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A New Approximation (ziti's δ-scheme) of the Entropic (Admissible) Solution of the Hyperbolic Problems in One and Several Dimensions: Applications to Convection, Burgers, Gas Dynamics and Some Biological Problems

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Abstract As it is well known in numerical analysis, most of the numerical schemes have undesirable oscillations, especially near the domain's border, or near the physical phenomena (empty region, collapse, boundary layer, among others) (mathematically invisible) eg: Burgers equation(the solution loses its regularity in finite time). In the case where the differential problem solution presents a singularity (shock, blow-up which cannot be numerically detected easily), the classical scheme cannot generally operate correctly and in the best case we are confronted with a very difficult algorithm, especially in several dimensions. Our objective here is to construct a less complicated scheme compared to the classical methods by keeping their advantages and obtained the admissible solution in the most difficult situations without complications obtained from the selected meshing. In this paper, we applied our new method called ziti's δ- scheme which is able to resist to such oscillations near the singularity and enables us to detect a lot of physical phenomena (eg: shock waves, rarefaction waves, conservation of the matter quantity ...). We depict the ziti's δ- scheme for the multidimensional partial differential equations and systems on any meshing with simple numbering. We apply our method to some models and compare its results with the exact one and other classical numerical methods. We can conclude that our results are very striking. The ziti's δ-method that we obtained is faster and more efficient and robust.

Keywords: *hyperbolic, Riemann problem, entropy solution, shock, Gibbs phenomena, Godunov, Grimm, oscillation, ziti's δ- scheme*

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1. Introduction

Hyperbolic systems of conservations laws can usually be obtained by assuming that the phenomena under consideration evolves on the advection time scale and that other effects, like viscosity, dispersion, capillary, etc, can neglect. This leads to discontinuities, non-uniqueness and "unphysical" solutions. To keep the discontinuities but to avoid the other two possibilities solutions are considered in a weak sense together with some admissibility conditions. Following are the most common admissibility criteria for shocks waves in case of strictly hyperbolic systems:

a. Linearized stability analysis. (Lax conditions)
b. Existence of a stable viscous profile. (Liu's conditions)
c. Physical entropy derived from the second law of thermodynamics.
d. Requirement for hyperbolic equations to be a limit of the same equations perturbed by linear viscosity terms with the multiple of identity viscosity matrix.
e. Solution should be admissible for the equations derived as a weakly nonlinear asymptotic limit of the full physical system of equations.

For example, in case of polytropic gas dynamics and other similar systems all of above criteria reject the expansion shocks waves.

As it is well known in numerical analysis that most of the numerical schemes have undesirable oscillations, especially near the domain's border, or near the physical phenomena (empty region, collapse, boundary layer, among others)(mathematically invisible) eg: the heat equation with a bad sign, Burgers equation(the solution loses its regularity).

In the case where the differential problem solution presents a singularity (shock, blow-up which cannot be numerically detected easily), the classical scheme cannot generally operate correctly and in the best case we are confronted with a very difficult algorithm, especially in several dimensions.

Generally, well using classical methods such as Finite differences, Finite elements, Particle methods and Spectral methods in hyperbolic problems usually give some oscillation or Gibs phenomena that are not detect and admissible shocks. Therefore we can't obtain any entropy solution. An analysis of such schemes was made in [8-13,18], but the extension to several dimensions (n = 2,3,...) is not encouraging [18]. As against the schemes of Godunov and Glimm give entropy solutions in one dimension but unfortunately they are based on solving the Riemann problem involve a complication state, adding that the extension to several dimension does not give satisfactory results. The problem becomes more complicated if the problem is not strictly hyperbolic.

Our objective here is to construct a less complicated scheme compared to the classical methods by keeping their advantages and obtained the admissible solution in the most difficult situations without complications obtained from the selected meshing.

In this paper, we apply a new approximation method called ziti's δ-scheme to strictly or not strictly hyperbolic problem in one or more dimensions which is able to resist to such oscillation near the singularity and enables us to detect a lot of physical phenomena. We test our method to some models and compare its results with the exact one and other classical methods eg: burgers equation [7,14,15], gas dynamic [6,7,14], advection convection problems, biology chemotactic problems [16,17].

We can conclude that our results are very striking. The ziti's δ- scheme that we obtained is faster, more efficient, rebut, easy to handle in several dimension and gives entropy solutions.

2. Numerical Approximation of the Linear Hyperbolic Equation.

The advection equation is the prototype problem of the linear hyperbolic. In this section, we will study the one and two dimensional case. The comparison will be happen between the exact and the approximate solution obtained by ziti's δ-method.

2.1. The One Dimensional Case

We consider, here, the Cauchy problem associated with the advection equation:

$$\begin{cases} U_t(x,t)+c.U_x(x,t)=0 \quad x\in[a,b] \text{ and } t\in[0,T_f], \\ U(x,0)=U_0(x) & x\in[a,b], \\ \dfrac{\partial U}{\partial x}(a,t)=0 & t\in[0,T_f], \\ \dfrac{\partial U}{\partial x}(a,t)=0 & t\in[0,T_f], \\ \dfrac{\partial U}{\partial x}(b,t)=0 & t\in[0,T_f], \end{cases} \quad (1)$$

where, c and T_f are constants , U_0 is the initial data function.

The ziti's δ- method is based on the Galerkin method: First, we approximate the weak solution U(x,t) of (1) by:

$$U(x,t)\approx\sum_{i=1}^{i=m+1}\alpha_i(t)\psi_i(x) \quad (2)$$

where, (ψ_i) is the orthonormal family satisfies the following relation:

$$\psi_i(x)=\frac{\tilde{\psi}_i(x)}{\|\tilde{\psi}_i(x)\|}$$

where, $(\tilde{\psi}_k)$ is the orthogonal family satisfies the following recurrence relation:

$$\begin{cases} \tilde{\psi}_1(x)=\varphi_1(x)\forall x\in[x_1,x_{m+1}] \\ \tilde{\psi}_i=\varphi_i(x)+\lambda_{i-1}\tilde{\psi}_{i-1}\forall x\in[x_1,x_{m+1}];i=2,\ldots,m+1 \\ whers, \lambda_{i-1}=-\dfrac{<\varphi_i,\varphi_{i-1}>}{\tilde{\psi}_{i-1}{}^2} \end{cases} \quad (3)$$

and (φ_k) is obtained from the test function (Figure 1):

$$\Phi(x)=\begin{cases}\exp\left(\dfrac{1}{x^2-R^2}\right);|x|<R \\ 0 \qquad otherwise\end{cases} \quad (4)$$

where R is a positive constant, m is an integer such $x_1 = a$ and $x_{m+1} = b$.

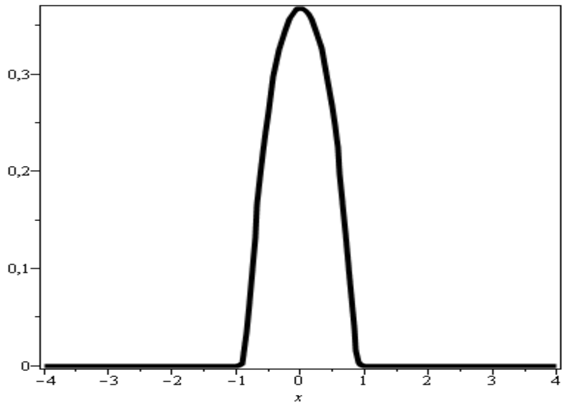

Figure 1. Illustration of the test function for R=1

Multiplying (a) of (1) by ψ_k and integrating it over [a,b] we obtain:

$$\int_a^b U_t(x,t)\psi_k(x)\mathrm{dx}+c\int_a^b U_x(x,t)\psi_k(x)\mathrm{dx}=0.$$

We use r_k as points of meshing instead of the x_k, where, r_k is the k^{th} root of ψ_{m+1} on the interval [a,b] and $r_{m+1} = x_{m+1}$ [1].

