On Fixed Points for Chatterjea’s Maps in b-Metric Spaces

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Abstract In this paper we find sufficient conditions for the existence and uniqueness of fixed points of Chatterjea’s maps in b-metric space. These conditions do not involve the b-metric constant. We establish a priori error estimate for the sequence of successive iterations. The error estimate, which we present is better that the well-known one for a wide class of Chatterjea’s maps in metric spaces.

Keywords: fixed point, Chatterjea’s map, b-Metric space, a priori error estimate


1. Introduction

Fixed point theory has got wide applications in different branches of mathematics. Since the work of S. Banach [3] known as the Banach Contraction Principle, many mathematicians have extended and generalized the results in [3]. Some of the classical generalizations of [3] are presented in [14]. The concept of a b-metric space as a generalization of a metric space is introduced in [2] and a contraction mapping theorem is proved there. Since then results about fixed points, variational principles and applications were obtained in b-metric spaces. We will cite just a few recent results in these directions [1,5,7,8,9,10,11,12,13,16].

We recall some definitions and properties for b-metric spaces [12,13,16].

Definition 1.1. Let $X$ be a non-empty set, $s \geq 1$. A functional $\rho: X \times X \to \mathbb{R}$ is called a b-metric if it satisfies the following conditions:

$\rho(x,y) \geq 0$ for all $x,y \in X$ and $\rho(x,y) = 0$ iff $x = y$;

$\rho(x,y) = \rho(y,x)$ for all $x,y \in X$;

$\rho(x,y) \leq s(\rho(x,z) + \rho(z,y))$ for all $x,y,z \in X$.

The ordered pair $(X, \rho)$ is called a b-metric space (with constant $s$).

Any metric space is a b-metric space with $s = 1$. An example of b-metric space is $\mathbb{R}$ endowed with the b-metric function $\rho_p(x,y) = |x-y|^p$ for $p \in [1, +\infty)$. It is easy to see that in this case $s = 2^{p-1}$.

Other classical example of b-metric space is $\mathbb{R}$ endowed with the b-metric function $\rho_p(x,y) = |x-y|^p$ for $p \in [1, +\infty)$. It is easy to see that in this case $s = 2^{p-1}$ and for $p = 1$ we get the metric space of the real numbers with a metric $\rho_1(x,y) = |x-y|$.

Definition 1.2. Let $(X, \rho)$ be a b-metric space.

A sequence $\{x_n\}_{n=1}^\infty$ is called $\rho$-convergent if there exists $x \in X$, such that for any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that the inequality $\rho(x_n, x) < \varepsilon$ holds true for all $n \geq N$.

A sequence $\{x_n\}_{n=1}^\infty$ is called $\rho$-Cauchy sequence if for any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that the inequality $\rho(x_n, x_m) < \varepsilon$ holds true for all $n > m \geq N$.

The b-metric space $(X, \rho)$ is called complete b-metric space if any Cauchy sequence is convergent;

A subset $A \subseteq X$ is called $\rho$-bounded if $\sup \{\rho(x,y) : x,y \in A\} < \infty$;

If the set $A$ is $\rho$-bounded then the number $\sup \{\rho(x,y) : x,y \in A\}$ is called its $\rho$-diameter and is denoted with $\delta_b(A)$.

A subset $A \subseteq X$ is called $\rho$-closed if for any convergent sequence $\{x_n\}_{n=1}^\infty \subseteq A$ the convergence $\lim_{n \to \infty} x_n = x$ implies $x \in A$.

A b-metric function $\rho$ is called continuous if for any $y \in X$ and any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon,y) > 0$ such that there holds the inequality $|\rho(x,y) - \rho(y,z)| < \varepsilon$ provided that $\rho(x,z) < \delta$. It is easy to observe that if $\rho$ is continuous and $x_n$ is $\rho$-convergent to $x$ then $\rho(y,x_n) \to \rho(y,x)$.
Every $b$-convergent sequence in $b$-metric space is a $b$-Cauchy sequence. If a sequence is a $b$-convergent in $b$-metric space then its limit is unique. In general a $b$-metric function is not continuous [5,10].

As far as we will consider only $b$-metrics we will omit the letter $b$ in the above definitions.

**Definition 1.3.** ([14]) Let $(X, \rho)$ be a metric space. A map $T : X \to X$ is a Hardy Rogers map is there exist nonnegative constants $a_i$ , $i=1,2,3,4,5$ satisfying

$$\sum_{i=1}^{5} a_i < 1$$

such that for each $x,y \in X$ the inequality

$$\rho(Tx,Ty) \leq a_1 \rho(x,y) + a_2 \rho(x,Tx) + a_3 \rho(y,Ty) + a_4 \rho(x,Ty) + a_5 \rho(y,Ty)$$

holds for all $x,y \in X$.

