-k-1}] + At^m \\ &= pG_{m+1} - qG_m + At^m. \end{aligned}$$

Hence the theorem.

2. Some Identities for G_n

In this section, we obtain some fundamental identities for GPF sequence $\{G_n\}$.

- **Binet type Formula:**

Let $z = z(t) = At^2 - pt + q$.

Then the Binet form of G_n is given by

$$G_n = c_1\alpha^n + c_2\beta^n + zt^n, \tag{2.1}$$

where

$$\begin{cases} c_1 = \frac{(b-a\beta) - z(t-\beta)}{\alpha - \beta}, \\ c_2 = \frac{(a\alpha - b) - z(\alpha - t)}{\alpha - \beta}, \end{cases} \tag{2.2}$$

α, β are the distinct roots of the equation $x^2 - px + q = 0$ given by

$$\alpha = \frac{p+d}{2}, \beta = \frac{p-d}{2}, \tag{2.3}$$

writing $d = \sqrt{p^2 - 4q}$.

Note that

$$\alpha + \beta = p, \alpha\beta = q, \alpha - \beta = d. \tag{2.4}$$

We can deduce from (2.2) that

$$\begin{cases} c_1 + c_2 = a - z, \\ c_1 - c_2 = ((2b - ap) - z(2t - p))d^{-1}, \\ c_1c_2 = ed^{-2}, \end{cases} \tag{2.5}$$

where

$$c = abp - b^2 - a^2q - z\{bp - 2bt - 2aq + atp + A\}. \tag{2.6}$$

- **Generating function:**

Generating function $G^*(x)$ for G_n is given by

$$G^*(x) = \frac{1}{(1 - px + qx^2)} \left\{ \frac{Ax^2}{1-tx} + (a + bx - apx) \right\}.$$

We have the following result for sum of first n GPF numbers. Proofs follow from recurrence relation and Binet formula.

Proposition 3. For $p - q \neq \pm 1$

$$i) \sum_{k=0}^{n-1} G_k = 1(p - q - 1) \left\{ G_{n+1} - b + (p - 1)(a - G_n) - A \sum_{k=0}^{n-1} t^k \right\}.$$

$$ii) \sum_{k=0}^{n-1} (-1)^k G_k = 1(p - q + 1)[(-1)^{n+1} G_{n+1} + b - (p + 1)(a + (-1)^{n+1} G_n) + A(1 - t) \sum_{k=0}^{n-1} (-1)^k t^{2k}].$$

Using the recurrence relation (1.2), we have the following result. The same can also be obtained by induction.

Proposition 4.

$$\sum_{k=0}^n G_k t^{k-1} = \frac{1}{(1 - pt + qt^2)} \left[\begin{array}{l} a/t - ap + b \\ -t^n(G_{n+1} - qtG_n) \\ + A \sum_{k=0}^{n-1} t^{2k+1} \end{array} \right].$$

Proof. Using the recurrence relation (1.2), we have

$$\begin{aligned}
 G_k t^{k-1} &= [pG_{k-1} - qG_{k-2} + At^{k-2}]t^{k-1} \\
 G_{k-1} t^{k-2} &= [pG_{k-2} - qG_{k-3} + At^{k-3}]t^{k-2} \\
 &\vdots \\
 &\vdots \\
 G_2 t^1 &= [pG_1 - qG_0 + At^0]t^1.
 \end{aligned}$$

On summing both sides of these $(n-1)$ equations, we get,

$$\sum_{k=2}^n G_k t^{k-1} = p \sum_{k=1}^{n-1} G_k t^k - q \sum_{k=0}^{n-2} G_k t^{k+1} + A \sum_{k=0}^{n-2} t^{2k+1}.$$

i.e.

$$\begin{aligned}
 \sum_{k=0}^n G_k t^{k-1} - G_0 t^{-1} - G_1 &= p \sum_{k=0}^n G_k t^k - p(G_0 + G_n t^n) \\
 -q \sum_{k=0}^n G_k t^{k+1} + q(G_{n-1} t^n + G_n t^{n+1}) &+ A \sum_{k=0}^{n-2} t^{2k+1}.
 \end{aligned}$$

i.e.

