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On Semi-symmetric Para Kenmotsu Manifolds

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Abstract In this paper we study some remarkable properties of para Kenmotsu (briefly p-Kenmotsu) manifolds satisfying the conditions $R(X,Y).R=0$, $R(X,Y).P=0$ and $P(X,Y).R=0$, where R(X, Y) is the Riemannian curvature tensor and P(X, Y) is the Weyl projective curvature tensor of the manifold. It is shown that a semi-symmetric p-Kenmotsu manifold (M_n, g) is of constant curvature and hence is an sp-Kenmotsu manifold. Also, we obtain the necessary and sufficient condition for a p-Kenmotsu manifold to be Weyl projective semi-symmetric and shown that the Weyl projective semi-symmetric p-Kenmotsu manifold is projectively flat. Finally we prove that if the condition $P(X,Y).R=0$ is satisfied on a p-Kenmotsu manifold then its scalar curvature is constant.

Keywords: *para Kenmotsu manifolds, curvature tensor, projective curvature tensor, scalar curvature*

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1. Introduction

The notion of an almost para-contact Riemannian manifold was introduced by Sato [7] in 1976. After that, T. Adati and K. Matsumoto [1] defined and studied *p*-Sasakian and *sp*-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu [6] defined a class of almost contact Riemannian manifolds. In 1995, Sinha and Sai Prasad [9] have defined a class of almost para-contact metric manifolds namely para-Kenmotsu (briefly *p*-Kenmotsu) and special para Kenmotsu (briefly *sp*-Kenmotsu) manifolds. In a recent paper, the authors Satyanarayana and Sai Prasad [8] studied conformally symmetric p-Kenmotsu manifolds, that is the *p*-Kenmotsu manifolds satisfying the condition $R(X,Y).C=0$, and they prove that such a manifold is conformally flat and hence is an *sp*-Kenmotsu manifold, where R is the Riemannian curvature and C is the conformal curvature tensor defined by

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\begin{bmatrix} g(Y,Z)QX - g(X,Z)QY \\ +S(Y,Z)X - S(X,Z)Y \end{bmatrix} + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \quad (1.1)$$

Here S is the Ricci tensor, r is the scalar curvature and Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S [3] i.e.,

$$g(QX,Y) = S(X,Y). \quad (1.2)$$

A Riemannian manifold M is locally symmetric if its curvature tensor R satisfies ∇ R = 0, where ∇ is Levi-Civita connection of the Riemannian metric [4]. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M_n is said to be semi-symmetric if its curvature tensor R satisfies $R(X,Y).R=0$ where $R(X,Y)$ acts on R as derivation [10]. Locally symmetric and semi-symmetric *p*-Sasakian manifolds are widely studied by many geometers [2,5].

In this study, we consider the *p*-Kenmotsu manifolds satisfying the conditions $R(X,Y).R=0$, known as semi-symmetric *p*-Kenmotsu manifolds, where $R(X,Y)$ is considered as a derivation of tensor algebra at each point of the manifold for tangent vectors X and Y and the *p*-Kenmotsu manifolds (M_n, g) (n > 2) satisfying the condition $R(X,Y).P=0$, where P denotes the Weyl projective curvature tensor [12] defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\begin{bmatrix} g(Y,Z)QX \\ -g(X,Z)QY \end{bmatrix}. \quad (1.3)$$

Here we consider the *p*-Kenmotsu manifolds M_n for $n > 2$; as if for $n = 2$, the projective curvature tensor identically vanishes.

In section 3, it is shown that a semi-symmetric *p*-Kenmotsu manifold (M_n, g) of constant curvature is an *sp*-Kenmotsu manifold. In the next section we obtain the necessary and sufficient condition for a *p*-Kenmotsu manifold to be Weyl projective semi-symmetric and shown that the Weyl projective semi-symmetric p-Kenmotsu manifold is projectively flat. Finally we prove that if the condition $P(X,Y).R=0$ is satisfied on a p-Kenmotsu manifold then its scalar curvature is constant.

2. *p*-Kenmotsu Manifolds

Let M_n be an *n*-dimensional differentiable manifold equipped with structure tensors (Φ, ξ, η) where Φ is a tensor of type (1,1), ξ is a vector field, η is a 1-form such that

$$\eta(\xi) = 1 \tag{2.1}$$

$$\Phi^2(X) = X - \eta(X)\xi; \overline{X} = \Phi X. \tag{2.2}$$

Then M_n is called an almost para contact manifold.

