New Extensions of Some Known Special Polynomials under the Theory of Multiple q -Calculus

Mehmet Acikgoz¹, Serkan Araci^{2*}, Uğur Duran³

¹Department of Mathematics, Faculty of Arts and Science, University of Gaziantep, Gaziantep, Turkey

²Department of Economics, Faculty of Economics, Administrative and Social Science, Hasan Kalyoncu University, Gaziantep, Turkey

³Department of Mathematics, Faculty of Arts and Science, University of Gaziantep, Gaziantep, Turkey

*Corresponding author: mtsrkn@hotmail.com

Received June 12, 2015; Revised September 25, 2015; Accepted October 03, 2015

Abstract In the year 2014, the idea of multiple q -calculus was formulated and introduced in the book of Nalci and Pashaev [9] in which this idea is simple but elegant method in order to derive new generating functions of some special polynomials that are generalizations of known q -polynomials. In this paper, we will use Nalci and Pashaev's method in order to find a systematic study of new types of the Bernoulli polynomials, Euler polynomials and Genocchi polynomials. Also we will obtain recursive formulas for these polynomials.

Keywords: Quantum calculus, Multiple quantum calculus, q -Bernoulli polynomials, q -Euler polynomials, q -Genocchi polynomials, Generating function.

Cite This Article: Mehmet Acikgoz, Serkan Araci, and Uğur Duran, "New Extensions of Some Known Special Polynomials under the Theory of Multiple q -Calculus." *Turkish Journal of Analysis and Number Theory*, vol. 3, no.5 (2015): 128-139. doi: 10.12691/tjant-3-5-4.

$$[xy]_q = [x]_q [y]_q x \quad (q\text{-product rule}) \quad (1.4)$$

1. Introduction

1.1. q -Calculus. The usual quantum calculus (or recalled q -calculus) has been extensively studied for a long time by many mathematicians, physicists and engineers. The development of q -calculus stems from the applications in many fields such as engineering, economics, mathematics, and so on. One of the important branches of q -calculus is q -special polynomials. For example, Kim [18] constructed q -generalized Euler polynomials based on q -exponential function. Moreover, Srivastava *et al* investigated Apostol q -Bernoulli, Apostol q -Euler polynomials and Apostol q -Genocchi polynomials. This is why q -calculus is thought as one of the useful tools to study with special numbers and polynomials. For more information related these issues, see, e.g. [1,2,3,5,6,8-13,16-21].

Before starting at multiple q -calculus, we first give some basic notations about q -calculus which can be found in [3].

For a real number (or complex number) x , q -number (quantum number) is known as

$$[x]_q := \begin{cases} \frac{1-q^x}{1-q}, & \text{if } q \neq 1, \\ x, & \text{if } q = 1 \end{cases} \quad (1.1)$$

which is also called non-symmetrical q -number. The followings can be easily derived using (1.1):

$$[x+y]_q = [x]_q + q^x [y]_q \quad (q\text{-addition formula}) \quad (1.2)$$

$$[x-y]_q = -q^{x-y} [y]_q + [x]_q \quad (q\text{-subtraction formula}) \quad (1.3)$$

$$\left[\frac{x}{y} \right]_q = \frac{[x]_q}{[y]_q} = \frac{[x]_q q^{\frac{1}{y}}}{[y]_q q^{\frac{1}{y}}} \quad (q\text{-division rule}) \quad (1.5)$$

where x, y are real or complex numbers.

The q -binomial coefficients are defined for positive integer n, k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad (1.6)$$

where $[n]_q! = [n]_q [n-1]_q [n-2]_q \dots [1]_q$, $n = 1, 2, \dots$; $[0]_q! = 1$.

The q -derivative $D_q f(x)$ of a function f is given as

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad q \neq 1, x \neq 0, D_q f(0) = f'(0),$$

provided $f'(0)$ exists.

For any $z \in \mathbb{C}$ with $|z| < 1$,

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \text{ and } E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.$$

For the q -commuting variables x and y such as $yx = qx y$, we know that

$$e_q(x+y) = e_q(x) e_q(y).$$

The q -integral was defined by Jackson as follows:

$$\int_0^x f(y) d_q y = (1-q)x \sum_{n=0}^{\infty} f(q^n x) q^n$$

provided that the series on right hand side converges absolutely.

1.2. Multiple q -calculus. All notations and all corollaries written in this part have been taken from the Book of Nalci and Pashaev [9].

Consider basis vector $\vec{q}$ with coordinates $q_1, q_2, \dots, q_N$ so that the multiple q -number can be defined as

$$[n]_{q_i, q_j} := \frac{q_i^n - q_j^n}{q_i - q_j} = [n]_{q_j, q_i}, \quad (1.7)$$

which is symmetric. Hence, we can write $N \times N$ matrix with q -numbers elements in the following form:

$$\left([n]_{q_i, q_j} \right) = \begin{pmatrix} [n]_{q_1, q_1} & [n]_{q_1, q_2} & \dots & [n]_{q_1, q_N} \\ [n]_{q_2, q_1} & [n]_{q_2, q_2} & \dots & [n]_{q_2, q_N} \\ \dots & \dots & \dots & \dots \\ [n]_{q_N, q_1} & [n]_{q_N, q_2} & \dots & [n]_{q_N, q_N} \end{pmatrix}.$$

Diagonal terms of this matrix are defined in the limit $q_j \rightarrow q_i$ as

$$\lim_{q_j \rightarrow q_i} [n]_{q_i, q_j} = \lim_{q_j \rightarrow q_i} \frac{q_i^n - q_j^n}{q_i - q_j} = nq_i^{n-1}. \quad (1.8)$$

So, by (1.8), we see that this symmetric matrix can be shown as

$$\left([n]_{q_i, q_j} \right) = \begin{pmatrix} nq_1^{n-1} & [n]_{q_1, q_2} & \dots & [n]_{q_1, q_N} \\ [n]_{q_2, q_1} & nq_2^{n-1} & \dots & [n]_{q_2, q_N} \\ \dots & \dots & \dots & \dots \\ [n]_{q_N, q_1} & [n]_{q_N, q_2} & \dots & nq_N^{n-1} \end{pmatrix}.$$

The followings can be easily derived using (1.7):

$$[n+m]_{q_i, q_j} = q_i^n [m]_{q_i, q_j} + q_j^m [n]_{q_i, q_j}$$

(q -multiple addition formula)

$$[n-m]_{q_i, q_j} = -q_j^{-m} \left([n]_{q_i, q_j} - q_i^{n-m} [m]_{q_i, q_j} \right)$$

(q -multiple subtraction formula)

$$[nm]_{q_i, q_j} = [m]_{q_i, q_j} [n]_{q_i^m, q_j^m}$$

(q -multiple product rule)

$$\left[\frac{n}{m} \right]_{q_i, q_j} = \frac{[n]_{q_i, q_j} \frac{1}{m}}{[m]_{q_i^m, q_j^m} \frac{1}{m}} = \frac{[n]_{q_i^m, q_j^m} \frac{1}{m}}{[m]_{q_i^m, q_j^m} \frac{1}{m}}$$

(q -multiple division rule)

where n, m are real or complex numbers.

