Fixed Points Results for Graphic Contraction on Closed Ball

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Abstract In this paper, we introduce a new class of ciric fixed point theorem of $(\alpha, \psi)$-contractive mappings on a closed ball in complete metric space. As an application, we have derived some new fixed point theorems for ciric $\psi$-graphic contractions defined on a metric space endowed with a graph in metric space. Our results provide extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

Keywords: fixed point, $\alpha$ - admissible, $(\alpha, \psi)$ – contraction, closed ball


1. Introduction

In 2012, Samet et al. [18], introduced a concept of $\alpha-\psi$ - contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards Karapinar and Samet [6], refined the notions and obtained various fixed point results. Hussain et al. [9], enlarged the concept of $\alpha$-admissible mappings and obtained useful fixed point theorems. Subsequently, Abdeljawad [4] introduced pairs of $\alpha$-admissible mappings satisfying new sufficient contractive conditions different from [9] and [18], and proved fixed point and common fixed point theorems. Lately, Salimi et al. [17], modified the concept of $\alpha-\psi$ - contractive mappings and established fixed point results. Mohammadi et al. [7] introduced a new notion of $\alpha-\psi$ - contractive mappings and show that this is a real generalization for some old results. Arshad et al. [2] established fixed point results of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space. Hussain et al. [8], introduced the concept of an $\alpha$-admissible map with respect to $\eta$ and modify the $\alpha-\psi$ - contractive condition for a pair of mappings and established common fixed point results for two, three, and four mappings in a closed ball in complete dislocated metric spaces. Over the years, fixed point theory has been generalized in multi-directions by several mathematicians(see [1-18]).

Let $\Psi$ be a family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=0}^{+\infty} \psi^n(t) < +\infty$, for each $t > 0$.

Lemma 1. ([17]). If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

Definition 2. ([18]). Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is an $(\alpha, \psi)$-contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)),$

for all $x, y \in X$.

Definition 3. ([18]). Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that $T$ is $\alpha$-admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.

Example 4. Let $X = (0, \infty)$ and $T$ an identity mapping on $X$. Define $\alpha : X \times X \to [0, \infty)$ by

$\alpha(x, y) = \begin{cases} \frac{y}{x} & \text{if } x \geq y, x \neq 0 \\ 0 & \text{if } x < y. \end{cases}$

Then $T$ is $\alpha$-admissible.

Definition 5. ([17]). Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ two functions. We say that $T$ is $\alpha$-admissible mapping with respect to $\eta$ if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies that $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

If $\eta(x, y) = 1$, then above definition reduces to definition 3. If $\alpha(x, y) = 1$, then $T$ is called an $\eta$-subadmissible mapping.

Definition 6. ([7]). Let $T : X \to X$ and $\alpha_0 : X \times X \to [0, +\infty)$ by

$\alpha_0(x, y) = \begin{cases} 1 & \alpha(x, y) \geq \eta(x, y) \\ 0 & \text{otherwise} \end{cases}$

We say that $T$ is $\alpha_0$-admissible. If $\alpha_0(x, y) \geq 1$, then $\alpha(x, y) \geq \eta(x, y)$ and so $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$. This implies $\alpha_0(Tx, Ty) = 1$. Also $\alpha_0(x_0, Tx_0) = 1$. 


2. Main Results

We prove ciric fixed point results for $(\alpha, \psi)$-contraction mappings on a closed ball in complete metric space.

Theorem 7. Let $(X, d)$ be a complete metric space and $T$ is $\alpha$-admissible mapping with respect to $\eta$. For $r > 0$, $x_0 \in \overline{B(x_0, r)}$ and $\psi \in \Psi$, assume that,

$$x, y \in B(x_0, r), \alpha(x, y) \geq \eta(x, y) \Rightarrow d(Tx, Ty) \leq \psi(M(x, y)), \quad (1)$$

where

$$M(x, y) = \max \left\{ \frac{d(x, y), d(x, Tx), d(y, Ty)}{d(x, Ty) + d(y, Tx)} \right\},$$

and

$$\sum_{i=0}^{j} \psi^i(d(x_0, Tx_0)) \leq r, \text{ for all } j \in N. \quad (2)$$

Suppose that the following assertions hold:

- $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
- for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$ and $x_n \to u \in B(x_0, r)$ as $n \to +\infty$ then $\alpha(x_n, u) \geq \eta(x_n, u)$ for all $n \in N \cup \{0\}$.