$$\begin{aligned}
 \sum_{k=0}^n G_k t^{k-1} (1 - pt + qt^2) &= G_0 / t + G_1 - pG_0 \\
 -(pG_n - q(G_{n-1})t^n + qG_n t^{n+1}) &+ A \sum_{k=0}^{n-1} t^{2k+1}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \sum_{k=0}^n G_k t^{k-1} &= 1(1 - pt + qt^2) \\
 \left\{ at - ap + b - t^n (G_{n+1} - qtG_n) + A \sum_{k=0}^{n-1} t^{2k+1} \right\}.
 \end{aligned}$$

We now obtain the sum of the squares of GPFs and sum of the product of two consecutive GPFs.

For simplicity, let $X = \sum_{k=0}^n G_k^2$, $Y = \sum_{k=0}^n G_k G_{k+1}$,

$$S_1 = \sum_{k=0}^n G_k t^{k-1} \text{ and } S_2 = \sum_{k=0}^n t^{2k}.$$

Let $v_1 = 1 - p^2 - q^2$ and $v_2 = 1 + p^2 - q^2$.

Further let

$$\begin{aligned}
 P_1 &= (1 - p^2)[G_0^2 - G_{n+1}^2] + (G_1^2 - G_{n+2}^2) \\
 &+ 2A(G_0 t^{-2} + G_1 t^{-1} - G_{n+1} t^{n-1} - G_{n+2} t^n), \\
 P_2 &= (1 + p^2)[G_0^2 - G_{n+1}^2] + (G_1^2 - G_{n+2}^2) \\
 &- 2p(G_{n+1} G_{n+2} - G_0 G_1).
 \end{aligned}$$

We have the following results:

Proposition 5. For $p \neq 0, v_1 + qv_2 \neq 0$

$$\text{i) } X = \sum_{k=0}^n G_k^2 = \frac{\begin{pmatrix} P_1 + qv_2 + 2(t^{-1} - q^2 t)AS_1 \\ + (q-1)A^2 S_2 + qP_2 \end{pmatrix}}{v_1},$$

$$\begin{aligned}
 \text{ii) } Y &= \sum_{k=0}^n G_k G_{k+1} = v_2 P_1 - v_1 P_2 \\
 &+ 2(t^{-1} v_2 + tqv_1)AS_1 - (v_1 + v_2)A^2 S_2 - 2p(v_1 + qv_2).
 \end{aligned}$$

Proof. Consider

$$\begin{aligned}
 G_{k+2}^2 - p^2 G_{k+1}^2 &= (G_{k+2} - pG_{k+1})(G_{k+2} + pG_{k+1}) \\
 &= 2AG_{k+2}^2 - 2pqG_k G_{k+1} + q^2 G_k^2 - A^2 t^{2k}.
 \end{aligned}$$

Hence, summing up to $n+1$ terms on both sides, we get,

$$\begin{aligned}
 \sum_{k=0}^n G_{k+2}^2 - p^2 \sum_{k=0}^n G_k^2 &= 2A \sum_{k=0}^n G_{k+2}^2 + q^2 \sum_{k=0}^n G_k^2 - 2pq \sum_{k=0}^n G_k G_{k+1} - \sum_{k=0}^n A^2 t^{2k}.
 \end{aligned}$$

Adjusting the variables of summation and simplifying, we get

$$v_1 X + 2pqY = P_1 + At^{-1}S_1 - A^2 S_2.$$

Similarly, starting with

$$\begin{aligned}
 G_{k+2}^2 - q^2 G_k^2 &= 2pG_{n+1}G_{n+2} - 2AqG_n t^n - p^2 G_{n+1}^2 + A^2 t^{2n}
 \end{aligned}$$

and simplifying as above, we get

$$v_2 X - 2pY = P_2 - 2qAtS_1 + A^2 S_2.$$

Solving these two equations for X and Y , we get the required results.