In multiple q -calculus, multiple q -derivative with base q_i, q_j is given by

$$D_{q_i, q_j} f(x) = \frac{f(q_i x) - f(q_j x)}{(q_i - q_j)x}$$

representing $N \times N$ matrix of multiple q -derivative operators $D := (D_{q_i, q_j})$ which is sym-metric:

$D_{q_i, q_j} = D_{q_j, q_i}$ where i and $j = 1, 2, \dots, N$.

$$D = (D_{q_i, q_j}) = \begin{pmatrix} D_{q_1, q_1} & D_{q_1, q_2} & \dots & D_{q_1, q_N} \\ D_{q_2, q_1} & D_{q_2, q_2} & \dots & D_{q_2, q_N} \\ \dots & \dots & \dots & \dots \\ D_{q_N, q_1} & D_{q_N, q_2} & \dots & D_{q_N, q_N} \end{pmatrix}.$$

Corollary 1. For $N = 1$ case and $q_1 = q_2 \equiv q$, we have

$$[n]_{q, q} = nq^{n-1} \text{ and } D_{q, q} = M_q \frac{d}{dx}$$

where $M_q = q^x \frac{d}{dx}$. Also, in the case $q = 1$, we have the standard number $[n]_{1,1} = n$ and the usual derivative

$$D_{1,1} = \frac{d}{dx}.$$

Corollary 2. For $N = 2$ case, we have

$$[n]_{q_1, q_1} = nq_1^{n-1}, [n]_{q_1, q_2} = [n]_{q_2, q_1} = \frac{q_1^n - q_2^n}{q_1 - q_2},$$

$$[n]_{q_2, q_2} = nq_2^{n-1}$$

$$D_{q_1, q_1} = M_{q_1} \frac{d}{dx}, D_{q_1, q_2} = D_{q_2, q_1} = \frac{M_{q_1} - M_{q_2}}{(q_1 - q_2)x},$$

$$D_{q_2, q_2} = M_{q_2} \frac{d}{dx}$$

Corollary 3. Choosing $q_1 = 1$ and $q_2 = q$ gives non-symmetrical case as

$$[n]_{1,1} = n, [n]_{1,q} = [n]_{q,1} = [n]_q, [n]_{q,q} = nq^{n-1},$$

$$D_{1,1} = \frac{d}{dx}, D_{1,q} = D_{q,1} = \frac{1 - M_q}{(1-q)^x}, D_{q,q} = M_q \frac{d}{dx}.$$

Corollary 4. Taking $q_1 = q$ and $q_2 = \frac{1}{q}$ gives symmetrical case as

$$[n]_{q,q} = nq^{n-1}, [n]_{\frac{1}{q}, \frac{1}{q}} = [n]_{\tilde{q}}, [n]_{\frac{1}{q}, q} = n \left(\frac{1}{q} \right)^{n-1},$$

$$D_{q,q} = M_q \frac{d}{dx}, D_{\frac{1}{q}, q} = D_{q, \frac{1}{q}} = \frac{M_{\frac{1}{q}} - M_q}{\left(\frac{1}{q} - q \right)^x},$$

$$D_{\frac{1}{q}, \frac{1}{q}} = M_{\frac{1}{q}} \frac{d}{dx}.$$

The multiple q -analogue of $(x-a)^n$ is the polynomial

$$(x+a)_{q_i, q_j}^n := \begin{cases} \left[\begin{matrix} (x+q_i^{n-1}a)(x+q_i^{n-2}q_ja) \dots \\ (x+q_iq_j^{n-2}a)(x+q_j^{n-2}a) \end{matrix} \right], & \text{if } n \geq 1 \\ 1, & \text{if } n = 1 \end{cases}$$

or equivalently

$$(x+a)_{q_i, q_j}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} (q_i q_j)^{\frac{k(k-1)}{2}} x^{n-k} a^k$$

where x and a is commutative, $xa = ax$. q -multiple Binomial coefficients and multiple q -factorial are defined by

$$\begin{aligned} [n]_{q_i, q_j}! &= \begin{cases} [n]_{q_i, q_j} [n-1]_{q_i, q_j} \dots [2]_{q_i, q_j} [1]_{q_i, q_j} & \text{if } n \geq 1 \\ 1 & \text{if } n \geq 1 \end{cases} \quad (n \in \mathbb{N}). \end{aligned}$$

Two types of multiple q -exponential functions are defined by

$$\begin{aligned} e_{q_i, q_j}(x) &= \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q_i, q_j}!} \\ E_{q_i, q_j}(x) &= \sum_{n=0}^{\infty} (q_i q_j)^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_{q_i, q_j}!} \end{aligned}$$

which satisfy the following condition for commutative x and y , $xy = yx$

$$e_{q_i, q_j}(x+y)_{q_i q_j} = e_{q_i, q_j}(x) E_{q_i, q_j}(y).$$

The generalization of Jackson's integral (called multiple q -integral) is given by

$$\int f\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x = (q_i - q_j) \sum_{k=0}^{\infty} \frac{q_j^k x}{q_i^{k+1}} f\left(\frac{q_j^k}{q_i^{k+1}} x\right).$$

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be formal power series.

Applying multiple q -integral to the both sides of $f(x)$ gives

$$\int f\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x = \sum_{k=0}^{\infty} q_i^{k+1} a_k \frac{x^{k+1}}{[k+1]_{q_i, q_j}} + C$$

where C is constant.

In the next section, we will use Nalci and Pashaev's method in order to find a systematic study of new types of the Bernoulli polynomials, Euler polynomials and Genocchi polynomials. Also we will obtain recursive formulas for these polynomials.