Then, there exists a point $x^*$ in $B(x_0, r)$ such that $Tx^* = x^*$.

Proof. Let $x_1$ in $X$ be such that $x_1 = Tx_0$, $x_2 = Tx_1$. Continuing this process, we construct a sequence $x_n$ of points in $X$ such that $x_n = Tx_{n-1}$. By assumption $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and $T$ is $\alpha$-admissible mapping with respect to $\eta$. We have $\alpha(Tx_0, Tx_1) \geq \eta(Tx_0, Tx_1)$ from which we deduce that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ which also implies that $\alpha(Tx_1, Tx_2) \geq \eta(Tx_1, Tx_2)$. Continuing in this way we obtain $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$. First, we show that $x_n \in B(x_0, r)$ for all $n \in N$. Using inequality (2), we have

$$d(x_0, Tx_0) \leq r.$$

It follows that,

$$x_1 \in B(x_0, r).$$

Let $x_2, \cdots, x_j \in B(x_0, r)$ for some $j \in N$. Using inequality (1), we obtain

$$d(x_i, x_{i+1}) = d(Tx_i-1, Tx_i) \leq \psi(M(x_{i-1}, x_i))$$

$$M(x_{i-1}, x_i) = \max \left\{ \frac{d(x_{i-1}, x_i), d(x_i, x_{i+1})}{2} \right\} \leq \max \left\{ \frac{d(x_{i-1}, x_i), d(x_i, x_{i+1})}{2} \right\}.$$ 

So

$$M(x_{i-1}, x_i) \leq \max \{d(x_{i-1}, x_i), d(x_i, x_{i+1})\}. \quad (3)$$

the case $M(x_{i-1}, x_i) = d(x_i, x_{i+1})$ is impossible

$$d(x_i, x_{i+1}) \leq \psi(d(x_{i-1}, x_i)) < d(x_{i-1}, x_i).$$

Which is a contradiction. Otherwise, in other case $M(x_{i-1}, x_i) = d(x_{i-1}, x_i)$

$$d(x_i, x_{i+1}) \leq \psi(d(x_{i-1}, x_i)) \leq \psi^2(d(x_{i-2}, x_{i-1})) \leq \cdots \leq \psi^j(d(x_0, x_1)).$$

Thus we have,

$$d(x_i, x_{i+1}) \leq \psi^j(d(x_0, x_1)). \quad (4)$$

Now,

$$d(x_0, x_{j+1}) \leq d(x_0, x_j) + d(x_j, x_{j+1}) \leq \sum_{i=0}^{j} \psi^i(d(x_0, x_j)) \leq r.$$

Thus $x_{j+1} \in B(x_0, r)$. Hence $x_n \in B(x_0, r)$ for all $n \in N$. Now inequality (3.4) can be written as

$$d_j(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)), \text{ for all } n \in N. \quad (5)$$

Fix $\varepsilon > 0$ and let $N \in N$ such that $n \geq N \Rightarrow \psi^n(d_j(x_0, x_1)) < \varepsilon$. Let $m, n \in N$ with $m > n$. Then, by the triangle inequality, we have

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k(d_j(x_0, x_1)) \leq \sum_{n \geq \varepsilon} \psi^k(d_j(x_0, x_1)) < \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in $(B(x_0, r), d)$. As every closed ball in a complete metric space is complete, so there exists $x^* \in B(x_0, r)$ such that $x_n \to x^*$. Also

$$\lim_{n \to +\infty} d(x_n, x^*) = 0. \quad (6)$$

So by given assumption from (ii), we have $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$, for all $n \in N \cup \{0\}$. Now from (1), we obtain

$$d(x_n, Tx_n) \leq \psi(M(x_n, x^*). \quad (7)$$
where
\[
M(x_n, x^*) = \max \left\{ \frac{d(x_n, x^*)_1, d(x_n, x^*)}{2} \right\}.
\]

If \( d(x^*, Tx^*) \neq 0 \), then \( M(x_n, x^*) > 0 \) for every \( n \).

Thus \( d(x_{n+1}, Tx^*) \leq \psi(M(x_n, x^*)) < M(x_n, x^*) \).