From [1] we have,

$$\int_a^b U_t(x,t)\psi_k(x)dx\approx\frac{U_t(r_k,t)}{\psi_k(r_k)},$$

therefore,

$$\int_a^b\sum_{i=1}^{m}\alpha_i'(t)\psi_i(x)\psi_k(x)dx+c\frac{U_x(r_k,t)}{\psi_k(r_k)}=0 \quad (5)$$

By using the orthonormal property of the (ψ_i) we obtain,

$$\int_a^b \sum_{i=1}^{m} \alpha_i^{'}(t)\psi_i(x)\psi_k(x)dx$$
$$= \sum_{i=1}^{m} \alpha_i^{'}(t)\int_a^b \psi_i(x)\psi_k(x)dx = \alpha_k^{'}(t).$$

Therefore the quantity (5) becomes,

$$\alpha_k^{'}(t) + c\frac{U_x(r_k,t)}{\psi_k(r_k)} = 0. \quad (6)$$

To approximate (6), we denote by: U_k^n the approximate value of the $U(r_k, n.dt)$, α_k^n the approximate value of the $\alpha_k(n.dt)$.

For example, if we take the center finite difference approximation which is second order accuracy:

$$U_x(r_k,t) \approx \frac{U(r_{k+1},t) - U(r_{k-1},t)}{r_{k+1} - r_{k-1}}$$

then, for t=ndt,

$$U_x(r_k, n.dt) \approx \frac{U_{k+1}^n - U_{k-1}^n}{r_{k+1} - r_{k-1}}.$$

From [1] we have,

$$U(r_k,t) = \alpha_k(t)\psi_k(r_k)$$

and,

$$\alpha_k(t) = \frac{U(r_k,t)}{\psi_k(r_k)} \approx \frac{U(r_{k+1},t) + U(r_{k-1},t)}{2.\psi_k(r_k)}.$$

By taking the time semi discretization as following:

$$\alpha_k^{'}(t) = \frac{\alpha_k^{n+1} - \alpha_k^n}{dt} \, at \; t = n.dt.$$

Therefore (6) becomes,

$$\alpha_k^{n+1} = \frac{U_{k+1}^n + U_{k-1}^n}{2.\psi_k(r_k)} - \beta_k \frac{U_{k+1}^n - U_{k-1}^n}{\psi_k(r_k)}; \quad (7)$$
$$k = 2, \ldots, m,$$

where, $\beta_k = \frac{c.dt}{r_{k+1} - r_{k-1}}$.

Boundary and initial conditions treatment.

From [1], the initial data U_0 can be approximated by,

$$U_0(x) \approx \sum_{i=1}^{m+1} \frac{U_0(r_i)}{\psi_i(r_i)}\psi_i(x),$$

therefore,

$$\alpha_k^1 = \frac{U_0(r_k)}{\psi_k(r_k)}; k = 1, \ldots, m+1. \quad (8)$$

The Neumann conditions (c) and (d) of (1) are approximated by,

$$\begin{cases} U_1^{n+1} = U_2^{n+1} \\ U_m^{n+1} = U_{m+1}^{n+1} \end{cases}$$

therefore,

$$\begin{cases} \alpha_1^{n+1} = \dfrac{U_2^{n+1}}{\psi_1(r_1)} \\ \alpha_{m+1}^{n+1} = \dfrac{U_m^{n+1}}{\psi_{m+1}(r_{m+1})} \end{cases} \quad (9)$$

By combining (7) with (8) and (9), we build an algorithm which enable to compute α_k^n at each level n (n ≥1) in accordance with the following scheme,

$$\begin{cases} \alpha_k^1 = \dfrac{U_0(r_k)}{\psi_i(r_k)}; k = 1, \ldots, m+1 \\ \alpha_k^{n+1} = \dfrac{U_{k+1}^n + U_{k-1}^n}{2.\psi_k(r_k)} - \beta_k \dfrac{U_{k+1}^n - U_{k-1}^n}{\psi_k(r_k)}; k = 2, \ldots, m, \\ \alpha_1^{n+1} = \dfrac{U_2^{n+1}}{\psi_1(r_1)}; \\ \alpha_{m+1}^{n+1} = \dfrac{U_m^{n+1}}{\psi_{m+1}(r_{m+1})} \end{cases} \quad (10)$$

where, $\beta_k = \frac{c.dt}{r_{k+1} - r_{k-1}}$.

2.1.1. Application.

Our scheme (10) gives an approximate solution which coincide with the exact solution of the problem (1) in three initial datas:

1) Gaussian function: (Figure 2).

2) Rectangular function: (Figure 3).

3) The initial function: has one or more singularity: (Figure 4 and Figure 5).

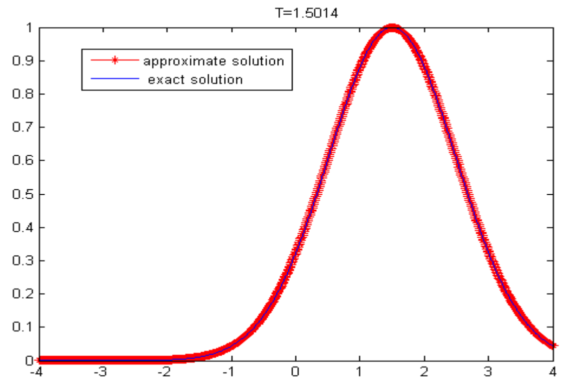

Figure 2. Comparison our approximate solution with the exact one. The initial condition is Gaussian.

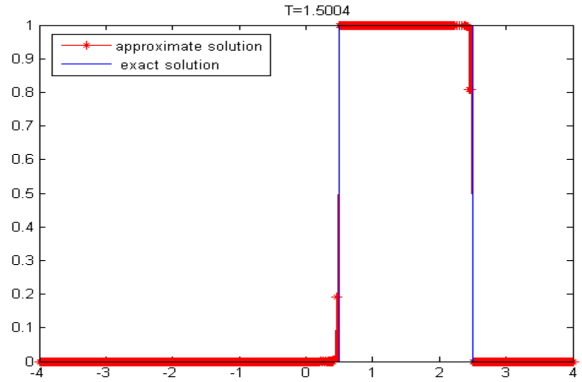

Figure 3. Comparison our approximate solution with the exact one. The initial condition is rectangular

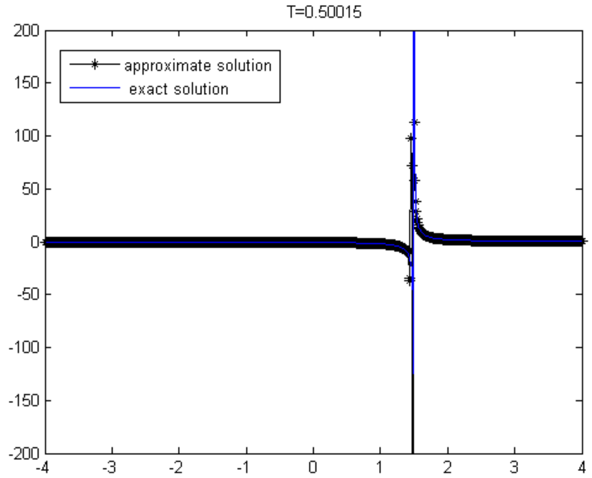

Figure 4. Comparison our approximate solution with the exact one. The initial condition has one singularity.