2. Main Results

Recently, analogues of Bernoulli, Euler and Genocchi polynomials were studied by many mathematicians

[1,2,5,6,11,12,13,17,18,19,20,21]. We are now ready to give the definition of generating functions, corresponding to multiple q -calculus, of Bernoulli type, Euler type and Genocchi type polynomials.

Definition 1. Let n be positive integer, we define

$$\begin{aligned} \mathcal{S}(x, z : q_i, q_j) &= \sum_{n=0}^{\infty} \mathcal{B}_n(x : q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \\ &= \frac{z}{e_{q_i, q_j}(z) - 1} e_{q_i, q_j}(xz) \quad (|z| < 2\pi) \\ \mathcal{U}(x, z : q_i, q_j) &= \sum_{n=0}^{\infty} \mathcal{E}_n(x : q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \\ &= \frac{[2]_{q_i, q_j}}{e_{q_i, q_j}(z) + 1} e_{q_i, q_j}(xz) \quad (|z| < \pi) \\ \mathcal{M}(x, z : q_i, q_j) &= \sum_{n=0}^{\infty} \mathcal{G}_n(x : q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \\ &= \frac{[2]_{q_i, q_j} z}{e_{q_i, q_j}(z) + 1} e_{q_i, q_j}(xz) \quad (|z| < \pi) \end{aligned}$$

where $\mathcal{B}_n(x : q_i, q_j)$, $\mathcal{E}_n(x : q_i, q_j)$ and $\mathcal{G}_n(x : q_i, q_j)$ are called, respectively, Bernoulli-type, Euler-type and Genocchi-type polynomials.

Corollary 5. Taking $q_i = q_j = 1$ for indexes i and j in the case $N = 1$ in Definition 1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} &= \frac{z}{e^z - 1} e^{xz} \quad (|z| < 2\pi) \\ \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} &= \frac{2}{e^z + 1} e^{xz} \quad (|z| < \pi) \\ \sum_{n=0}^{\infty} G_n(x) \frac{z^n}{n!} &= \frac{2z}{e^z + 1} e^{xz} \quad (|z| < \pi) \end{aligned}$$

where $B_n(x)$, $E_n(x)$ and $G_n(x)$ are called Bernoulli polynomials, Euler polynomials and Genocchi polynomials, respectively (see [4,7,14,15,19]).

Corollary 6. As a special case of Definition 1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_n(x|q) \frac{z^n}{[n]_q!} &= \frac{z}{e_q(z) - 1} e_q(xz) \quad (|z| < 2\pi) \\ \sum_{n=0}^{\infty} E_n(x|q) \frac{z^n}{[n]_q!} &= \frac{[2]_q}{e_q(z) + 1} e_q(xz) \quad (|z| < \pi) \\ \sum_{n=0}^{\infty} G_n(x|q) \frac{z^n}{[n]_q!} &= \frac{[2]_q z}{e_q(z) + 1} e_q(xz) \quad (|z| < \pi) \end{aligned}$$

where $B_n(x|q)$, $E_n(x|q)$ and $G_n(x|q)$ are called q -Bernoulli polynomials, q -Euler polynomials and q -Genocchi polynomials, respectively (see [18,20,21]).

Taking $x = 0$ in the above definition, we have

$$\mathcal{B}_n(0 : q_i, q_j) := \mathcal{B}_n(q_i, q_j) \quad (\text{Bernoulli-type number})$$

$$\mathcal{E}_n(0: q_i, q_j) := \mathcal{E}_n(q_i, q_j) \quad (\text{Euler-type number})$$

$$\mathcal{G}_n(0: q_i, q_j) := \mathcal{G}_n(q_i, q_j) \quad (\text{Genocchi-type number})$$

and from the above, we write

$$\begin{aligned} \mathcal{S}(0, z: q_i, q_j) &:= \mathcal{S}(z: q_i, q_j), \\ \mathcal{U}(0, z: q_i, q_j) &:= \mathcal{U}(z: q_i, q_j), \\ \mathcal{M}(0, z: q_i, q_j) &:= \mathcal{M}(z: q_i, q_j). \end{aligned} \quad (2.1)$$

From Definition 1 and (2.1), we get the following corollary.

Corollary 7. *The following functional equations hold true:*

$$\begin{aligned} \mathcal{S}(x, z: q_i, q_j) &:= \mathcal{S}(z: q_i, q_j) e_{q_i, q_j}(xz), \\ \mathcal{U}(x, z: q_i, q_j) &:= \mathcal{U}(z: q_i, q_j) e_{q_i, q_j}(xz), \\ \mathcal{M}(x, z: q_i, q_j) &:= \mathcal{M}(z: q_i, q_j) e_{q_i, q_j}(xz). \end{aligned}$$

By using Definition 1 and Corollary 7, it becomes

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathcal{B}_n(x: q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \\ &= \left(\sum_{n=0}^{\infty} \mathcal{B}_n(q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \right) \left(\sum_{n=0}^{\infty} x^n \frac{z^n}{[n]_{q_i, q_j}!} \right). \end{aligned}$$

From the rule of Cauchy product, we get

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathcal{B}_n(x: q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(q_i, q_j) x^{n-k} \right) \frac{z^n}{[n]_{q_i, q_j}!}. \end{aligned} \quad (2.2)$$

Comparing the coefficients of $\frac{z^n}{[n]_{q_i, q_j}!}$ in (2.2), we have

$$\mathcal{B}_n(x: q_i, q_j) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(q_i, q_j) x^{n-k}. \quad (2.3)$$

From this, we can get similar identities for Euler-type and Genocchi-type polynomials. Therefore, we state the following theorem.

Theorem 1. *The following identities hold true:*

$$\begin{aligned} \mathcal{B}_n(x: q_i, q_j) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(q_i, q_j) x^{n-k}, \\ \mathcal{E}_n(x: q_i, q_j) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{E}_k(q_i, q_j) x^{n-k}, \\ \mathcal{G}_n(x: q_i, q_j) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{G}_k(q_i, q_j) x^{n-k}. \end{aligned}$$

Now we are in a position to investigate some properties of Bernoulli-type numbers and polynomials, Euler-type numbers and polynomials and Genocchi-type numbers and polynomials as follows.