Next result deals with sum of even and odd terms of GPF sequence. Again for simplicity, let $E = \sum_{i=1}^n G_{2i}$ and

$$O = \sum_{i=1}^n G_{2i-1}, \quad O_A = A \sum_{i=1}^n t^{2i-1}, \quad E_A = A \sum_{i=0}^n t^{2i} \text{ so that}$$

$$E_A + O_A = A \sum_{i=0}^{2n} t^i.$$

Proposition 6. The sum of the even (odd) indexed terms of $\{G_n\}$ is given by

$$\begin{aligned}
 \text{(i) } \sum_{k=1}^n G_{2k} &= \frac{1}{\{p^2 - (1+q^2)\}} [p(G_{2n+1} - G_1) \\
 &+ (1+q)(qG_0 + G_{2n+2} - pG_{2n+1}) \\
 &- pA \sum_{k=1}^n t^{2k-1} - (1+q) \sum_{k=0}^n t^{2k}]
 \end{aligned}$$

and

$$\begin{aligned}
 \text{(ii) } \sum_{k=1}^n G_{2k-1} &= \frac{1}{\{p^2 - (1+q^2)\}} \\
 &[pqG_0 + pG_{2n+2} - p^2 G_{2n+1} + (1+q)(G_{2n+1} - G_1) \\
 &- (1+q)A \sum_{k=1}^n t^{2k-1} - Ap \sum_{k=0}^n t^{2k}]
 \end{aligned}$$

Provided $p^2 \neq 1 + q^2$.

Proof. From the recurrence relation (1.1), we have,

$$pG_{2k} = G_{2k+1} + qG_{2k-1} - At^{2k-1}.$$

Summing up to n terms, we get

$$p \sum_{k=1}^n G_{2k} = \sum_{k=1}^n G_{2k+1} + q \sum_{k=1}^n G_{2k-1} - A \sum_{k=1}^n t^{2k-1},$$

which on simplifying yields,

$$pE = (1 + q)O + G_{2n+1} - G_1 - O_A. \tag{2.7}$$

Similarly using the relation

$$pG_{2k+1} = G_{2k+2} + qG_{2k} - At^{2k},$$

and summing up to n terms, we get,

$$pO = (1 + q)E + G_{2n+2} - pG_{2n+1} + qG_0 - E_A. \tag{2.8}$$

Solving (2.7) and (2.8), we get the required results.

We have following identity.

Proposition 7. For $m, n > 0$,

$$\begin{aligned} G_m G_n - qG_{m-1} G_{n-1} &= (b - zt)G_{m+n-1} \\ &+ (z - a)qG_{m+n-2} + (aq - bt)zt^{m+n-2} \\ &+ z[G_m t^n + G_n t^m - G_{n-1} q t^{m-1} - G_{m-1} q t^{n-1}]. \end{aligned}$$

Proof. Using Binets formula (2.1),

$$\begin{aligned} \text{L.H.S.} &= (c_1 \alpha^m + c_2 \beta^m + zt^m)(c_1 \alpha^n + c_2 \beta^n + zt^n) \\ &- q(c_1 \alpha^{m-1} + c_2 \beta^{m-1} + zt^{m-1})(c_1 \alpha^{n-1} + c_2 \beta^{n-1} + zt^{n-1}). \end{aligned}$$

On simplification, we get,

$$\begin{aligned} \text{L.H.S.} &= [(b - a\beta) - z(t - \beta)]c_1 \alpha^{m+n-1} \\ &- [(a\alpha - b) - z(\alpha - t)]c_2 \beta^{m+n-1} - z^2 t^{m+n} \\ &+ zt^n G_m + zt^m G_n - qz[t^{m-1} G_{n-1} + t^{n-1} G_{m-1} - zt^{m+n-2}] \end{aligned}$$

$$\text{Since, } \alpha - \beta = d, c_1 d = (b - a\beta) - z(t - \beta) \text{ and } c_2 d = (a\alpha - b) - z(\alpha - t),$$

$$\begin{aligned} G_m G_n - qG_{m-1} G_{n-1} &= (b - zt)G_{m+n-1} + (z - a)qG_{m+n-2} \\ &+ (aq - bt)zt^{m+n-2} + z[G_m t^n + G_n t^m - G_{n-1} q t^{m-1} - G_{m-1} q t^{n-1}]. \end{aligned}$$

By letting $m = n$ we have the following result.