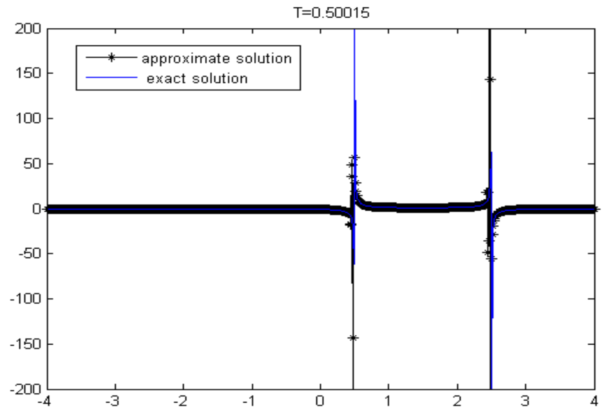

Figure 5. Comparison our approximate solution with the exact one. The initial condition has two singularities

2.2. The Bi-dimensional Advection Equation

Let us take, Q in the form $Q = \Omega \times [0, T_f]$, where, $\Omega = [a,b] \times [c,d]$, In two dimension, the Cauchy problem associated with the advection equation is written as:

$$\begin{cases} \begin{bmatrix} U_t(x,y,t) + c_1.U_x(x,y,t) \\ +c_2.U_y(x,y,t) \end{bmatrix} = 0; (x,y,t) \in Q, \\ U(x,y,0) = U_0(x,y); (x,y) \in \Omega, \\ \dfrac{\partial U}{\partial n}(a,y,t) = \dfrac{\partial U}{\partial n}(b,y,t) = 0; y \in [c,d] \text{ and } t \in [0,T_f] \\ \dfrac{\partial U}{\partial n}(x,c,t) = \dfrac{\partial U}{\partial n}(x,d,t) = 0; x \in [a,b] \text{ and } t \in [0,T_f], \end{cases} \quad (11)$$

where, c_1, c_2 and T_f are constants, U_0 is the initial data function.

With the same spirit as previously in section (2.1), we use [1] for approximate U(x,y,t) by:

$$U(x,y,t) = \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_{i,j}(t)\psi_{i,j}(x,y)$$

where, $(\psi_{i,j})$ is the orthonormal family defined in [1] and m is an integer such $x_1 = a$ and $x_{m+1} = b$, $y_1 = c$ and $y_{m+1} = d$.

We build an algorithm which enable to compute $\alpha_{i,j}^n$ at each level n (n $\geq$1) in accordance with the following scheme,

$$\begin{cases} \alpha_{k,l}^1 = \dfrac{U_0(r_k^1, r_l^2)}{\psi_{k,l}\left(r_k^1, r_l^2\right)}; k = 1,\ldots,m+1 \text{ and } l = 1,\ldots,m+1 \\ \alpha_{k,l}^{n+1} = \begin{bmatrix} \dfrac{U_{k+1,l}^n + U_{k-1,l}^n + U_{k,l+1}^n + U_{k,l-1}^n}{4.\psi_{k,l}\left(r_k^1, r_l^2\right)} \\ -\gamma_k^1 \dfrac{U_{k+1,l}^n - U_{k-1,l}^n}{\psi_{k,l}\left(r_k^1, r_l^2\right)} - \gamma_k^2 \dfrac{U_{k,l+1}^n - U_{k,l-1}^n}{\psi_{k,l}\left(r_k^1, r_l^2\right)} \end{bmatrix}; \\ \quad k = 2,\ldots,m \text{ and } l = 2,\ldots,m \\ \alpha_{1,l}^{n+1} = \dfrac{U_{2,l}^{n+1}}{\psi_{1,l}\left(r_1^1, r_l^2\right)}; l = 1,\ldots,m+1 \\ \alpha_{m+1,l}^{n+1} = \dfrac{U_{m,l}^{n+1}}{\psi_{m+1,l}\left(r_{m+1}^1, r_l^2\right)}; l = 1,\ldots,m+1 \\ \alpha_{k,1}^{n+1} = \dfrac{U_{k,2}^{n+1}}{\psi_{k,1}\left(r_k^1, r_1^2\right)}; k = 1,\ldots,m+1 \\ \alpha_{k,m+1}^{n+1} = \dfrac{U_{k,m}^{n+1}}{\psi_{k,m+1}\left(r_k^1, r_{m+1}^2\right)}; k = 1,\ldots,m+1 \end{cases} \quad (12)$$

where, $\gamma_j^i = \frac{c_i.dt}{r_{j+1}^i - r_{j-1}^i}$, i=1,2 ; j=2,…,m and r_k^1 is the k^{th} root of ψ_{m+1}^1 on [a,b] and r_k^2 is the k^{th} root of ψ_{m+1}^2 on [c,d].

2.2.1. Application

We compute the solution of (11) by taking two different initial conditions:

1. The first test: The initial condition is Gaussian.

We consider, in this case, an initial condition under the form:

$$U_0(x,y) = \exp\left(\frac{-\left(x^2+y^2\right)}{2}\right) \quad (13)$$

At final time Tf= 1, under the CFL condition, our scheme (12) gives an exact solution (Figure 6) and (Figure 7).

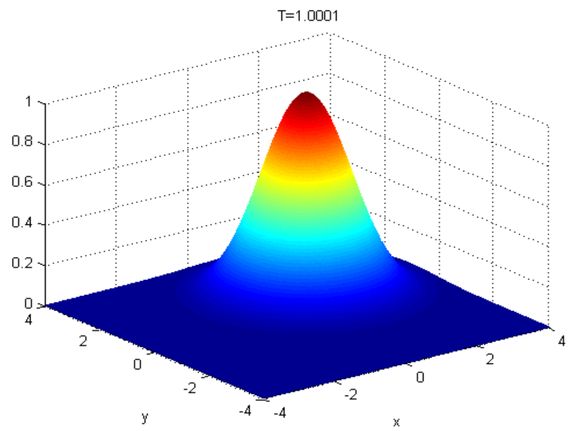

Figure 6. Approximate solution. The initial condition is Gaussian

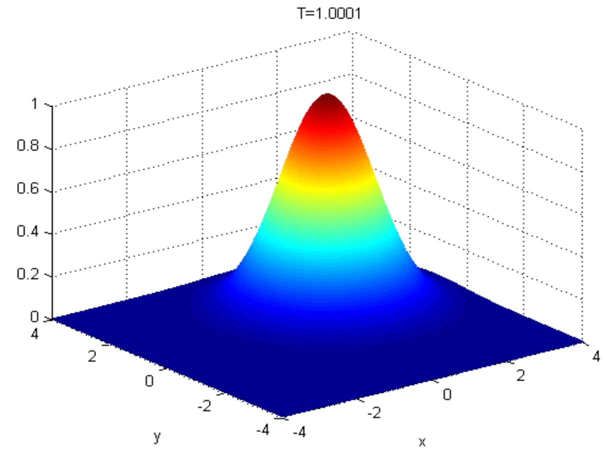

Figure 7. Exact solution. The initial condition is Gaussian

2. **The second test: The initial condition is rectangular function.**

We consider, in this case, an initial condition under the form:

$$U_0(x,y)=\begin{cases}1\; if\; -1\le x\le 1 \text{ and } -1\le y\le 1\\ 0 \qquad\qquad\qquad otherwise\end{cases}. \qquad (14)$$

At final time Tf= 1, under the CFL condition, our scheme (12) gives an exact solution (Figure 8) and (Figure 9).

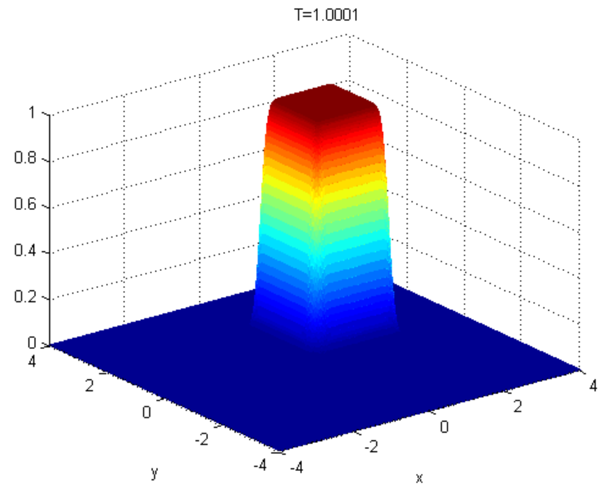

Figure 8. Approximate solution. The initial condition is rectangular function

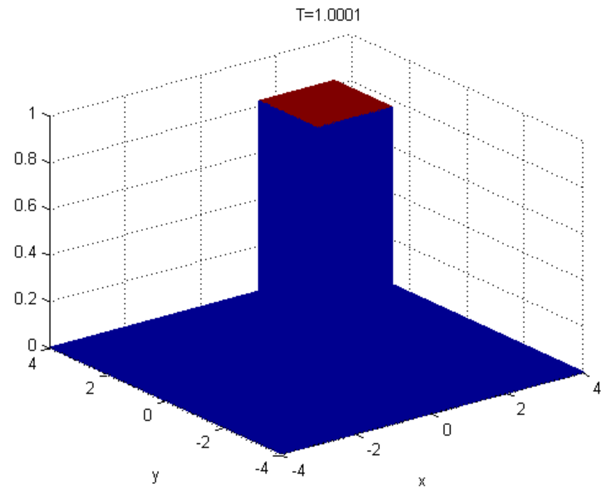

Figure 9. Exact solution. The initial condition is rectangular function

3. Numerical Approximation of the Nonlinear Hyperbolic Equation

In this section, we consider the numerical approximation of the hyperbolic problem solution with initial value. We start with the mono-dimensional case.