From Definition 1 and by using Cauchy product, we get

$$\begin{aligned} &\frac{z}{e_{q_i, q_j}(z) - 1} e_{q_i, q_j}(z) = z + \frac{z}{e_{q_i, q_j}(z) - 1} \\ &\left(\sum_{n=0}^{\infty} \mathcal{B}_n(q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \right) \left(\sum_{n=0}^{\infty} \frac{z^n}{[n]_{q_i, q_j}!} \right) \\ &= z + \sum_{n=0}^{\infty} \mathcal{B}_n(q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \\ &\sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(q_i, q_j) \right) \frac{z^n}{[n]_{q_i, q_j}!} \\ &= z + \sum_{n=0}^{\infty} \mathcal{B}_n(q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!}. \end{aligned}$$

If we compute both of side and then compare coefficient

of $\frac{z^n}{[n]_{q_i, q_j}!}$, then for $n > 1$, we acquire

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{\mathcal{B}_k(q_i, q_j)}{[k]_{q_i, q_j}! [n-k]_{q_i, q_j}!} - \frac{\mathcal{B}_n(q_i, q_j)}{[n]_{q_i, q_j}!} = 0 \\ &\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(q_i, q_j) = \mathcal{B}_n(q_i, q_j). \end{aligned} \quad (2.4)$$

From this, we can get similar identities for Euler-type numbers and Genocchi-type numbers. The following theorem is an immediate consequence of Eq. (2.4).

Theorem 2. *(Recurrence Formula) For $n > 1$, we have*

$$\begin{aligned} &\sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(q_i, q_j) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases} \\ &\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{E}_k(q_i, q_j) + \mathcal{E}_n(q_i, q_j) = \begin{cases} [2]_{q_i, q_j}, & n = 0 \\ 0, & n \geq 1 \end{cases} \\ &\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{G}_k(q_i, q_j) + \mathcal{G}_n(q_i, q_j) = \begin{cases} [2]_{q_i, q_j}, & n = 1 \\ 0, & n > 1 \end{cases}. \end{aligned}$$

It is not difficult to show the following equality:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \begin{bmatrix} n-k \\ m \end{bmatrix}_{q_i, q_j} = \begin{bmatrix} n \\ m+k \end{bmatrix}_{q_i, q_j} \begin{bmatrix} m+k \\ m \end{bmatrix}_{q_i, q_j}. \quad (2.5)$$

By (2.5), we get readily the following theorem.

Theorem 3. *For $n \in \mathbb{N}$, the followings hold true*

$$\begin{aligned} \mathcal{B}_n(x+y: q_i, q_j) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(x: q_i, q_j) y^{n-k}, \\ \mathcal{E}_n(x+y: q_i, q_j) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{E}_k(x: q_i, q_j) y^{n-k}, \\ \mathcal{G}_n(x+y: q_i, q_j) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{G}_k(x: q_i, q_j) y^{n-k}. \end{aligned}$$

Proof. If we change x by $x + y$ in $\mathcal{S}_n(x, z: q_i, q_j)$, then we acquire

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{B}_n(x+y: q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \\ &= \frac{z}{e_{q_i, q_j}(z) - 1} e_{q_i, q_j}((x+y)z) \\ &= \sum_{n=0}^{\infty} \mathcal{B}_n(q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q_i, q_j}!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(x: q_i, q_j) y^{n-k} \right) \frac{z^n}{[n]_{q_i, q_j}!}. \end{aligned}$$

By computing the coefficient $\frac{z^n}{[n]_{q_i, q_j}!}$ of both of side,

then we have

$$\mathcal{B}_n(x+y: q_i, q_j) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(x: q_i, q_j) y^{n-k}.$$

The others can be proved in a like manner.

Now we consider the special cases of Theorem 3 as Corollary 8 and Corollary 9.

Corollary 8. Letting $y = 1$ in the Theorem 3, we then get

$$\begin{aligned} \mathcal{B}_n(x+1: q_i, q_j) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(x: q_i, q_j), \\ \mathcal{E}_n(x+1: q_i, q_j) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{E}_k(x: q_i, q_j), \\ \mathcal{G}_n(x+1: q_i, q_j) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{G}_k(x: q_i, q_j). \end{aligned}$$

Corollary 9. Letting $x = 0$ in the Theorem 3, we then get

$$\begin{aligned} \mathcal{B}_n(x: q_i, q_j) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{B}_k(q_i, q_j) x^{n-k}, \\ \mathcal{E}_n(x: q_i, q_j) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{E}_k(q_i, q_j) x^{n-k}, \\ \mathcal{G}_n(x: q_i, q_j) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i, q_j} \mathcal{G}_k(q_i, q_j) x^{n-k}. \end{aligned}$$

Theorem 4. The following expressions hold true for $n \in \mathbb{N}$

$$\begin{aligned} \mathcal{B}_n(x+1: q_i, q_j) - \mathcal{B}_n(x: q_i, q_j) &= [n]_{q_i, q_j} x^{n-1} \\ \mathcal{E}_n(x+1: q_i, q_j) + \mathcal{E}_n(x: q_i, q_j) &= [2]_{q_i, q_j} x^n \\ \mathcal{G}_n(x+1: q_i, q_j) + \mathcal{G}_n(x: q_i, q_j) &= [2]_{q_i, q_j} [n]_{q_i, q_j} x^{n-1}. \end{aligned}$$

Proof. By using definitions of these polynomials and numbers, one can easily obtain these relations.

Theorem 5. (Identity of Symmetry) The followings hold true for $n \in \mathbb{N}$:

$$\begin{aligned} \mathcal{B}_n(1-x: q_i, q_j) &= (-1)^n \mathcal{B}_n(x: q_i, q_j), \\ \mathcal{E}_n(1-x: q_i, q_j) &= (-1)^n \mathcal{E}_n(x: q_i, q_j), \\ \mathcal{G}_n(1-x: q_i, q_j) &= (-1)^{n+1} \mathcal{G}_n(x: q_i, q_j). \end{aligned}$$

Proof. Setting $1-x$ instead of x in $\mathcal{S}_n(x, z: q_i, q_j)$, we then get

$$\begin{aligned} & \frac{z}{e_{q_i, q_j}(z) - 1} e_{q_i, q_j}((1-x)z) \\ &= \sum_{n=0}^{\infty} \mathcal{B}_n(1-x: q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \\ &= \frac{(-z)}{e_{q_i, q_j}(-z) - 1} e_{q_i, q_j}(x(-z)) \\ &= \sum_{n=0}^{\infty} (-1)^n \mathcal{B}_n(x: q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!}. \end{aligned}$$

Comparing coefficients $\frac{z^n}{[n]_{q_i, q_j}!}$ both of side in above

equality, we have desired the result. Similar to that of this proof, it can be proved for Euler-type polynomials and Genocchi-type polynomials. So we completed this proof.