As pointed in [15] from the symmetry of the function $\rho$ it follows that $a_3 = a_2$ and $a_4 = a_5$. Therefore if $T$ is a Hardy-Rogers contraction then exist $k_1, k_2, k_3 \geq 0$, such that $k_1 + 2k_2 + 2k_3 < 1$ and there holds the inequality

$$\rho(Tx,Ty) \leq k_1 \rho(x,y) + k_2 (\rho(x,Tx) + \rho(y,Ty)) + k_3 (\rho(x,Ty) + \rho(y,Tx))$$

Generalizations of Hardy Rogers map in $b$-metric space are investigated in [8,13].

If $k_1 = k_2 = 0$ and $k_3 \in (0,1/2)$ in the above inequality we get a generalization of Chatterjea’s map [6] in $b$-metric space.

**Definition 1.4.** Let $(X, \rho)$ be a $b$-metric space. A map $T : X \to X$ is called Chatterjea’s map if there exists $k \in [0,1/2)$ such that the inequality

$$\rho(Tx,Ty) \leq k (\rho(Tx,y) + \rho(Ty,x))$$

holds for all $x,y \in X$.

We will denote for the rest of the article $\alpha = \frac{k}{1-k}$, where $k$ is the constant from the definition of Chatterjea’s map. From $k \in [0,1/2)$ it follows that $\alpha \in [0,1)$.

**2. Fixed Points for Chatterjea’s Maps in $b$-Metric Spaces**

**Theorem 2.1.** Let $(X, \rho)$ be a complete $b$-metric space, $\rho$ be a continuous function, $T : X \to X$ be a Chatterjea’s map, such that the inequality $\sup_{n \in \mathbb{N}} \{ \rho(T^n x,x) \} < \infty$ holds for any $x \in X$. Then

(i) there exists a unique fixed point say $\xi$ of $T$;

(ii) for any $x_0 \in A$ the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $\xi$, where $x_{n+1} = Tx_n$ , $n = 0,1,2,...$;

(iii) there holds the a priori error estimate

$$\rho(\xi,T^m x) \leq \alpha^m \sup_{j \in \mathbb{N}} \rho(T^j x,x).$$

**Lemma 2.2.** Let $(X, \rho)$ be a $b$-metric space and let $T : X \to X$ be a Chatterjea’s map. Then for any $x \in X$ there holds the inequality

$$\rho(T^n x,x) \leq \left( \frac{k}{1-k} \right)^m \sup_{2 \leq j \leq n} \{ \rho(T^j x,x) \}$$

(2.2)

for any $n > m \geq 1$.

**Proof.** Let us denote $r_n(x) = \rho(T^n x,x)$ and $x_{m,n} = \rho(T^m x,T^n x)$. We consider the sequence

$$x_{2,1}, x_{3,1}, x_{3,2}, ..., x_{n-1,n-2}, x_{n-1,n}, x_{n,2}, ..., x_{n-1,n-1}, x_{n+1,1}, ...$$

(2.3)

We will prove inequality (2.2) by induction on the sequence (2.3). Let us denote by $i$ the sum of the indices of the sequence in (2.3).

Let $i = 3$ , i.e. $n = 2$ and $m = 1$. Then

$$x_{2,1} \leq k(2 \rho(T^2 x,x)).$$

Let $i = 4$ , i.e. $n = 3$ and $m = 1$. Then

$$x_{3,1} \leq k^2 (x_{3,1} + x_{2,1}) \leq k (1 + \frac{k}{1-k}) \sup_{2 \leq j \leq 3} r_j(x)$$

$$= \frac{k}{1-k} \sup_{2 \leq j \leq 3} \rho(T^j x,x).$$

Let inequality (2.2) holds for $i = p$.

We will prove that (2.2) holds true for $i = p + 1$. Let $n + m = p$. There are two cases: If $m < n$ then we consider $x_{n,m+1}$ , if $m = n - 1$ then we consider $x_{n+1,1}$.

Case I) There are two subcases: $m < n - 2$ and $m = n - 2$.

Let first $m < n - 2$. Then

$$x_{n,m+1} \leq k (x_{n,m} + x_{n-1,m+1})$$

$$\leq k \left( \frac{k}{1-k} \right)^m \sup_{2 \leq j \leq n} r_j(x)$$

$$+ \left( \frac{k}{1-k} \right)^{m+1} \sup_{2 \leq j \leq n-1} r_j(x)$$

$$= k \left( \frac{k}{1-k} \right)^m \left( 1 + \frac{k}{1-k} \right) \sup_{2 \leq j \leq n} r_j(x)$$