Corollary 8.

$$\begin{aligned} G_n^2 - qG_{n-1}^2 &= (b - z)G_{2n-1} + (z - a)qG_{2n-2} \\ &+ (aq - bt)zt^{2n-2} + 2z[t^n G_n - qt^{n-1} G_{n-1}]. \end{aligned}$$

Note that the above corollary along with Proposition

3.(i) can be used to find $\sum_{k=0}^n G_k^2$ obtained in Proposition 5(i).

Next we prove a version of Catalan's Identity for GPF numbers.

Proposition 9.

$$\begin{aligned} G_{n+r} G_{n-r} - G_n^2 &= eq^{n-r} u_{r-1}^2 \\ &+ zt^n [t^r G_{n-r} + t^{-r} G_{n+r} - 2G_n] \end{aligned}$$

where $u_r = W_r(1, p; p, q)$ and e is as defined by (2.6).

Proof. Using (2.3)

L.H.S.

$$\begin{aligned} &= (c_1 \alpha^{n+r} + c_2 \beta^{n+r} + zt^{n+r})(c_1 \alpha^{n-r} + c_2 \beta^{n-r} + zt^{n-r}) \\ &- (c_1 \alpha^n + c_2 \beta^n + zt^n)^2 \\ &= c_1^2 \alpha^{2n} + c_2^2 \beta^{2n} + z^2 t^{2n} + c_1 c_2 (\alpha^{n+r} \beta^{n-r} + \alpha^{n-r} \beta^{n+r}) \\ &+ zt^{n+r} (c_1 \alpha^{n-r} + c_2 \beta^{n-r}) + zt^{n-r} (c_1 \alpha^{n+r} + c_2 \beta^{n+r}) \\ &- \{c_1^2 \alpha^{2n} + c_2^2 \beta^{2n} + z^2 t^{2n} \\ &+ 2(c_1 c_2 \alpha^n \beta^n + zc_2 e t a^n t^n + zc_1 t^n \alpha^n)\} \\ &= c_1 c_2 \alpha^n \beta^n (\alpha^r \beta^{-r} + \alpha^{-r} \beta^r - 2) + zt^{n+r} G_{n-r} \\ &- z^2 t^{2n} + zt^{n-r} G_{n+r} - z^2 t^{2n} + zt^n G_n + 2z^2 t^{2n} \\ &= c_1 c_2 (\alpha \beta)^{n-r} (\alpha^r - \beta^r)^2 + zt^n [(t^r G_{n-r} + t^{-r} G_{n+r} + G_n)] \\ &= eq^{n-r} \frac{(\alpha^r - \beta^r)^2}{(\alpha - \beta)^2} + zt^n [(t^r G_{n-r} + t^{-r} G_{n+r} + G_n)] \\ &= eq^{n-r} u_{r-1}^2 + zt^n [(t^r G_{n-r} + t^{-r} G_{n+r} + G_n)]. \end{aligned}$$

From this result we immediately have a version of Cassini's identity for GPF numbers.

Corollary 10.

$$G_{n+1} G_{n-1} - G_n^2 = eq^{n-1} + zt^n [t G_{n-1} + t^{-1} G_{n+1} - 2G_n].$$

Next we have an expression for G_{2n} in terms of binomial coefficients.

Proposition 11.

$$G_{2n} = (-q)^n \sum_{i=0}^n \binom{n}{i} (-p/q)^{n-i} G_{n-i} - z[(pt - q)^n - t^{2n}].$$