Theorem 6. (Raabe's Formula) For $n \in \mathbb{N}$, the followings hold true

$$\begin{aligned} \mathcal{B}_n(dx: q_i, q_j) &= d^{n-1} \sum_{k=0}^{d-1} \mathcal{B}_n\left(x + \frac{k}{d}: q_i, q_j\right) \quad (d \in \mathbb{Z}^+) \\ \mathcal{E}_n(dx: q_i, q_j) &= d^n \sum_{k=0}^{d-1} (-1)^k \mathcal{E}_n\left(x + \frac{k}{d}: q_i, q_j\right) \quad (d \equiv 1 \pmod{2}) \\ \mathcal{G}_n(dx: q_i, q_j) &= d^{n-1} \sum_{k=0}^{d-1} (-1)^k \mathcal{G}_n\left(x + \frac{k}{d}: q_i, q_j\right) \quad (d \equiv 1 \pmod{2}). \end{aligned}$$

Proof. By using Definition 1, then we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(d^{n-1} \sum_{k=0}^{d-1} \mathcal{B}_n\left(x + \frac{k}{d}: q_i, q_j\right) \right) \frac{z^n}{[n]_{q_i, q_j}!} \\ &= \frac{1}{d} \sum_{k=0}^{d-1} \sum_{n=0}^{\infty} \mathcal{B}_n\left(x + \frac{k}{d}: q_i, q_j\right) \frac{(dz)^n}{[n]_{q_i, q_j}!} \\ &= \frac{1}{d} \sum_{k=0}^{d-1} \frac{dz}{e_{q_i, q_j}(dz) - 1} e_{q_i, q_j}\left(\left(x + \frac{k}{d}\right)dz\right) \\ &= \frac{z}{e_{q_i, q_j}(dz) - 1} e_{q_i, q_j}(xdz) \sum_{k=0}^{d-1} e_{q_i, q_j}(kz) \\ &= \sum_{n=0}^{\infty} \mathcal{B}_n(dx: q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!}. \end{aligned}$$

Similarly, we can prove this theorem for Euler-type numbers and Genocchi-type polynomials. So we omit them. Hence, we complete the proof of this theorem.

Theorem 7. The three relations between Euler-type numbers and polynomials and Genocchi-type numbers and polynomials are given by

$$\begin{aligned}\mathcal{E}_n(q_i, q_j) &= \frac{\mathcal{G}_{n+1}(q_i, q_j)}{n+1}, \\ \mathcal{E}_n(x: q_i, q_j) &= \frac{\mathcal{G}_{n+1}(x: q_i, q_j)}{n+1}, \\ \mathcal{E}_n(x: q_i, q_j) &= \sum_{k=0}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right]_{q_i, q_j} \frac{\mathcal{G}_{k+1}(q_i, q_j)}{k+1} x^{n-k}.\end{aligned}$$

Proof. By Definition 1, we can easily obtain these relations. So we omit the proof.

Let us now apply the multiple q -derivative D_{q_i, q_j} , with respect to x , on the both sides of Definition 1,

$$\begin{aligned}D_{q_i, q_j} \left(\sum_{n=0}^{\infty} \mathcal{E}_n(x: q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \right) \\ = \sum_{n=0}^{\infty} D_{q_i, q_j} \mathcal{E}_n(x: q_i, q_j) \frac{z^n}{[n]_{q_i, q_j}!} \\ = \frac{z}{e_{q_i, q_j}(z) - 1} D_{q_i, q_j} e_{q_i, q_j}(xz) \\ = \frac{z^2}{e_{q_i, q_j}(z) - 1} e_{q_i, q_j}(xz) \\ = \sum_{n=0}^{\infty} \mathcal{E}_n(x: q_i, q_j) \frac{z^{n+1}}{[n]_{q_i, q_j}!}.\end{aligned}$$

Matching the coefficients of $\frac{z^n}{[n]_{q_i, q_j}!}$ gives us

$$D_{q_i, q_j} \mathcal{E}_n(x: q_i, q_j) = [n]_{q_i, q_j} \mathcal{E}_{n-1}(x: q_i, q_j).$$

Thus we procure the following theorem.

Theorem 8. The following identities hold true:

$$\begin{aligned}D_{q_i, q_j} \mathcal{E}_n(x: q_i, q_j) &= [n]_{q_i, q_j} \mathcal{E}_{n-1}(x: q_i, q_j), \\ D_{q_i, q_j} \mathcal{E}_n(x: q_i, q_j) &= [n]_{q_i, q_j} \mathcal{E}_{n-1}(x: q_i, q_j)\end{aligned}$$

and

$$D_{q_i, q_j} \mathcal{G}_n(x: q_i, q_j) = [n]_{q_i, q_j} \mathcal{G}_{n-1}(x: q_i, q_j).$$

Applying k -times the operator D_{q_i, q_j} denoted by D_{q_i, q_j}^k and the limit $t \rightarrow 0$, respectively, to the Definition 1, we derive that

$$\mathcal{B}_n(x: q_i, q_j) = \lim_{t \rightarrow 0} D_{q_i, q_j}^k \frac{ze_{q_i, q_j}(xz)}{e_{q_i, q_j}(z) - 1}.$$

So we conclude the following theorem.