3.1. The One Dimensional Case

In one dimension, Burgers equation take the form:

$$U_t(x,t)+\left(\frac{U^2}{2}\right)_x(x,t)=0$$

which is a nonlinear hyperbolic equation in conservation laws.

The Cauchy problem is written under the form:

$$\begin{cases}U_t(x,t)+\left(\frac{U^2}{2}\right)_x(x,t)=0; x\in[a,b] \, and\; t\in\left[0,T_f\right],\\ U(x,0)=U_0(x); x\in[a,b],\\ \frac{\partial U}{\partial x}(a,t)=0; t\in\left[0,T_f\right]\\ \frac{\partial U}{\partial x}(b,t)=0; t\in\left[0,T_f\right],\end{cases}, (15)$$

where, T_f is a constant , U_0 is the initial data function.

The ziti's δ- method is based on the Galerkin method: First, we approximate the weak solution U(x,t) of (15) by:

$$U(x,t)\approx\sum_{i=1}^{i=m+1}\alpha_i(t)\psi_i(x).$$

We build an algorithm which enable to compute α_k^n at each level n (n ≥1) in accordance with the following scheme,

$$\begin{cases}\alpha_k^1=\frac{U_0(r_k)}{\psi_i(r_k)}; k=1,\ldots,m+1\\ \alpha_k^{n+1}=\begin{pmatrix}\frac{U_{k+1}^n+U_{k-1}^n}{2.\psi_k(r_k)}\\ -\beta_k\frac{\left(U_{k+1}^n\right)^2-\left(U_{k-1}^n\right)^2}{2.\psi_k(r_k)}\end{pmatrix}; k=2,\ldots,m,\\ \alpha_1^{n+1}=\frac{U_2^{n+1}}{\psi_1(r_1)};\\ \alpha_{m+1}^{n+1}=\frac{U_m^{n+1}}{\psi_{m+1}(r_{m+1})}\end{cases}, \quad (16)$$

where, $\beta_k=\frac{dt}{r_{k+1}-r_{k-1}}$, k=2,…,m.

3.1.1. Application.

As we said in the introduction the numerical result obtained by the ziti's δ- scheme very significantly.

Indeed:

For the Riemann problem, our scheme gives the entropy solution (Figure 10) and (Figure 11) in each case eg: Rarefaction wave, shock wave.

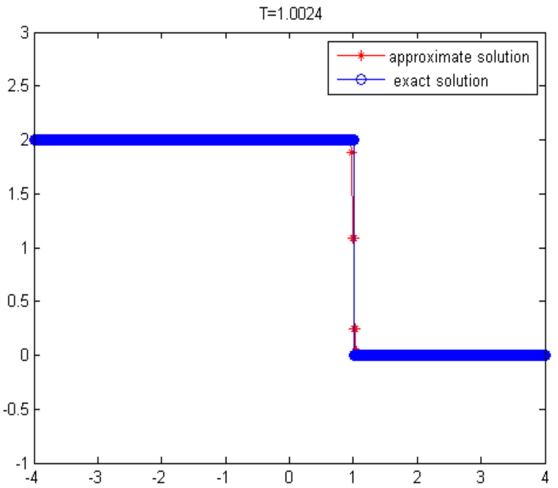

Figure 10. Comparison our approximate solution with the exact one in presence the shock wave

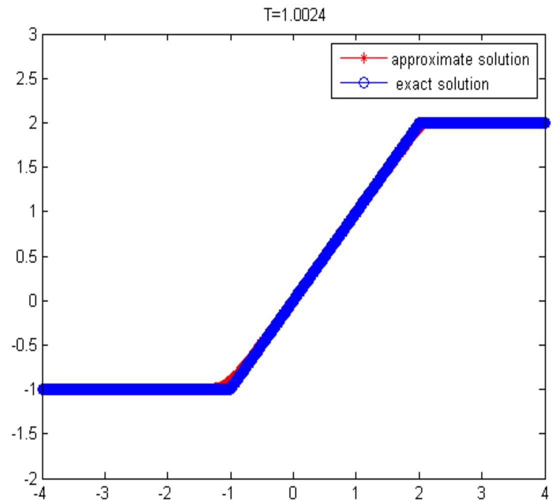

Figure 11. Comparison our approximate solution with the exact one in presence the rarefaction wave

3.2. The Bi-dimensional Advection Equation

Let us take, $\mathcal{Q}$ in the form $\mathcal{Q} = \Omega \times [0, T_f]$, where, $\Omega = [a,b] \times [c,d]$, In two dimension of epace we write the problem (15) as following:

$$\begin{cases} \left(U_t(x,y,t) + \left(\frac{U^2}{2}\right)_x (x,y,t) + \left(\frac{U^2}{2}\right)_y (x,y,t) \right) = 0; (x,y,t) \in \mathcal{Q}, \\ U(x,y,0) = U_0(x,y); (x,y) \in \Omega, \\ \frac{\partial U}{\partial n}(a,y,t) = \frac{\partial U}{\partial n}(b,y,t) = 0; y \in [c,d] \text{ and } t \in [0,T_f] \\ \frac{\partial U}{\partial n}(x,c,t) = \frac{\partial U}{\partial n}(x,d,t) = 0; x \in [a,b] \text{ and } t \in [0,T_f], \end{cases} \quad (17)$$

where, T_f is constant, U_0 is the initial data function.

With the same spirit as previously in section (2.2), we use [1] for approximate U(x,y,t) by:

$$U(x,y,t) = \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_{i,j}(t)\psi_{i,j}(x,y)$$

where, $(\psi_{i,j})$ is the orthonormal family defined in [1] and m is an integer such $x_1 = a$ and $x_{m+1} = b$, $y_1 = c$ and $y_{m+1} = d$.

We build an algorithm which enable to compute $\alpha_{i,j}^n$ at each level n (n ≥1) in accordance with the following scheme,

$$\begin{cases} \alpha_{k,l}^1 = \dfrac{U_0(r_k^1, r_l^2)}{\psi_{k,l}(r_k^1, r_l^2)}; k = 1,\dots,m+1 \text{ and } l = 1,\dots,m+1 \\ \left(\alpha_{k,l}^{n+1} = \dfrac{U_{k+1,l}^n + U_{k-1,l}^n + U_{k,l+1}^n + U_{k,l-1}^n}{4.\psi_{k,l}(r_k^1, r_l^2)} - \chi_1 \dfrac{(U_{k+1,l}^n)^2 - (U_{k-1,l}^n)^2}{2.\psi_{k,l}(r_k^1, r_l^2)} - \chi_2 \dfrac{(U_{k,l+1}^n)^2 - (U_{k,l-1}^n)^2}{2.\psi_{k,l}(r_k^1, r_l^2)} \right), \\ \quad k = 2,\dots,m \text{ and } l = 2,\dots,m \\ \alpha_{1,l}^{n+1} = \dfrac{U_{2,l}^{n+1}}{\psi_{1,l}(r_1^1, r_l^2)}; l = 1,\dots,m+1 \\ \alpha_{m+1,l}^{n+1} = \dfrac{U_{m,l}^{n+1}}{\psi_{m+1,l}(r_{m+1}^1, r_l^2)}; l = 1,\dots,m+1 \\ \alpha_{k,1}^{n+1} = \dfrac{U_{k,2}^{n+1}}{\psi_{k,1}(r_k^1, r_1^2)}; k = 1,\dots,m+1 \\ \alpha_{k,m+1}^{n+1} = \dfrac{U_{k,m}^{n+1}}{\psi_{k,m+1}(r_k^1, r_{m+1}^2)}; k = 1,\dots,m+1 \end{cases} \quad (18)$$

where, $\chi_1 = \frac{dt}{r_{j+1}^1 - r_{j-1}^1}$, $\chi_2 = \frac{dt}{r_{j+1}^2 - r_{j-1}^2}$; j=2,…,m and r_k^1 is the k^{th} root of ψ_{m+1}^1 on [a,b] and r_k^2 is the k^{th} root of ψ_{m+1}^2 on [c,d].