Let g be the Riemannian metric in an n-dimensional almost para-contact manifold M_n such that

$$g(X,\xi) = \eta(X) \tag{2.3}$$

$$\Phi\xi = 0, \eta(\Phi X) = 0, \text{rank } \Phi = n-1 \tag{2.4}$$

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.5}$$

for all vector fields X and Y on M_n. Then the manifold M_n [7] is said to admit an almost para-contact Riemannian structure (Φ, ξ, η, g) and the manifold is called an almost para-contact Riemannian manifold.

A manifold of dimension $'n'$ with Riemannian metric $'g'$ admitting a tensor field $'\Phi'$ of type (1, 1), a vector field $'\xi'$ and a 1-form $'\eta'$ satisfying (2.1), (2.3) along with

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \tag{2.6}$$

$$\begin{aligned}(\nabla_X \nabla_Y \eta)Z &= [-g(X,Z) + \eta(X)\eta(Z)]\eta(Y) \\ &\quad + [-g(X,Y) + \eta(X)\eta(Y)]\eta(Z)\end{aligned} \tag{2.7}$$

$$\nabla_X \xi = \Phi^2 X = X - \eta(X)\xi \tag{2.8}$$

$$(\nabla_X \Phi)Y = g(\Phi X, Y)\xi - \eta(Y)\Phi X \tag{2.9}$$

is called a para-Kenmotsu manifold or briefly p-Kenmotsu manifold [9].

A p-Kenmotsu manifold admitting a 1-form $'\eta'$ satisfying

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) \tag{2.10}$$

$$\begin{aligned}&g(X,\xi) = \eta(X) \text{ and } (\nabla_X \eta)Y = \phi(\overline{X}, Y), \\ &\text{where } \phi \text{ is an associate of } \Phi,\end{aligned} \tag{2.11}$$

is called a special p-Kenmotsu manifold or briefly sp-Kenmotsu manifold [9].

It is known that [9] in a p-Kenmotsu manifold the following relations hold:

$$S(X,\xi) = -(n-1)\eta(X) \text{ where } g(QX,Y) = S(X,Y) \tag{2.12}$$

$$\begin{aligned}&g[R(X,Y)Z,\xi] = \eta[R(X,Y,Z)] \\ &= g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\end{aligned} \tag{2.13}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi \tag{2.14}$$

$$\begin{aligned}&R(X,Y,\xi) = \eta(X)Y - \eta(Y)X; \\ &\text{when } X \text{ is orthogonal to } \xi\end{aligned} \tag{2.15}$$

where S is the Ricci tensor and R is the Riemannian curvature.

Moreover, it is also known that if a p-Kenmotsu manifold is projectively flat then it is an Einstein manifold and the scalar curvature has a negative constant value $-n(n-1)$. Especially, if a p-Kenmotsu manifold is of constant curvature, the scalar curvature has a negative constant value $-n(n-1)$ [9]. In this case,

$$S(Y,Z) = -(n-1)g(Y,Z) \tag{2.16}$$

and hence

$$S(\Phi Y, \Phi Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z). \tag{2.17}$$

Also, if a p-Kenmotsu manifold is of constant curvature, we have

$$'R(X,Y,Z,P) = \frac{1}{(n-1)}\begin{bmatrix} S(Y,Z)g(X,P) \\ -S(X,Z)g(Y,P)\end{bmatrix}. \tag{2.18}$$

The above results will be used further in the next sections.

3. *p*-Kenmotsu Manifolds Satisfying $R(X,Y).R = 0$

In this section, we consider semi-symmetric p-Kenmotsu manifolds, i.e., p-Kenmotsu manifolds satisfying the conditions $R(X,Y).R = 0$ where $R(X,Y)$ is considered as a derivation of tensor algebra at each point of the manifold for tangent vectors X and Y. Now

$$\begin{aligned}&(R(X,Y)\cdot R)(U,V)W \\ &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &\quad - R(U,R(X,Y)V)W - R(U,V)R(X,Y)W.\end{aligned} \tag{3.1}$$

Putting $X = \xi$ in (3.1), and on using the condition $R(X,Y).R = 0$, we get

$$\begin{aligned}&g(R(\xi,Y)R(U,V)W,\xi) - g(R(R(\xi,Y)U,V)W,\xi) \\ &-g(R(U,R(\xi,Y)V)W,\xi) - g(R(U,V)R(\xi,Y)W,\xi) \\ &= 0.\end{aligned} \tag{3.2}$$