Theorem 9. For $k \geq 0$ and $n \geq 0$, we have

$$\mathcal{B}_n(x: q_i, q_j) = \lim_{t \rightarrow 0} D_{q_i, q_j}^k \frac{ze_{q_i, q_j}(xz)}{e_{q_i, q_j}(z) - 1},$$

$$\mathcal{E}_n(x: q_i, q_j) = \lim_{t \rightarrow 0} D_{q_i, q_j}^k \frac{[2]_{q_i, q_j}}{e_{q_i, q_j}(z) + 1} e_{q_i, q_j}(xz)$$

and

$$\mathcal{G}_n(x: q_i, q_j) = \lim_{t \rightarrow 0} D_{q_i, q_j}^k \frac{[2]_{q_i, q_j} z}{e_{q_i, q_j}(z) + 1} e_{q_i, q_j}(xz).$$

Definition 2. Let $0 < a < b$. The definite multiple q -integral has the following representation:

$$\int_a^b f\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x = (q_i - q_j) b \sum_{k=0}^{\infty} \frac{q_j^k}{q_i^{k+1}} f\left(\frac{q_j^k}{q_i^{k+1}} b\right)$$

and

$$\int_a^b f\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x = \int_0^b f\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x - \int_0^a f\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x.$$

Theorem 10. The following holds true:

$$\begin{aligned}\int_0^b f\left(\frac{x}{q_i}\right) D_{q_i, q_j} g\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x \\ = \sum_{n=0}^{\infty} f\left(\frac{q_j^k}{q_i^{k+1}} b\right) \left(g\left(\frac{q_j^k}{q_i^k} b\right) - g\left(\frac{q_j^{k+1}}{q_i^{k+1}} b\right) \right).\end{aligned}$$

Proof. From Definition 2, we write that

$$\begin{aligned}\int_0^b f\left(\frac{x}{q_i}\right) D_{q_i, q_j} g\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x \\ = (q_i - q_j) b \sum_{k=0}^{\infty} \frac{q_j^k}{q_i^{k+1}} f\left(\frac{q_j^k}{q_i^{k+1}} b\right) D_{q_i, q_j} g\left(\frac{q_j^k}{q_i^{k+1}} b\right),\end{aligned}\tag{2.6}$$

where $D_{q_i, q_j} g\left(\frac{q_j^k}{q_i^{k+1}} b\right)$ equals to

$$\frac{g\left(\frac{q_j^k}{q_i^k} b\right) - g\left(\frac{q_j^{k+1}}{q_i^{k+1}} b\right)}{(q_i - q_j) b}.\tag{2.7}$$

Combining the Eq. (2.6) with the Eq. (2.7) gives us the proof of the theorem.

Theorem 11. $\left| \frac{q_j}{q_i} \right| < 1$ and $i, j \in \{1, 2, \dots, N\}$. Then we have

$$\begin{aligned}\int_0^1 \mathcal{B}_n\left(\frac{x}{q_i}: q_i, q_j\right) d_{\frac{q_j}{q_i}} x &= \frac{\mathcal{B}_{n+1}(1: q_i, q_j) - \mathcal{B}_{n+1}(q_i, q_j)}{[n+1]_{q_i, q_j}} \\ \int_0^1 \mathcal{E}_n\left(\frac{x}{q_i}: q_i, q_j\right) d_{\frac{q_j}{q_i}} x &= \frac{\mathcal{E}_{n+1}(1: q_i, q_j) - \mathcal{E}_{n+1}(q_i, q_j)}{[n+1]_{q_i, q_j}}\end{aligned}$$

$$\int_0^1 \mathcal{G}_n \left(\frac{x}{q_i} : q_i, q_j \right) d_{\frac{q_j}{q_i}} x = \frac{\mathcal{G}_{n+1}(1 : q_i, q_j) - \mathcal{G}_{n+1}(q_i, q_j)}{[n+1]_{q_i, q_j}}.$$

Proof. By using Theorem 1, Definition 2 and for $\left| \frac{q_j}{q_i} \right| < 1$,

we have

$$\begin{aligned} & \int_0^1 \mathcal{B}_n \left(\frac{x}{q_i} : q_i, q_j \right) d_{\frac{q_j}{q_i}} x \\ &= \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{q_i, q_j} \mathcal{B}_{n-l}(q_i, q_j) \left((q_i - q_j) \sum_{k=0}^{\infty} \frac{q_j^k}{q_i^{k+1}} \left(\frac{q_j^k}{q_i^{k+1}} \right)^l \right) \\ &= \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{q_i, q_j} \mathcal{B}_{n-l}(q_i, q_j) \frac{1}{[l+1]_{q_i, q_j}} \\ &= \frac{\mathcal{B}_{n+1}(1 : q_i, q_j) - \mathcal{B}_{n+1}(q_i, q_j)}{[n+1]_{q_i, q_j}}. \end{aligned}$$

Similarly, the identities of Euler-type polynomials and Genocchi-type polynomials can be shown. Therefore, we complete the proof of theorem.

3. Further Remarks

Here we list a few values of Bernoulli-type, Euler-type and Genocchi-type numbers as follows:

Bernoulli-type number:

Table 1.

$\mathcal{B}_0(q_i, q_j) = 1$
$\mathcal{B}_1(q_i, q_j) = -\frac{1}{[2]_{q_i, q_j}}$
$\mathcal{B}_2(q_i, q_j) = -\frac{1}{[3]_{q_i, q_j}} + \frac{1}{[2]_{q_i, q_j}}$
$\mathcal{B}_3(q_i, q_j) = -\frac{1}{[4]_{q_i, q_j}} + \frac{2}{[2]_{q_i, q_j}} - \frac{[3]_{q_i, q_j}}{[2]_{q_i, q_j}^2}$
$\mathcal{B}_4(q_i, q_j) = -\frac{1}{[5]_{q_i, q_j}} + \frac{1}{[2]_{q_i, q_j}} + \frac{[4]_{q_i, q_j}}{[2]_{q_i, q_j}} \left(\frac{1}{[3]_{q_i, q_j}} - \frac{1}{[2]_{q_i, q_j}} \right)$
$+\frac{[4]_{q_i, q_j}}{[2]_{q_i, q_j}} \left(\frac{1}{[4]_{q_i, q_j}} - \frac{2}{[2]_{q_i, q_j}} + \frac{[3]_{q_i, q_j}}{[2]_{q_i, q_j}^2} \right)$

As a special case of Table 1, we get

$\mathfrak{B}_0(q) = 1$
$\mathfrak{B}_1(q) = -\frac{1}{[2]_q}$

$\mathfrak{B}_2(q) = -\frac{1}{[3]_q} + \frac{1}{[2]_q}$
$\mathfrak{B}_3(q) = -\frac{1}{[4]_q} + \frac{2}{[2]_q} - \frac{[3]_q}{[2]_q^2}$
$\mathfrak{B}_4(q) = -\frac{1}{[5]_q} + \frac{1}{[2]_q} + \frac{[4]_q}{[2]_q} \left(\frac{1}{[3]_q} - \frac{1}{[2]_q} \right)$
$+\frac{[4]_q}{[2]_q} \left(\frac{1}{[4]_q} - \frac{2}{[2]_q} + \frac{[3]_q}{[2]_q^2} \right)$