(8)

which on taking limit as \( n \to \infty \) gives
\[
d(x^*, Tx^*) = \lim_{n \to \infty} d(x_{n+1}, Tx^*) \\
\leq \lim_{n \to \infty} M(x_n, x^*) = d(x^*, Tx^*).
\]

Hence \( d(x^*, Tx^*) = 0 \). The result follows.

**Example 8.** Let \( X = [0, \infty] \) with metric on \( X \) defined by
\[
d(x, y) = |x - y|.
\]

Let \( T : X \to X \) be defined by,
\[
T_x = \begin{cases} 
4 \text{ if } x \in [0, 1] \\
x - \frac{1}{4} \text{ if } x \in (1, \infty).
\end{cases}
\]

Consider \( x_0 = 1, r = 2, \psi(t) = \frac{t}{2} \) and
\[
\alpha(x, y) = \begin{cases} 
1 \text{ if } x, y \in [0, 1] \\
0 \text{ otherwise.}
\end{cases}
\]

Now \( B(x_0, r) = [0, 1] \), then
\[
d(x_0, Tx_0) = d(1, T1) = d(1, \frac{1}{4}) = |1 - \frac{1}{4}| = \frac{3}{4}
\]

Also if \( x, y \in (1, \infty) \), then
\[
\frac{3}{4}x - \frac{3}{4}y > |x - y| \\
\left| x - y \right| > \frac{|x - y|}{3} \\
\left| x - \frac{1}{4} - \frac{1}{4}y \right| > \psi(|x - y|)
\]
\[
d(Tx, Ty) > \psi(d(x, y)) \\
d(Tx, Ty) > \psi(M(x, y))
\]

Then the contractive condition does not hold on \( X \).

Also if, \( x, y \in \overline{B(x_0, r)} \), then
\[
\frac{3}{4}x - \frac{3}{4}y \geq |x - y| \\
\left| x - y \right| \geq \frac{|x - y|}{3} \\
\frac{1}{4}|x - y| \leq \psi(|x - y|)
\]
\[
d(Tx, Ty) \leq \psi(d(x, y)) \leq \psi(M(x, y)).
\]

If \( \eta(x, y) = 1 \) in the Theorem 7, we have the following corollary.

**Corollary 9.** Let \( (X, d) \) be a complete metric space and \( T \) is \( \alpha \) - admissible mapping. For \( r > 0 \), \( x_0 \in \overline{B(x_0, r)} \) and \( \psi \in \Psi \), assume that,
\[
x, y \in \overline{B(x_0, r)}, \alpha(x, y) \geq 1 \\
\Rightarrow d(Tx, Ty) \leq \psi(M(x, y)).
\]

where
\[
M(x, y) = \max \left\{ \frac{d(x, y), d(x, Tx), d(y, Ty)}{2} \right\}.
\]

and
\[
\sum_{i=0}^{n} \psi(d(x_0, Tx_0)) \leq r, \text{ for all } j \in N.
\]

Suppose that the following assertions hold:
- \( \alpha(x_0, T_0) \geq 1 \);
- for any sequence \( \{x_n\} \) in \( \overline{B(x_0, r)} \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in N \cup \{0\} \) and \( x_n \to u \in \overline{B(x_0, r)} \) as \( n \to +\infty \) then \( \alpha(x_n, u) \geq 1 \) for all \( n \in N \cup \{0\} \).

Then, there exists a point \( x^* \) in \( \overline{B(x_0, r)} \) such that \( Tx^* = x^* \).

If \( \eta(x, y) = 1 \) in the Theorem 7, we have the following corollary.

**Corollary 10.** Let \( (X, d) \) be a complete metric space and \( T \) is \( \eta \) - subadmissible mapping. For \( r > 0 \), \( x_0 \in \overline{B(x_0, r)} \) and \( \psi \in \Psi \), assume that,
\[
x, y \in \overline{B(x_0, r)}, \eta(x, y) \leq 1 \\
\Rightarrow d(Tx, Ty) \leq \psi(M(x, y)).
\]

where
\[
M(x, y) = \max \left\{ \frac{d(x, y), d(x, Tx), d(y, Ty)}{2} \right\}.
\]

and
\[
\sum_{i=0}^{n} \psi(d(x_0, Tx_0)) \leq r, \text{ for all } j \in N.
\]

Suppose that the following assertions hold:
- \( \eta(x_0, T_0) \leq 1 \);
- for any sequence \( \{x_n\} \) in \( \overline{B(x_0, r)} \) such that \( \eta(x_n, x_{n+1}) \leq 1 \) for all \( n \in N \cup \{0\} \) and \( x_n \to u \in \overline{B(x_0, r)} \) as \( n \to +\infty \) then \( \eta(x_n, u) \leq 1 \) for all \( n \in N \cup \{0\} \).