3.2.1. Application: The Riemann Problem [15]

We consider an initial condition of the form: (Figure 12 and Figure 13).

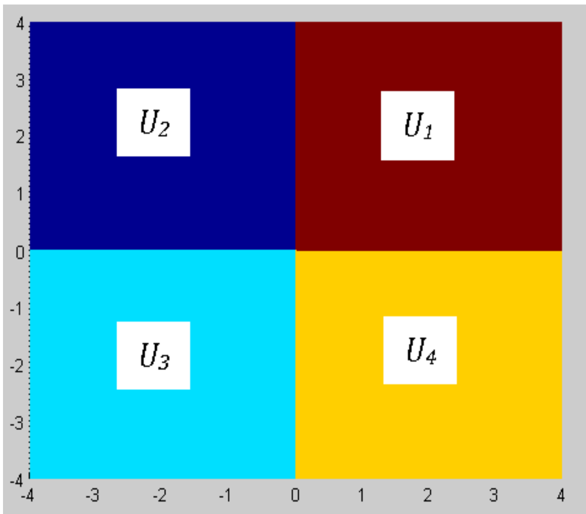

Figure 12. Initial condition for Riemann problem

$$U_0(x,t)=\begin{cases} u_1 & if\ x\in[0,4]\ and\ [0,4], \\ u_2 & if\ x\in[-4,0]\ and\ [0,4], \\ u_3 & if\ x\in[-4,0]\ and\ [-4,0], \\ u_4 & if\ x\in[0,4]\ and\ [-4,0], \end{cases} \quad (19)$$

where, u_i, i=1;2;3;4 are the given constants.

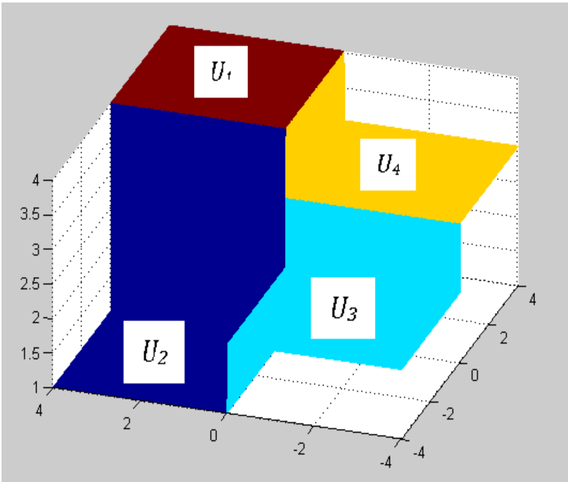

Figure 13. Initial condition for Riemann problem

We have 4!=24 cases to be treated according to the values of u_i, i=1;2;3;4 . We handle five different cases:

1) We have four rarefactions for initial conditions:
u_1=4; u_2=2; u_3=1 and u_4=3 (Figure 14; Figure 15).

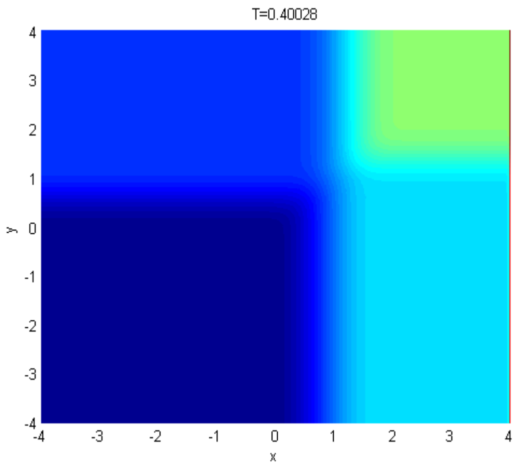

Figure 14. Four rarefactions

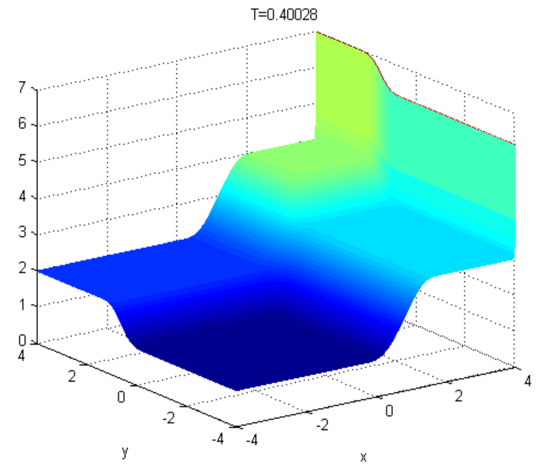

Figure 15. Four rarefactions

2) We have one shock and three rarefactions for initial conditions:
u_1=3; u_2=2; u_3=1 and u_4=4 (Figure 16; Figure 17).

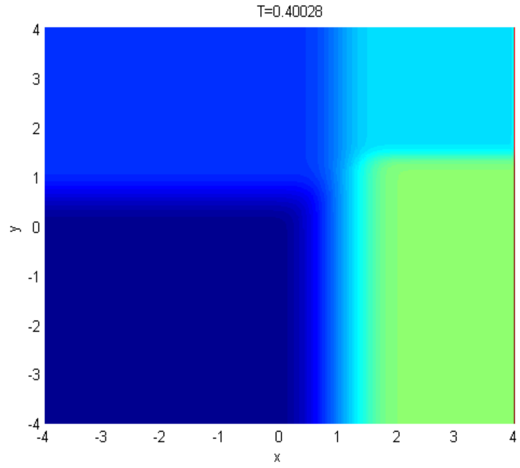

Figure 16. one shock and three rarefactions

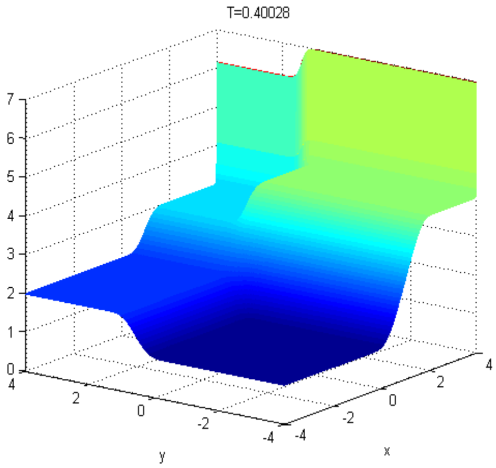

Figure 17. one shock and three rarefactions

3) We have two shocks and two rarefactions for initial conditions:
u_1=2; u_2=1; u_3=3 and u_4=4 (Figure 18; Figure 19).

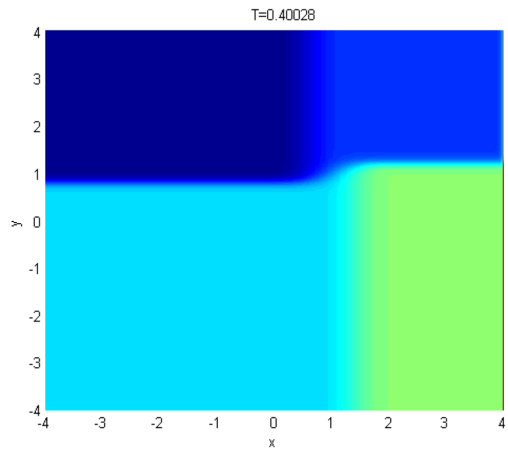

Figure 18. Two shocks and two rarefactions

4) We have three shocks and one rarefaction for initial conditions:
u_1=2; u_2=1; u_3=4 and u_4=3 (Figure 20; Figure 21).