As a special case of Table 1, we get

$B_0 = 1$
$B_1 = -\frac{1}{2}$
$B_2 = \frac{1}{6}$
$B_3 = 0$
$B_4 = -\frac{1}{30}$

Moreover the first few Bernoulli-type numbers can be shown $N \times N$ matrix with multiple q -numbers elements in the following form

$$\left(\mathcal{B}_0(q_i, q_j) \right) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

$$\left(\mathcal{B}_1(q_i, q_j) \right) = \begin{pmatrix} -\frac{1}{2q_1} & -\frac{1}{[2]_{q_1, q_2}} & \dots & -\frac{1}{[2]_{q_1, q_N}} \\ -\frac{1}{[2]_{q_2, q_1}} & -\frac{1}{2q_2} & \dots & -\frac{1}{[2]_{q_2, q_N}} \\ \dots & \dots & \dots & \dots \\ -\frac{1}{[2]_{q_N, q_1}} & -\frac{1}{[2]_{q_N, q_2}} & \dots & -\frac{1}{[2]_{q_N, q_N}} \end{pmatrix}$$

$$\left(\mathcal{B}_2(q_i, q_j) \right) = \begin{pmatrix} \left(-\frac{1}{3q_1^2} \right) & \left(-\frac{1}{[3]_{q_1, q_2}} \right) & \dots & \left(-\frac{1}{[3]_{q_1, q_N}} \right) \\ +\frac{1}{2q_1} & +\frac{1}{[2]_{q_1, q_2}} & \dots & +\frac{1}{[2]_{q_1, q_N}} \\ \left(-\frac{1}{[3]_{q_2, q_1}} \right) & \left(-\frac{1}{3q_2^2} \right) & \dots & \left(-\frac{1}{[3]_{q_2, q_N}} \right) \\ +\frac{1}{[2]_{q_2, q_1}} & +\frac{1}{2q_2} & \dots & +\frac{1}{[2]_{q_2, q_N}} \\ \dots & \dots & \dots & \dots \\ \left(-\frac{1}{[3]_{q_N, q_1}} \right) & \left(-\frac{1}{[3]_{q_N, q_2}} \right) & \dots & \left(-\frac{1}{3q_N^2} \right) \\ +\frac{1}{[2]_{q_N, q_1}} & +\frac{1}{[2]_{q_N, q_2}} & \dots & +\frac{1}{2q_N} \end{pmatrix}$$

From Definition 1 and the Table 1, we easily acquire the first few Bernoulli-type poly-nomials

Bernoulli-type polynomials

$$\mathcal{B}_0(x: q_i, q_j) = 1$$

$$\mathcal{B}_1(x: q_i, q_j) = x - \frac{1}{[2]_{q_i, q_j}}$$

$$\mathcal{B}_2(x: q_i, q_j) = x^2 - x - \frac{1}{[3]_{q_i, q_j}} + \frac{1}{[2]_{q_i, q_j}}$$

Usual Bernoulli polynomials

$$B_0 = 1$$

$$B_1 = x - \frac{1}{2}$$

$$B_2 = x^2 - x - \frac{1}{6}$$

Moreover, the first few Bernoulli-type polynomials can be shown $N \times N$ matrix with q -numbers elements in the following form

$$\left(\mathcal{B}_2(x: q_i, q_j) \right) = \begin{pmatrix} x^2 - x - \frac{1}{3q_1^2} + \frac{1}{2q_1} & x^2 - x - \frac{1}{[3]_{q_1, q_2}} + \frac{1}{[2]_{q_1, q_2}} & \dots & x^2 - x - \frac{1}{[3]_{q_1, q_N}} + \frac{1}{[2]_{q_1, q_N}} \\ x^2 - x - \frac{1}{[3]_{q_2, q_1}} + \frac{1}{[2]_{q_2, q_1}} & x^2 - x - \frac{1}{3q_2^2} + \frac{1}{2q_2} & \dots & x^2 - x - \frac{1}{[3]_{q_2, q_N}} + \frac{1}{[2]_{q_2, q_N}} \\ \dots & \dots & \dots & \dots \\ x^2 - x - \frac{1}{[3]_{q_N, q_1}} + \frac{1}{[2]_{q_N, q_1}} & x^2 - x - \frac{1}{[3]_{q_N, q_2}} + \frac{1}{[2]_{q_N, q_2}} & \dots & x^2 - x - \frac{1}{3q_N^2} + \frac{1}{2q_N} \end{pmatrix}$$

Euler-type Numbers and Polynomials:

We begin to compute the first few value of $\mathcal{E}_n(q_i, q_j)$ as follows:

Table 2.

$\mathcal{E}_0(q_i, q_j) = \frac{[2]_{q_i, q_j}}{2}$
$\mathcal{E}_1(q_i, q_j) = -\frac{[2]_{q_i, q_j}}{4}$
$\mathcal{E}_2(q_i, q_j) = -\frac{[2]_{q_i, q_j}}{4} + \frac{([2]_{q_i, q_j})^2}{8}$
$\mathcal{E}_3(q_i, q_j) = -\frac{[2]_{q_i, q_j}}{4} + \frac{[3]_{q_i, q_j} [2]_{q_i, q_j}}{4} \left(1 - \frac{[2]_{q_i, q_j}}{4} \right)$
$\mathcal{E}_4(q_i, q_j) = -\frac{[2]_{q_i, q_j}}{4} + \frac{[4]_{q_i, q_j} [2]_{q_i, q_j}}{4} + \frac{[4]_{q_i, q_j} [3]_{q_i, q_j} [2]_{q_i, q_j}}{8} \left(\frac{1}{[2]_{q_i, q_j}} - \frac{3}{2} + \frac{[2]_{q_i, q_j}}{4} \right)$
$\mathcal{E}_5(q_i, q_j) = -\frac{[2]_{q_i, q_j}}{4} + \frac{[5]_{q_i, q_j} [2]_{q_i, q_j}}{4} + \frac{[5]_{q_i, q_j} [4]_{q_i, q_j} [2]_{q_i, q_j}}{4} \left(-\frac{3}{4} - \frac{1}{[2]_{q_i, q_j}} \right)$
$+ \frac{[5]_{q_i, q_j} [4]_{q_i, q_j} [3]_{q_i, q_j} [2]_{q_i, q_j}}{16} \left(\frac{3}{[2]_{q_i, q_j}} - 2 + \frac{[2]_{q_i, q_j}}{4} \right)$