Then, there exists a point \( x^* \) in \( \overline{B(x_0, r)} \) such that \( Tx^* = x^* \).
Corollary 11. Let \((X, d)\) be a complete metric space and \(T\) is \(\alpha\)-admissible mapping with respect to \(\eta\). For \(r > 0\), \(x_0 \in B(x_0, r)\) and \(\psi \in \Psi\), assume that,

\[
x, y \in B(x_0, r), \alpha(x, y) \geq \eta(x, y) \\
\Rightarrow d(Tx, Ty) \leq \psi(N(x, y)),
\]

where

\[
N(x, y) = \max \left\{ \frac{d(x, y) + d(Tx, Ty)}{2}, \frac{d(x, Tx) + d(y, Ty)}{2} \right\},
\]

and

\[
\sum_{i=0}^{j} \psi_i(d(x_0, Tx_0)) \leq r, \text{ for all } j \in N.
\]

Suppose that the following assertions hold:

- \(\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)\);
- for any sequence \(\{x_n\}\) in \(B(x_0, r)\) such that \(\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})\) for all \(n \in N \cup \{0\}\) and \(x_n \to u \in B(x_0, r)\) as \(n \to +\infty\) then \(\alpha(x_n, u) \geq \eta(x_n, u)\) for all \(n \in N \cup \{0\}\).

Then, there exists a point \(x^*\) in \(B(x_0, r)\) such that \(Tx^* = x^*\).

If \(N(x, y) = \frac{d(x, Tx) + d(y, Ty)}{2}\) in the corollary 11, we have the following corollary.

Corollary 13. Let \((X, d)\) be a complete metric space and \(T\) is \(\alpha\)-admissible mapping. For \(r > 0\), \(x_0 \in B(x_0, r)\) and \(\psi \in \Psi\), assume that,

\[
x, y \in B(x_0, r), \alpha(x, y) \geq 1 \\
\Rightarrow d(Tx, Ty) \leq \psi(N(x, y)),
\]

and

\[
\sum_{i=0}^{j} \psi_i(d(x_0, Tx_0)) \leq r, \text{ for all } j \in N.
\]

Suppose that the following assertions hold:

- \(\alpha(x_0, Tx_0) \geq 1\);
- for any sequence \(\{x_n\}\) in \(B(x_0, r)\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n \in N \cup \{0\}\) and \(x_n \to u \in B(x_0, r)\) as \(n \to +\infty\) then \(\alpha(x_n, u) \geq 1\) for all \(n \in N \cup \{0\}\).

Then, there exists a point \(x^*\) in \(B(x_0, r)\) such that \(Tx^* = x^*\).

If \(N(x, y) = \frac{d(x, Tx) + d(y, Ty)}{2}\) in the corollary 11, we have the following corollary.

Corollary 14. Let \((X, d)\) be a complete metric space and \(T\) is \(\alpha\)-admissible mapping. For \(r > 0\), \(x_0 \in B(x_0, r)\) and \(\psi \in \Psi\), assume that,

\[
x, y \in B(x_0, r), \alpha(x, y) \geq 1 \\
\Rightarrow d(Tx, Ty) \leq \psi(N(x, y)),
\]

and

\[
\sum_{i=0}^{j} \psi_i(d(x_0, Tx_0)) \leq r, \text{ for all } j \in N.
\]

Suppose that the following assertions hold:

- \(\alpha(x_0, Tx_0) \geq 1\);
- for any sequence \(\{x_n\}\) in \(B(x_0, r)\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n \in N \cup \{0\}\) and \(x_n \to u \in B(x_0, r)\) as \(n \to +\infty\) then \(\alpha(x_n, u) \geq 1\) for all \(n \in N \cup \{0\}\).

Then, there exists a point \(x^*\) in \(B(x_0, r)\) such that \(Tx^* = x^*\).

If \(N(x, y) = d(x, y)\), we obtain the following corollary.