By using the equations (2.3) and (2.14), from (3.2) we get

$$\begin{aligned}&'R(U,V,W,Y) - \eta(Y)\eta(R(U,V)W) \\ &+\eta(U)\eta(R(Y,V)W) + \eta(V)\eta(R(U,Y)W) \\ &+\eta(W)\eta(R(U,V)Y) - g(Y,U)\eta(R(\xi,V)W) \\ &-g(Y,V)\eta(R(U,\xi)W) - g(Y,W)\eta(R(U,V)\xi) = 0\end{aligned} \tag{3.3}$$

where $'R(U,V,W,Y) = g(R(U,V)W,Y)$.

On putting $Y = U$ in (3.3), we get

$$\begin{aligned}&'R(U,V,W,U) + \eta(V)\eta(R(U,U)W) \\ &+\eta(W)\eta(R(U,V)U) - g(U,U)\eta(R(\xi,V)W) \\ &-g(U,V)\eta(R(U,\xi)W) - g(U,W)\eta(R(U,V)\xi) = 0.\end{aligned} \tag{3.4}$$

Now putting $U = e_i$, where $\{e_i\}, i = 1,2,\cdots n$ is an orthogonal basis of the tangent space at any point, and taking the summation of (3.4) over i, $1 \le i \le n$, we get (2.16).

Also, using the equations (2.12), (2.16) and (3.3) we get (2.18), shows that the manifold is of constant curvature.

Thus we state the following result.

Theorem 3.1: A semi-symmetric p-Kenmotsu manifold is of constant curvature.

Now, from (2.16) and (2.18) we have

$$'R(X,Y,Z,P) = g(X,Z)g(Y,P) - g(Y,Z)g(X,P), \quad (3.5)$$

and from equations (2.16) and (2.5), we have

$$S(\Phi X,\Phi Y) = -(n-1)[g(X,Y) - \eta(X)\eta(Y)]. \quad (3.6)$$

On contraction of (3.6) with covariant tensor $\phi(X,Y) = g(\overline{X},Y)$, we get

$$\phi(\overline{X},Y) = g(X,Y) - \eta(X)\eta(Y),$$

shows that the manifold is an sp-Kemotsu one.

Thus, we state the following theorem.

Theorem 3.2: If a semi-symmetric p-Kenmotsu manifold (M_n, g) is of constant curvature, the manifold is an sp-Kenmotsu one.

4. *p*-Kenmotsu Manifolds Satisfying $R(X,Y).P = 0$

In this section, we consider Weyl projective semi-symmetric p-Kenmotsu manifolds, i.e., p-Kenmotsu manifolds satisfying the condition $R(X,Y).P = 0$. Now

$$\begin{aligned}&(R(X,Y)\cdot P)(U,V)W\\ &= R(X,Y)P(U,V)W - P(R(X,Y)U,V)W\\ &-P(U,R(X,Y)V)W - P(U,V)R(X,Y)W.\end{aligned} \quad (4.1)$$

Put $X = \xi$ in (4.1). Then the condition $R(X,Y).P = 0$ implies that

$$\begin{aligned}&g(R(\xi,Y)P(U,V)W,\xi) - g(P(R(\xi,Y)U,V)W,\xi)\\ &-g(P(U,R(\xi,Y)V)W,\xi) - g(P(U,V)R(\xi,Y)W,\xi)\\ &= 0.\end{aligned} \quad (4.2)$$

Then on using equations (2.12), (2.13) and (1.3), we get

$$\eta(P(X,Y)Z) = 0. \quad (4.3)$$

On the other hand, by using (2.3), (2.4), and (4.3), we get

$$g(R(\xi,Y)P(U,V)W,\xi) = -g(P(U,V)W,Y). \quad (4.4)$$

Then from equations (4.2) and (4.3), the left hand side of (4.4) is zero, gives that $g(P(U,V)W,Y) = 0$ for all U, V, W and Y and hence $P(X,Y) = 0$. This leads to the following theorem:

Theorem 4.1: A Weyl projective semi-symmetric p-Kenmotsu manifold is projectively flat.

But it is known that [11], a projectively flat Riemannian manifold is of constant curvature. Also it can be easily seen that a manifold of constant curvature is projectively falt. Hence we have the following theorem.

Theorem 4.2: A p-Kenmotsu manifold is Weyl projective semi-symmetric if and only if the manifold is of constant curvature.