$$= \left( \frac{k}{1-k} \right)^{m+1} \sup_{2 \leq j \leq n} \rho(T^j x,x).$$

Let now $m = n - 2$. Then

$$x_{n,m+1} \leq k (x_{n,m} + x_{n-1,m+1}) = k x_{n,m}$$

$$\leq k \left( \frac{k}{1-k} \right)^m \sup_{2 \leq j \leq n} r_j(x)$$

$$= \left( \frac{k}{1-k} \right)^{m+1} \sup_{2 \leq j \leq n} \rho(T^j x,x).$$

Case II)
\[ x_{n+1} \leq k \left( r_{n+1}(x) + x_{n+1} \right) \]

\[ \leq k \left( \sup_{2 \leq j \leq n+1} r_j(x) + \frac{k}{1-k} \sup_{2 \leq j \leq n} r_j(x) \right) \]

\[ = k \left( \frac{1}{1-k} \sup_{2 \leq j \leq n+1} r_j(x) \right) \]

\[ = \frac{k}{1-k} \sup_{2 \leq j \leq n+1} \rho(T^jx, x). \]

**Proof of Theorem 2.1** (i) Let \( x \in X \) be arbitrary.

Let us put \( M = \sup_{j \geq 2} \rho(T^jx, x) \). From Lemma 2.2 we have that the inequality

\[ \rho(T^n, T^m) \leq \alpha^m \sup_{2 \leq j \leq n} \rho(T^j, x) \leq \alpha^m M \]

holds for every \( n > m \geq 1 \). Consequently the sequence \( \{T^n\}_{n=1}^{\infty} \) is a Cauchy sequence. From the assumption that \( X \) is complete b-metric space it follows that the sequence \( \{T^n\}_{n=1}^{\infty} \) is b-convergent. Therefore it follows that there exists \( \xi = \lim T^n x \in X \). Let us fix \( n \in \mathbb{N} \). After taking a limit on \( m \to \infty \) from the assumption that the b-metric is continuous and using that \( T \) is Chatterjea’s map we get the inequality

\[ \rho(T^n, T^m) = \lim_{m \to \infty} \rho(T^n, T^m) \leq \lim_{m \to \infty} \left( k \left( \rho(T^n, T^{m-1}) + \rho(T^{m}, x) \right) \right) \]

\[ = k \left( \rho(T^n, x) + \rho(T^{m}, x) \right) = k \rho(T^n, x) \]

and therefore \( \rho(T^n, x) = 0 \) i.e. \( \xi \) is a fixed point for \( T \).

Let suppose that there are two fixed points \( \xi \neq \eta \). Then from the inequality

\[ \rho(\xi, \eta) = \rho(T^n\xi, T^n\eta) \leq k(\rho(T^n\xi, \eta) + \rho(T^n\eta, \xi)) \]

\[ = 2k \rho(\xi, \eta) \]

and the assumption that \( k \leq 0.1/2 \) it follows that \( \xi = \eta \).

(ii) The proof follows from (i), because any sequence \( \{T^n x_0\}_{n=1}^{\infty} \) is convergent to the fixed point of \( T \), which is unique.

(iii) Let \( x \in X \) be arbitrary. From Lemma 2.2 we have the inequality

\[ \rho(T^n, T^m) \leq \alpha^m \sup_{j \in \mathbb{N}} \rho(T^j, x) \]

holds for every \( n > m \geq 1 \) and every \( x \in X \). From (ii) it follows that the sequence \( \{T^n x\}_{n=1}^{\infty} \) converges to the unique fixed point \( \xi \). Therefore using the continuity of \( \rho \) and Lemma 2.2 we get

\[ \rho(\xi, T^m x) = \lim_{n \to \infty} \rho(T^n, T^m x) \leq \alpha^m \sup_{j \in \mathbb{N}} \rho(T^j, x). \]

As far as any metric space is a b-metric space, then Theorem 2.1 holds true for arbitrary metric space. If \( (X, d) \) is a complete metric space and \( T \) be Chatterjea’s map then the a priori error estimate is well known [4]

\[ d(\xi, T^n x) \leq \frac{\alpha^m}{1 - k} d(Tx, x). \]

If we assume that \( \sup \rho(T^n x, x) \leq \rho(Tx, x) \) then we will get from Theorem 2.1 the a priori estimate

\[ \rho(\xi, T^n x) \leq \alpha^m \rho(Tx, x). \]

Let us mention that in this case the a priori estimate (2.5) is better than (2.4).

Let \( \epsilon \in (0, \rho(Tx, x)) \), \( m_{\alpha} \in \mathbb{N} \) be the smallest number, that satisfies (2.5) and \( n_{\alpha} \in \mathbb{N} \) be the smallest number, that satisfies (2.4). Then

\[ n_{\alpha} - m_{\alpha} \geq \frac{\log \left( \frac{1 - \alpha}{\rho(Tx, x)} \right)}{\log \alpha} + 1 \]

\[ = \frac{\log(1 - \alpha)}{\log \alpha} - 1. \]

If \( k \) gets close to \( 1/2 \) then \( \alpha \) gets closer to \( 1 \) and therefore \( n_{\alpha} - m_{\alpha} \) gets closer to infinity.