Corollary 15. Let \((X, d)\) be a complete metric space and \(T\) is \(\alpha\)-admissible mapping with respect to \(\eta\). For \(r > 0\), \(x_0 \in B(x_0, r)\) and \(\psi \in \Psi\), assume that,

\[
x, y \in B(x_0, r), \alpha(x, y) \geq \eta(x, y) \\
\Rightarrow d(Tx, Ty) \leq \psi(d(x, y)),
\]

and
Suppose that the following assertions hold:
• $\alpha(x_0, T x_0) \geq \eta(x_0, T x_0)$.
• for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0, r)}$ as $n \to +\infty$ then $\alpha(x_n, u) \geq \eta(x_n, u)$ for all $n \in N \cup \{0\}$.

Then, there exists a point $x^*$ in $\overline{B(x_0, r)}$ such that $T x^* = x^*$.

If $\eta(x, y) = 1$, $N(x, y) = d(x, y)$ in the corollary 11, we have the following corollary.

**Corollary 16.** Let $(X, d)$ be a complete metric space and $T$ is $\alpha$-admissible mapping. For $r > 0$, $x_0 \in \overline{B(x_0, r)}$ and $\psi \in \Psi$, assume that,

$$x, y \in \overline{B(x_0, r)}, \alpha(x, y) \geq 1$$

$$\Rightarrow d(T x, T y) \leq \psi(d(x, y))$$

and

$$\sum_{i=0}^{j} \psi^j(d(x_0, T x_0)) \leq r, \text{ for all } j \in N.$$ 

Suppose that the following assertions hold:
• $\alpha(x_0, T x_0) \geq 1$.
• for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0, r)}$ as $n \to +\infty$ then $\alpha(x_n, u) \geq 1$ for all $n \in N \cup \{0\}$.

Then, there exists a point $x^*$ in $\overline{B(x_0, r)}$ such that $T x^* = x^*$.

### 3. Fixed Point Results for Graphic Contractions

Consistent with Jachymski [13], let $(X, d)$ be a metric space and $\Delta$ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph $G$ such that the set $V(G)$ of its vertices coincides with $X$ and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume $G$ has no parallel edges, so we can identify $G$ with the pair $(V(G), E(G))$. Moreover, we may treat $G$ as a weighted graph (see [13]) by assigning to each edge the distance between its vertices. If $x$ and $y$ are vertices in a graph $G$, then a path in $G$ from $x$ to $y$ of length $m$ ($m \in N$) is a sequence $\{x_i\}_{i=0}^{m}$ of $m + 1$ vertices such that $x_0 = x, x_m = y$ and $(x_{i-1}, x_{i}) \in E(G)$ for $i = 1, ..., m$. A graph $G$ is connected if there is a path between any two vertices. $G$ is weakly connected if $\overline{G}$ is connected (see for details [1,5,12,13]).

**Definition 17.** ([13]). We say that a mapping $T : X \to X$ is a Banach $G$-contraction or simply $G$-contraction if $T$ preserves edges of $G$, i.e.,

$$\forall x, y \in X, (x, y) \in E(G) \Rightarrow (T x, T y) \in E(G)$$

and $T$ decreases weights of edges of $G$ in the following way:

$$\exists k \in (0, 1), \forall x, y \in X, (x, y) \in E(G)$$

$$\Rightarrow d(T x, T y) \leq k d(x, y).$$

Now we extend concept of $G$-contraction as follows.

**Definition 18.** Let $(X, d)$ be a metric space endowed with a graph $G$ and $T : X \to X$ be self-mappings. Assume that for $r > 0$, $x_0 \in \overline{B(x_0, r)}$ and $\psi \in \Psi$, following conditions hold,

$$\forall x, y \in \overline{B(x_0, r)}, (x, y) \in E(G) \Rightarrow (T x, T y) \in E(G)$$

$$\forall x, y \in \overline{B(x_0, r)}, (x, y) \in E(G) \Rightarrow d(T x, T y) \leq \psi(M(x, y)).$$

where

$$M(x, y) = \max \left\{ \frac{d(x, y) + d(x, T x) + d(y, T y)}{2}, \frac{d(x, T x) + d(y, T y)}{2} \right\}.$$ 

Then the mappings $T$ is called ciric $\psi$-graphic contractive mappings. If $\psi(t) = kt$ for some $k \in (0, 1)$, then we say $T$ is $G$-contractive mappings.