5) We have four shocks for initial conditions:
u_1=1; u_2=2; u_3=4 and u_4=3 (Figure 22; Figure 23).

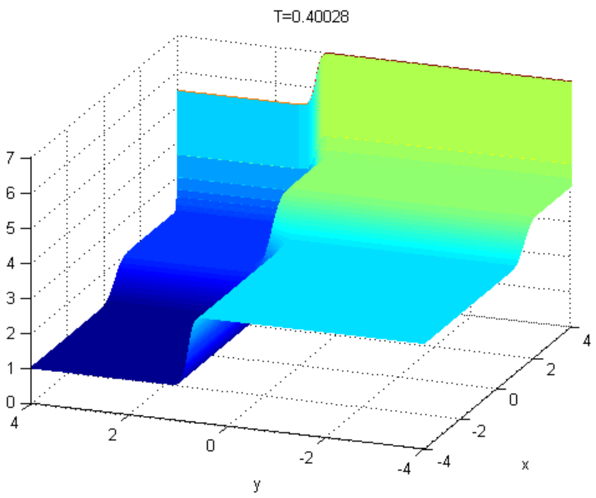

Figure 19. Two shocks and two rarefactions

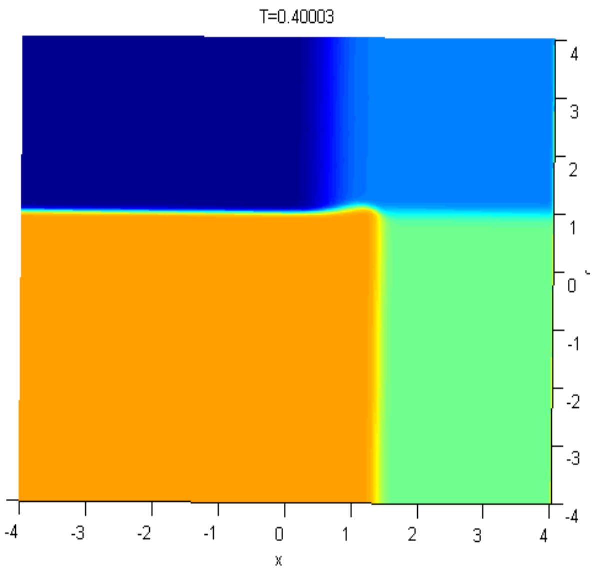

Figure 20. Three shocks and one rarefaction

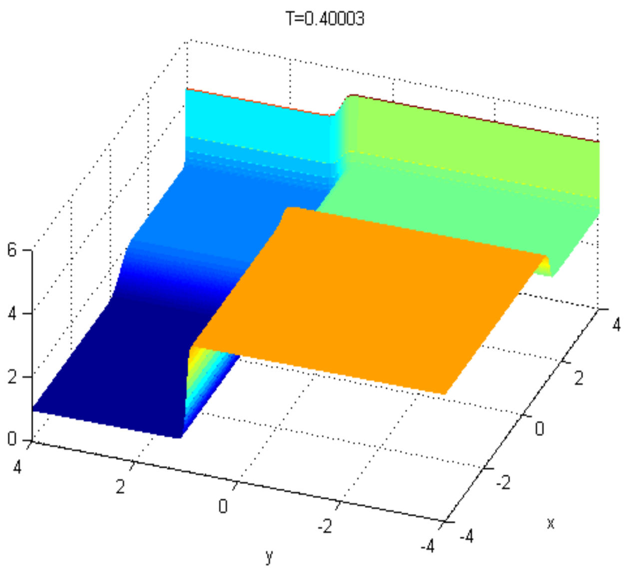

Figure 21. Three shocks and one rarefaction

We present, here, in Figure 14; Figure 15, Figure 16; Figure 17, Figure 18; Figure 19, Figure 20; Figure 21, Figure 22; Figure 23 the approximate solution obtained by ziti's δ- scheme.

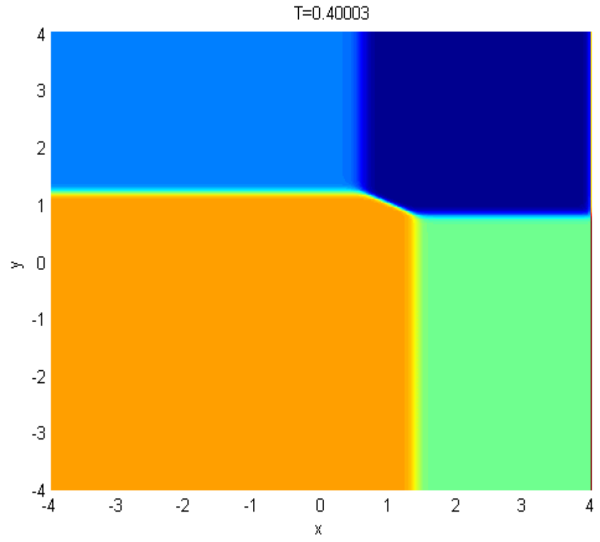

Figure 22. Four shocks

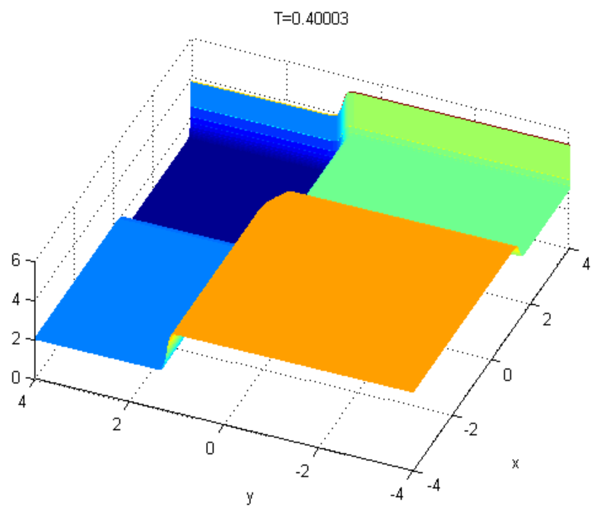

Figure 23. Four shocks

4. Numerical Approximation of the Euler Equations of Gas Dynamics

In this section we describe how to apply the ziti's δ-method to Euler equations of gas dynamics for a polytropic gas:

$$U_t\left(x,t\right)+F_x\left(U(x,t)\right)=0 x\in\left[a,b\right] and\ t\in\left[0,T_f\right] \quad (20)$$

Where,

$$U=\begin{pmatrix}\rho\\ m\\ E\end{pmatrix}, F\left(U\right)=\begin{pmatrix}m\\ \rho+\dfrac{m^2}{\rho}\\ \left(E+p\right)\dfrac{m}{\rho}\end{pmatrix} and\ \ p=\left(\gamma-1\right)\left(E-\frac{m^2}{2\rho}\right)$$

Where, T_f is constant, ρ the density, m the momentum $(m=\rho.v)$,v the velocity, E total energy, p the pressure and γ is the ratio of specific heats.

The ziti's δ-method is based on the Galerkin method.