We would like to point out that if the space is a metric space than using the triangle inequality we can obtain (2.5) from (2.1).

**Example 2.3.** Let us consider the b-metric space \( (\mathbb{R}, \rho_p) \) for \( p \geq 1 \). Let \( 0 < \alpha < \beta \) be two arbitrary positive real numbers. Let us define the map \( T_\alpha^\beta : [0, +\infty) \to [0, +\infty) \), by \( T_\alpha^\beta x = \left\{ \begin{array}{ll} \alpha, & x \in [\beta, +\infty) \\ 0, & x \in [0, \beta) \end{array} \right. \)

(Figure 1), which is a variation of the classical examples from [14]. It is well known that \( T_{\alpha/2}^\beta \) is Chatterjea’s map and \( T_{1/2}^\beta \) is not Chatterjea’s map in the metric space \( (\mathbb{R}, \rho_p) \) [14]. It is easy to observe that the Picard iteration sequence \( x_n = T_{\alpha/2}^\beta x_{n-1} \) converges to the fixed point \( x = 0 \) for any initial point \( x_0 \in [0, +\infty) \).

![Figure 1](image.png)

If \( x, y \in [0, \beta) \) or \( x, y \in [\beta, +\infty) \), then \( T_\alpha^\beta \) satisfies the condition in Definition 1.4 for any \( k \in [0, \frac{1}{2}) \), because
\[\rho_p(Tx, Ty) = |x - y|^p = 0.\] If \( y \in [0, \beta) \) and \( x \in [\beta, +\infty) \), then we get \( \rho_p(Tx, Ty) = |x - y|^p + \rho^p \) and \( \rho_p(Tx, Ty) = \alpha^p \). Using the inequality

\[
\inf \left\{ |x - y|^p + x^p : y \in [0, \beta), x \in [\beta, +\infty) \right\} = \beta^p
\]

we get that there holds \( \rho_p(Tx, Ty) = \alpha^p \leq k \beta^p \leq k \left( \rho_p(Tx, Ty) + \rho_p(Ty, x) \right) \).

Consequently for any map \( T_\alpha^\beta \) we can endow \((\mathbb{R}, \rho_\alpha)\) with a suitable b-metric \( \rho_p(x - y) = |x - y|^p \) so that \( T_\alpha^\beta \) to satisfy the condition in Definition 1.4 in \((\mathbb{R}, \rho_\alpha)\).

Let us consider the particular case \( 2\alpha \geq \beta \) and \( p > 1 \).

If we choose in this case \( k \geq \left( \frac{\alpha}{\beta} \right)^p \geq \left( \frac{1}{2} \right)^p \in \left[ 0, \frac{1}{2} \right) \), provided that we have considered the b-metric space \((\mathbb{R}, \rho_p)\), \( p > 1 \), then \( k \alpha \geq \frac{1}{2} \), because \( s = 2^{p-1} \) in \((\mathbb{R}, \rho_p)\). Consequently \( T_\alpha^\beta \) does not satisfy the conditions in (16) Theorem 3) for any \( p \in (0, +\infty) \) in \((\mathbb{R}, \rho_p)\) and thus Theorem 2.1 extends (12) Theorem 3) in the case when \( n \in \mathbb{N} \).

In the particular case \( T_{1/2} \) we get that \( k \alpha = \frac{1}{2} \), provided that \( k \) is chosen so that inequality (2.6) to hold in \((\mathbb{R}, \rho_p)\) and therefore (12) Theorem 3) could not be applied.

When applying fixed point theorems for approximating of a solution of the equation \( Tx = x \) we usually find an initial starting point \( x_0 \), which belongs to a neighborhood \( U \) of the solution \( \xi \), such that \( T : U \to U \) and \( U \) is bounded and closed. Thus the next Corollary can be applied in a wide class of problems.

**Corollary 2.3.** Let \((X, \rho)\) be a complete b-metric space, \( \rho \) be a continuous function, \( A \subseteq X \) be a b-bounded and b-closed set, \( T : A \to A \) be Chatterjea’s map. Then there exists a unique fixed point say \( \xi \) of \( T \); for any \( x_0 \in A \) the sequence \( \{x_n\}_{n=1}^{\infty} \) converges to \( \xi \), where \( x_{n+1} = Tx_n \), \( n = 0, 1, 2, \ldots \);

there holds the a priori error estimate

\[
\rho \left( \xi, T^n x \right) \leq a_\alpha^m \rho_b(A).
\]

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