**Definition 19.** Let $(X, d)$ be a metric space endowed with a graph $G$ and $T : X \to X$ be self-mappings. Assume that for $r > 0$, $x_0 \in \overline{B(x_0, r)}$ and $\psi \in \Psi$, following conditions hold,

$$\forall x, y \in \overline{B(x_0, r)}, (x, y) \in E(G) \Rightarrow (T x, T y) \in E(G)$$

$$\forall x, y \in \overline{B(x_0, r)}, (x, y) \in E(G) \Rightarrow d(T x, T y) \leq \psi(N(x, y)).$$

where

$$N(x, y) = \max \left\{ \frac{d(x, y) + d(x, T x) + d(y, T y)}{2}, \frac{d(x, T x) + d(y, T y)}{2} \right\}.$$ 

Then the mappings $T$ is called ciric $\psi$-graphic contractive mappings. If $\psi(t) = kt$ for some $k \in (0, 1)$, then we say $T$ is $G$-contractive mappings.

**Theorem 20.** Let $(X, d)$ be a complete metric space endowed with a graph $G$ and $T : X \to X$ be ciric $\psi$-graphic contractive mappings and $x_0 \in \overline{B(x_0, r)}$. Suppose that the following assertions hold:

• $(x_0, T x_0) \in E(G)$ and $\sum_{i=0}^{j} \psi^j(d(x_0, T x_0)) \leq r$ for all $j \in N$;

• if $\{x_n\}$ is a sequence in $\overline{B(x_0, r)}$ such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in N$.

Then $T$ has a fixed point.
Proof. Define, $\alpha : X^2 \to (-\infty, +\infty)$ by
\[
\alpha(x,y) = \begin{cases} 
1, & \text{if } (x,y) \in E(G) \\
0, & \text{otherwise} 
\end{cases}
\]

First we prove that the mapping $T$ is $\alpha$-admissible. Let $x,y \in \overline{B(x_0, r)}$ with $\alpha(x,y) \geq 1$, then $(x,y) \in E(G)$. As $T$ is ciric $\psi$-graphic contractive mappings, we have, $(Tx, Ty) \in E(G)$. That is, $\alpha(Tx, Ty) \geq 1$. Thus $T$ is $\alpha$-admissible mapping. From (i) there exists $x_0$ such that $(x_0, Tx_0) \in E(G)$. That is, $\alpha(x_0, Tx_0) \geq 1$. If $x, y \in \overline{B(x_0, r)}$ with $\alpha(x,y) \geq 1$, then $(x, y) \in E(G)$. Now, since $T$, is ciric $\psi$-graphic contractive mapping, so $d(Tx, Ty) \leq \psi(M(x,y))$. That is,
\[
\alpha(x,y) \geq 1 \Rightarrow d(Tx, Ty) \leq \psi(M(x,y)).
\]

Let $\{x_n\} \subset \overline{B(x_0, r)}$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N$. Then, $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$. So by (ii) we have, $(x_n, x) \in E(G)$ for all $n \in N$. That is, $\alpha(x_n, x) \geq 1$. Hence, all conditions of Corollary 9 are satisfied and $T$ has a fixed point.

**Corollary 21.** Let $(X,d)$ be a complete metric space endowed with a graph $G$ and and $T : X \to X$ be a mapping. Suppose that the following assertions hold:
- $T$ is Banach $G$-contraction on $\overline{B(x_0, r)}$;
- $(x_0, Tx_0) \in E(G)$ and $d(x_0, Tx_0) \leq (1-k)r$;
- if $\{x_n\}$ is a sequence in $\overline{B(x_0, r)}$ such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in N$.

Then $T$ has a fixed point.

**Corollary 22.** Let $(X,d)$ be a complete metric space endowed with a graph $G$ and and $T : X \to X$ be a mapping. Suppose that the following assertions hold:
- $T$ is Banach $G$-contraction on $X$ and there is $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;
- if $\{x_n\}$ is a sequence in $X$ such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$ and $x_n \to x$ as $n \to +\infty$, then $(x_n, x) \in E(G)$ for all $n \in N$.

Then $T$ has a fixed point.

**Conflict of Interests**

The authors declare that they have no competing interests.

**References**

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