First, we approximate the weak solution U(x,t) of (20) by:

$$\rho(x,t) \approx \sum_{i=1}^{m+1} \alpha_i(t)\psi_i(x)$$
$$m(x,t) \approx \sum_{i=1}^{m+1} \beta_i(t)\psi_i(x)$$
$$E(x,t) \approx \sum_{i=1}^{m+1} \gamma_i(t)\psi_i(x)$$

We build an algorithm which enable to compute α_k^n; β_k^n and γ_k^n at each level n (n ≥1) in accordance with the following scheme,

$$\begin{cases}
\alpha_k^1 = \dfrac{\rho(r_k,0)}{\psi_i(r_k)}; k = 1,\ldots,m+1 \\
\beta_k^1 = \dfrac{m(r_k,0)}{\psi_i(r_k)}; k = 1,\ldots,m+1 \\
\gamma_k^1 = \dfrac{E(r_k,0)}{\psi_i(r_k)}; k = 1,\ldots,m+1 \\
\alpha_k^{n+1} = \dfrac{\rho_{k+1}^n + \rho_{k-1}^n}{2.\psi_k(r_k)} + \lambda_k \dfrac{m_{k+1}^n - m_{k-1}^n}{\psi_k(r_k)}; k = 2,\ldots,m \\
\beta_k^{n+1} = \dfrac{m_{k+1}^n + m_{k-1}^n}{2.\psi_k(r_k)} \\
\quad -\lambda_k \dfrac{\left[(\gamma-1)(E_{k+1}^n - E_{k-1}^n) + \dfrac{3-\gamma}{2}\left(\dfrac{\left(m_{k+1}^n\right)^2}{\rho_{k+1}^n} - \dfrac{\left(m_{k-1}^n\right)^2}{\rho_{k-1}^n} \right) \right]}{\psi_k(r_k)}; \\
\quad k = 2,\ldots,m, \\
\alpha_k^{n+1} = \dfrac{E_{k+1}^n + E_{k-1}^n}{2.\psi_k(r_k)} \\
\quad -\lambda_k \dfrac{\left[\left(\gamma E_{k+1}^n + \dfrac{1-\gamma}{2}\dfrac{\left(m_{k+1}^n\right)^2}{\rho_{k+1}^n} \right)\dfrac{m_{k+1}^n}{\rho_{k+1}^n} - \left(\gamma E_{k-1}^n + \dfrac{1-\gamma}{2}\dfrac{\left(m_{k-1}^n\right)^2}{\rho_{k-1}^n} \right)\dfrac{m_{k-1}^n}{\rho_{k-1}^n} \right]}{\psi_k(r_k)}; \\
\quad k = 2,\ldots,m, \\
\alpha_1^{n+1} = \dfrac{\rho_2^{n+1}}{\psi_1(r_1)}; \beta_1^{n+1} = \dfrac{m_2^{n+1}}{\psi_1(r_1)}; \gamma_1^{n+1} = \dfrac{E_2^{n+1}}{\psi_1(r_1)} \\
\alpha_{m+1}^{n+1} = \dfrac{\rho_m^{n+1}}{\psi_{m+1}(r_{m+1})}; \beta_{m+1}^{n+1} = \dfrac{m_m^{n+1}}{\psi_{m+1}(r_{m+1})}; \\
\gamma_{m+1}^{n+1} = \dfrac{E_m^{n+1}}{\psi_{m+1}(r_{m+1})}
\end{cases} \tag{21}$$

where, $\lambda_k = \frac{dt}{r_{k+1}-r_{k-1}}$; $k = 2; \ldots; m$

4.1. Application

4.1.1. Example1: [14]

Applying the ziti's δ -scheme (21) to the Riemann problem (20):

$$\rho(x,0) = \begin{pmatrix} \rho_L = 1 \\ \rho_R = 0.1 \end{pmatrix};$$
$$m(x,0) = \begin{pmatrix} m_L = 0 \\ m_R = 0 \end{pmatrix};$$
$$E(x,0) = \begin{pmatrix} E_L = 2.5 \\ E_R = 0.25 \end{pmatrix}$$

and

$$\gamma = 1.4. \tag{22}$$

The ziti's **δ**-method gives the same results than founded in [14] (Figure 24).

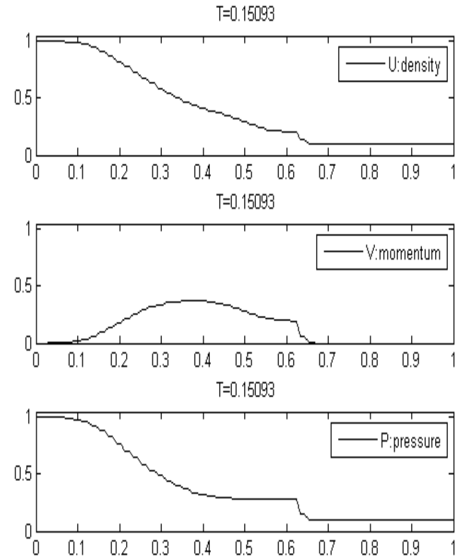

Figure 24. The numerical solutions of (20) by ziti's δ-scheme (21) at t=0.15 with initial data (22)

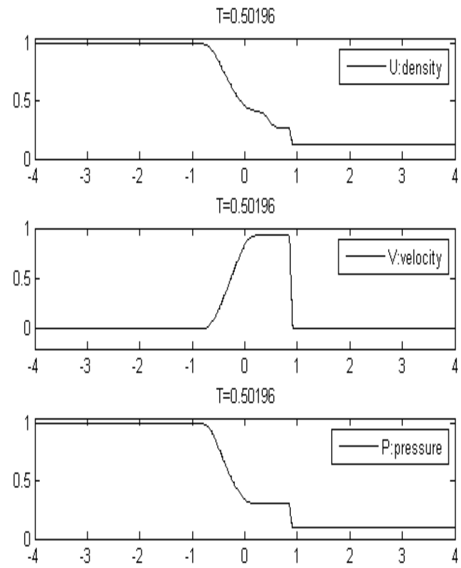

Figure 25. The numerical solutions of (20) by ziti's δ-scheme (21) at t=0.5 with initial data (23)

4.1.2. Example 2: [7]

Applying the ziti's **δ**-scheme (21) to the Riemann problem (20):

$$\rho(x,0)=\begin{pmatrix}\rho_L=1\\ \rho_R=0.125\end{pmatrix};$$

$$v(x,0)=\begin{pmatrix}v_L=0\\ v_R=0\end{pmatrix};p(x,0)=\begin{pmatrix}p_L=1\\ p_R=0.1\end{pmatrix}$$

and

$$\gamma=1.4. \tag{23}$$

The ziti's **δ**-method gives the same results than founded in [7] (Figure 25).

5. Numerical Approximation of the Biology Chemotactic Problem

In this section we apply the ziti's δ-method to system arising in biology [16]:

$$U_t(x,t)+F_x(U(x,t))=0, x\in[a,b] \text{ and } t\in[0,T_f] \tag{24}$$

Where, $U=\begin{pmatrix}u\\ v\end{pmatrix}$ and $F(U)=\begin{pmatrix}-uv\\ -u\end{pmatrix}$.

Remark 1. This problem is non-strictly hyperbolic system, and to study the Riemann problem associated, we must study more than 30 cases. The ziti's δ-method is based on the Galerkin method. First, we approximate the weak solution U(x,t) of (24) by:

$$u(x,t)\approx\sum_{i=1}^{m+1}\alpha_i(t)\psi_i(x)$$

$$v(x,t)\approx\sum_{i=1}^{m+1}\beta_i(t)\psi_i(x).$$

We build an algorithm which enable to compute α_k^n and β_k^n at each level n (n ≥1) in accordance with the following scheme,

$$\begin{cases}\alpha_k^1=\dfrac{u(r_k,0)}{\psi_i(r_k)};k=1,\ldots,m+1\\ \beta_k^1=\dfrac{v(r_k,0)}{\psi_i(r_k)};k=1,\ldots,m+1\\ \alpha_k^{n+1}=\begin{pmatrix}\dfrac{u_{k+1}^n+u_{k-1}^n}{2.\psi_k(r_k)}\\ +\lambda_k\dfrac{u_{k+1}^n v_{k+1}^n-u_{k-1}^n v_{k-1}^n}{\psi_k(r_k)}\end{pmatrix};k=2,\ldots,m\\ \beta_k^{n+1}=\dfrac{v_{k+1}^n+v_{k-1}^n}{2.\psi_k(r_k)}+\lambda_k\dfrac{u_{k+1}^n-u_{k-1}^n}{\psi_k(r_k)};k=2,\ldots,m\\ \alpha_1^{n+1}=\dfrac{u_2^{n+1}}{\psi_1(r_1)};\beta_1^{n+1}=\dfrac{v_2^{n+1}}{\psi_1(r_1)};\\ \alpha_{m+1}^{n+1}=\dfrac{u_m^{n+1}}{\psi_{m+1}(r_{m+1})};\beta_{m+1}^{n+1}=\dfrac{v_m^{n+1}}{\psi_{m+1}(r_{m+1})};\end{cases} \tag{25}$$

where, $\lambda_k=\frac{dt}{r_{k+1}-r_{k-1}};\ k=2;\ldots;m.$

5.1. Application

The ziti's δ-method gives the same results than founded in [16]:

5.1.1. Example 1: One Intermediate State

$$U^-=\begin{pmatrix}u_L=1\\ u_L=0\end{pmatrix};U^+=\begin{pmatrix}u_R=0.3\\ v_R=-0.6\end{pmatrix} \tag{26}$$

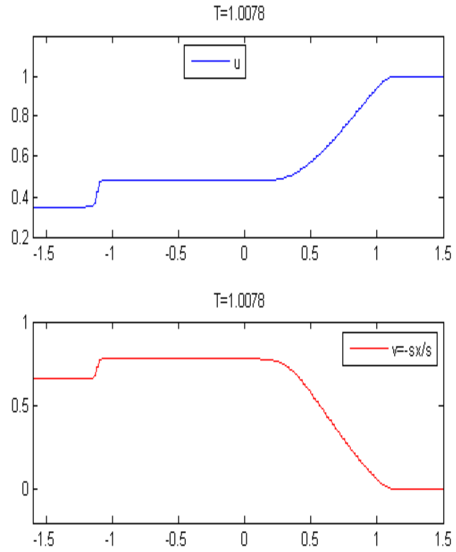

Figure 26. The numerical solutions of (24) by ziti's δ-scheme (25) at t=1 with initial data (26)

5.1.2. Example 2: Two Intermediate States

$$U^-=\begin{pmatrix}u_L=1\\ u_L=2\end{pmatrix};U^+=\begin{pmatrix}u_R=0.3\\ v_R=-0.7\end{pmatrix} \tag{27}$$

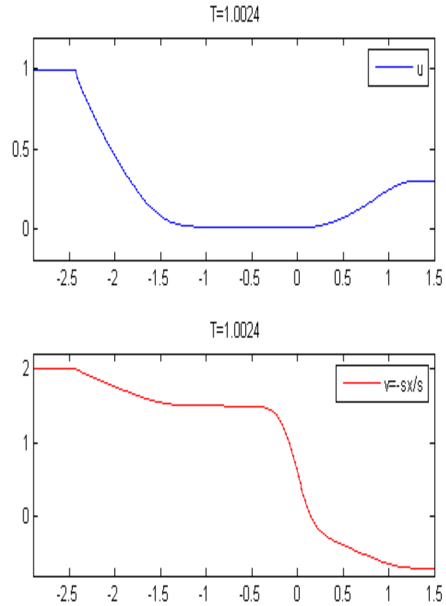

Figure 27. The numerical solutions of (24) by ziti's δ-scheme (25) at t=1 with initial data (27)

6. Conclusion

In this paper we are interested in the hyperbolic problems of conservation laws and the viscous problems associated (viscosity profil) to compare the solutions to the limit when the viscous term goes to zero. We have applied a new entropic conservation numerical scheme (in the general sense) called δ-ziti scheme. Hence we were able to reproduce at the discrete level an important property satisfied by the physical model. Also established that the proposed schemes may have sufficiently high order of accuracy. Indeed, we have compared our result for some models (travelling, Burgers, dynamics gas problems ...) in one dimension with the exact one or with another numerical result. It has also been testing the model Riemann problems cited in this paper, the results were efficient. Our method gives the result without numerical diffusion, dispersion and any oscillations. We also compared our results (obtained by our scheme) in two dimensions with famous numerical results [15]. Note here that we have at our disposal several choices meshes strategies and choice of basic functions, resulting all the best results easily. Note also that our method adapts well to mobile borders. Finally, all tested on our scheme, proved the detection of different singularities such as shock waves, the regularities such as the rarefaction waves, we could even detect more complex waves (eg instead of only one intermediate state, there are two or more intermediate states). We can conclude that our results are very striking. The ziti's δ-method that we obtained is faster and more efficient and robust.

References

[1] L.BSISS, C.ZITI, “A new numerical method for the integral approximation and solving the differential problems: Non-oscillating scheme, detecting the singularity in one and several dimensions”, *J. Afrika Matematika*, Submetted Springer, M.N AFMA-D-1500277, Oct. 2015.

[2] H.GILQUIN, *Analyse numérique d'un problème hyperbolique multidimensionnel en dynamique des gaz avec frontières mobiles*, Thèse de Doctorat. U.Saint-Etienne, 1984.

[3] J.Glimm, “Solutions in the large for nonlinear hyperbolic systems of equations”, *Comm. Pure and Applied Mathematics*, Vol XVII, 1959, 697-715.

[4] S.K.Godunov, *A finite difference method for the numerical computition of discontinous solutions of the equations of fluid dynamics*. Mat. Sb. 47, 1959.

[5] S.K.Godunov, A.Zabrodine, M.Ivanov, A.Kraiko, and G.Prokopov. *Résolution numérique des problèmes multidimensionnels de la dynamique des gaz*. Edition Mir, Moscou, 1979.

[6] G.-S Jiang, D.Levy, G.-T Lin, S.Osher, and E.Tadmor, “High Resolution nonoscillatory central Schemes with nonstaggerd drids for Hyperbolic Conservation Laws”, *SIAM J. NUMER. ANAL*. Vol. 35 No. 6, 1998, 2147-2168.

[7] A.Hartin, “High Resolution Schemes for Hyperbolic Conservation Laws”, *Journal of Computational Physics 49*, 1983, 357-393.

[8] A.Lerat and R.Peyret, "Sur le choix de schémas aux différences du second ordre fournissant des profils de choc sans oscillation", in *C.R.A.S*, 1973, 363-366.

[9] A.Lerat and R.Peyret, "Sur l'origine des oscillations apparaissant dans les profils de choc calculés par des méthodes aux différences", in C.R.A.S, 1973, 759-762

[10] A.Lerat and R.Peyret, *Noncentered schemes and shock propagation problems*, Computers and Fluids, 2, 1974, 35-52.

[11] A.Lerat and R.Peyret, "Mèthodes numériques en dynamique des fluides", in 4th conference International, Boulder (Colorado), juin. 1974.

[12] A.Lerat and R.Peyret, "Systèmes hyperboliques non linèaire", *Rech. Aérosp. N.1974-2,* Mars-Avril .1974.

[13] A.Lerat and R.Peyret, "Proprités dispersives et dissipatives d'une classe de schemas aux differences pour les systèmes hyperboliques non linèaires", *Rech. Aérosp. N.2:61-79*, 1975.

[14] B.Tessieras, "Résolution numérique, par des méthodes de poursuite de fronts, de systèmes hyperboliques de lois de conservation non lineaires", *These de Docteur, U.Bordeaux I*, 1983.

[15] D.Yoon and W.Hwang, "Two-dmensional Riemann problem for Burgers' equation", *Bull.Korean Math.Soc.45,* 2008, 191-205.

[16] C.Ziti, "Analyse et simulation numérique d'un système hyperbolique modélisant le comportement d'une population bactérienne", *Thèse de Docteur, U.Saint-Etienne*, 1987.

[17] C.Ziti, "Analysis and numerical simulation for a non-strictly hyperbolic system arising in biology", *C.R. Acad. Sci. Paris, t.319, Série I*, 1994, 751-756.

[18] C.Ziti, "Problèmes hyperboliques non linèaires en dynamique des populations chimiotactiques", *These de Docteur d'Etat, U.S.M Fes*, 1996.