Proof. We have

$$\begin{aligned} \text{R.H.S.} &= (-q)^n \sum_{i=0}^n \binom{n}{i} (-p/q)^{n-i} G_{n-i} - z[(pt - q)^n - t^{2n}] \\ &= (-q)^n \sum_{i=0}^n \binom{n}{i} (-p/q)^{n-i} (c_1 \alpha^{n-i} + c_2 \beta^{n-i} + zt^{n-i}) \\ &- z[(pt - q)^n - t^{2n}] \\ &= c_1 \sum_{i=0}^n \binom{n}{n-i} (p\alpha)^{n-i} (-q)^i + c_2 \sum_{i=0}^n \binom{n}{n-i} (p\beta)^{n-i} (-q)^i \\ &+ z \sum_{i=0}^n \binom{n}{n-i} (pt)^{n-i} (-q)^i - z[(pt - q)^n - t^{2n}] \\ &= c_1 (p\alpha - q)^n + c_2 (p\beta - q)^n + z(pt - q)^n \\ &- z[(pt - q)^n - t^{2n}]. \end{aligned}$$

Since, $\alpha^2 = p\alpha - q$ and $\beta^2 = p\beta - q$, we get,

$$\text{R.H.S.} = c_1 \alpha^{2n} + c_2 \beta^{2n} + zt^{2n} = G_{2n}.$$

Hence,

$$G_{2n} = (-q)^n \sum_{i=0}^n \binom{n}{i} (-p/q)^{n-i} G_{n-i} - z[(pt - q)^n - t^{2n}]$$

which is the required result.

3. Examples

In this section we present some examples in support of some results obtained in Section 2.

Examples: Consider $G_{n+2} = G_{n+1} - 2G_n + (-1)^n$, with $G_0 = 0, G_1 = 1$.

Here, $p = 1, q = 2, t = -1, A = 1$.

First few terms of G_n are $G_0 = 0, G_1 = 1, G_2 = 2, G_3 = -1, G_4 = -4, G_5 = -3, G_6 = 6, G_7 = 11, G_8 = 0, G_9 = -23, G_{10} = -22$.

(1) Verification of Proposition 5(i).

When $n = 5$, we have $S_1 = -1, S_2 = 6, P_1 = -128, P_2 = -6, v_1 + qV_2 = -8$. Then

$$\text{L.H.S.} = \sum_{k=0}^5 G_k^2 = G_0^2 + G_1^2 + \dots + G_5^2 = 31.$$

$$\text{R.H.S.} = -128 - 6 + 6 - 120 - 8 = -248 - 8 = 31.$$

Result is verified.

(2) Verification of Proposition 5 (ii).

Here let $n = 6$. We have, $P_1 = -19, P_2 = -241, S_1 = -7, S_2 = 7$. Then

$$\text{L.H.S.} = \sum_{k=0}^6 G_k G_{k+1} = G_0 G_1 + G_1 G_2 + \dots + G_6 G_7 = 64.$$

$$\text{R.H.S.} = 38 - 964 - 140 + 42 - 16 = 64.$$

Result is verified.

(3) Verification of Proposition 7.

Let $m = 2$ and $n = 3$. Then $z = 14$.

$$\text{L.H.S.} = G_2 G_3 - 2G_1 G_2 = 2(-1) - 2(2) = -6.$$

$$\text{R.H.S.} = -5 - \frac{1}{2} - \frac{1}{4} - \frac{3}{4} + \frac{1}{2} = -6.$$

Result is verified.

(4) Verification of Proposition 9.

Let $n = 6$ and $r = 2$. $z = \frac{1}{4}, e = -2, u_1 = 1$

$$\text{L.H.S.} = G_8 G_4 - G_6^2 = -36.$$

$$\text{R.H.S.} = -32 - 14X(4 + 12) = -32 - 4 = -36.$$

Result is verified.

(5) Verification of Proposition 11.

Let $n = 5$.

$$\text{L.H.S.} = G_{2n} = -22$$

$$\begin{aligned} \text{R.H.S.} &= (-q)^n \sum_{i=0}^n \binom{n}{i} (-p/q)^{n-i} G_{n-i} - z[(pt-q)^n - t^{2n}] \\ &= -32(83/32) - 1/4(-244) = -83 + 61 = -22. \end{aligned}$$

Result is verified.

4. Conclusion

The well known Horadam sequence is generalized via non homogeneous recurrence relation to obtain a Fibonacci like sequence. All the usual identities and properties of Fibonacci like sequences are obtained for the new generalization of Horadam sequence.

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