Also it is known that a p-Kenmotsu manifold of constant curvature is an sp-Kenmotsu manifold [8]. Hence we conclude the following result:

Theorem 4.3: A Weyl projective semi-symmetric p-Kenmotsu manifold is of constant curvature and hence is an sp-Kenmotsu manifold.

It is trivial that in case of a projective symmetric Riemannian manifold the condition $R(X,Y).P = 0$ hold good.

5. *p*-Kenmotsu Manifolds Satisfying $P(X,Y).R = 0$

It is known that the condition $R(X,Y).P = 0$ does not imply $P(X,Y).R = 0$. In this section, we study the remarkable property of p-Kenmotsu manifolds satisfying the condition $P(X,Y).R = 0$.

Now, we have

$$\begin{aligned}&(P(X,Y)\cdot R)(U,V)W\\ &= P(X,Y)R(U,V)W - R(P(X,Y)U,V)W\\ &-R(U,P(X,Y)V)W - R(U,V)P(X,Y)W.\end{aligned} \quad (5.1)$$

Put $X = \xi$ in (5.1). Then the condition $P(X,Y).R = 0$ implies that

$$\begin{aligned}&g(P(\xi,Y)R(U,V)W,\xi) - g(R(P(\xi,Y)U,V)W,\xi)\\ &-g(R(U,P(\xi,Y)V)W,\xi) - g(R(U,V)P(\xi,Y)W),\xi\\ &= 0.\end{aligned} \quad (5.2)$$

Putting $X = \xi$, $Z = U$ in (1.3) and on using (2.12) and (2.13), we get

$$\begin{aligned}&\eta(R(P(\xi,Y)U,V))W\\ &= \eta(U)[\eta(R(Y,V)W) - \frac{1}{(1-n)}\eta(R(QY,V)W)].\end{aligned} \quad (5.3)$$

Similarly, by putting $X = \xi$, $Z = V$ in (1.3) and on using (2.12) and (2.13), we get

$$\begin{aligned}&\eta(R(U,P(\xi,Y)V)W\\ &= \eta(V)[\eta(R(U,Y)W) - \frac{1}{(1-n)}\eta(R(U,QY)W)].\end{aligned} \quad (5.4)$$

In similar by putting $X = \xi$, $Z = W$ in (1.3) and on using (2.12) and (2.13), we get

$$\begin{aligned}&\eta(R(U,V)P(\xi,Y)W)\\ &= \eta(W)[\eta(R(U,V)Y) - \frac{1}{(1-n)}\eta(R(U,V)QY)].\end{aligned} \quad (5.5)$$

On using (4.3), (5.3), (5.4) and (5.5), we get from eqn (5.2) that

$$\begin{aligned}&\eta(U)[\eta(R(Y,V)W) - \frac{1}{(1-n)}\eta(R(QY,V)W)]\\ &+\eta(V)[\eta(R(U,Y)W) - \frac{1}{(1-n)}\eta(R(U,QY)W)]\\ &+\eta(W)[\eta(R(U,V)Y) - \frac{1}{(1-n)}\eta(R(U,V)QY)] = 0.\end{aligned} \quad (5.6)$$

By putting $Y = U$ in eqn (5.6), we get

$$\eta(U)[\eta(R(U,V)W) - \frac{1}{(1-n)}\eta(R(QU,V)W)]$$
$$+\eta(V)[\eta(R(U,U)W) - \frac{1}{(1-n)}\eta(R(U,QU)W)] \quad (5.7)$$
$$+\eta(W)[\eta(R(U,V)U) - \frac{1}{(1-n)}\eta(R(U,V)QU)] = 0.$$

Then on using (2.12) and (2.13), we get

$$\eta(W)\begin{bmatrix} g(U,W)\eta(V) - g(V,W)\eta(U) \\ -\frac{1}{(1-n)}[S(U,U)\eta(V) - S(V,U)\eta(U)] \end{bmatrix} = 0. \quad (5.8)$$

Now putting $U = e_i$, where $i = 1, 2, \cdots n$ and taking the summation of (5.8) over i, $1 \le i \le n$, we get $r = n(n-1)$, since $\eta(V) \ne 0$, shows that the scalar curvature is constant.

Hence we have the following theorem.

Theorem 5.1: If a p -Kenmotsu manifold satisfies the condition $P(X,Y).R = 0$ then its scalar curvature is constant.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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