As a special case of Table 2, we have

$$\left(\mathcal{B}_0(x: q_i, q_j) \right) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix} = \left(\mathcal{B}_0(q_i, q_j) \right)$$

$$\left(\mathcal{B}_1(x: q_i, q_j) \right) = \begin{pmatrix} x - \frac{1}{2q_1} & x - \frac{1}{[2]_{q_1, q_2}} & \dots & x - \frac{1}{[2]_{q_1, q_N}} \\ x - \frac{1}{[2]_{q_2, q_1}} & x - \frac{1}{2q_2} & \dots & x - \frac{1}{[2]_{q_2, q_N}} \\ \dots & \dots & \dots & \dots \\ x - \frac{1}{[2]_{q_N, q_1}} & x - \frac{1}{[2]_{q_N, q_2}} & \dots & x - \frac{1}{[2]_{q_N, q_N}} \end{pmatrix}$$

$E_0(q) = \frac{[2]_q}{2}$
$E_1(q) = -\frac{[2]_q}{4}$
$E_2(q) = -\frac{[2]_q}{4} + \frac{([2]_q)^2}{8}$
$E_3(q) = -\frac{[2]_q}{4} + \frac{[3]_q [2]_q}{4} \left(1 - \frac{[2]_q}{4}\right)$
$E_4(q) = -\frac{[2]_q}{4} + \frac{[4]_q [2]_q}{4} + \frac{[4]_q [3]_q [2]_q}{8} \left(\frac{1}{[2]_q} - \frac{3}{2}\right) + \frac{[2]_q}{4}$
$E_5(q) = -\frac{[2]_q}{4} + \frac{[5]_q [2]_q}{4} + \frac{[5]_q [4]_q [2]_q}{4} \left(-\frac{3}{4} - \frac{1}{[2]_q}\right) + \frac{[5]_q [4]_q [3]_q [2]_q}{16} \left(\frac{3}{[2]_q} - 2 + \frac{[2]_q}{4}\right)$

As a special case of Table 2, we have

$E_0(q) = 1$
$E_1(q) = -\frac{1}{2}$
$E_2(q) = 0$
$E_3(q) = \frac{1}{4}$
$E_4(q) = 0$
$E_5(q) = -\frac{1}{2}$

Moreover the first few Euler-type numbers can be shown $N \times N$ matrix with q -numbers elements in the following form

$$(\mathcal{E}_0(q_i, q_j)) = \begin{pmatrix} q_1 & \frac{[2]_{q_1, q_2}}{2} & \dots & \frac{[2]_{q_1, q_N}}{2} \\ \frac{[2]_{q_2, q_1}}{2} & q_2 & \dots & \frac{[2]_{q_2, q_N}}{2} \\ \dots & \dots & \dots & \dots \\ \frac{[2]_{q_N, q_1}}{2} & \frac{[2]_{q_N, q_2}}{2} & \dots & q_N \end{pmatrix}$$

$$(\mathcal{E}_1(q_i, q_j)) = \begin{pmatrix} -\frac{q_1}{2} & -\frac{[2]_{q_1, q_2}}{4} & \dots & -\frac{[2]_{q_1, q_N}}{4} \\ -\frac{[2]_{q_2, q_1}}{4} & -\frac{q_2}{2} & \dots & -\frac{[2]_{q_2, q_N}}{4} \\ \dots & \dots & \dots & \dots \\ -\frac{[2]_{q_N, q_1}}{4} & -\frac{[2]_{q_N, q_2}}{4} & \dots & -\frac{q_N}{2} \end{pmatrix}$$

$$(\mathcal{E}_2(x : q_i, q_j)) = \begin{pmatrix} -\frac{q_1}{2} + \frac{q_1^2}{2} & \begin{pmatrix} \frac{[2]_{q_1, q_2}}{4} \\ + \frac{([2]_{q_1, q_2})^2}{8} \end{pmatrix} & \dots & \begin{pmatrix} \frac{[2]_{q_1, q_N}}{4} \\ + \frac{([2]_{q_1, q_N})^2}{8} \end{pmatrix} \\ \begin{pmatrix} \frac{[2]_{q_2, q_1}}{4} \\ + \frac{([2]_{q_2, q_1})^2}{8} \end{pmatrix} & -\frac{q_2}{2} + \frac{q_2^2}{2} & \dots & \begin{pmatrix} \frac{[2]_{q_2, q_N}}{4} \\ + \frac{([2]_{q_2, q_N})^2}{8} \end{pmatrix} \\ \dots & \dots & \dots & \dots \\ \begin{pmatrix} \frac{[2]_{q_N, q_1}}{4} \\ + \frac{([2]_{q_N, q_1})^2}{8} \end{pmatrix} & \begin{pmatrix} \frac{[2]_{q_N, q_2}}{4} \\ + \frac{([2]_{q_N, q_2})^2}{8} \end{pmatrix} & \dots & -\frac{q_N}{2} + \frac{q_N^2}{2} \end{pmatrix}$$

From Definition 1 and the Table 2, we easily acquire the first few Euler-type polynomials

Euler-type polynomials

$$\mathcal{E}_0(x : q_i, q_j) = \frac{[2]_{q_i, q_j}}{2}$$

$$\mathcal{E}_1(x : q_i, q_j) = \frac{[2]_{q_i, q_j}}{2} x - \frac{[2]_{q_i, q_j}}{4}$$

$$\mathcal{E}_2(x : q_i, q_j) = \frac{[2]_{q_i, q_j}}{2} x^2 - \frac{([2]_{q_i, q_j})^2}{4} x - \frac{[2]_{q_i, q_j}}{4} + \frac{([2]_{q_i, q_j})^2}{8}$$

Usual Euler polynomials

$$E_0 = 1$$

$$E_1 = x - \frac{1}{2}$$

$$E_2 = x^2 - x$$

Moreover the first few Euler-type polynomials can be shown $N \times N$ matrix with multiple q -numbers elements in the following form

$$(\mathcal{E}_0(q_i, q_j)) = \begin{pmatrix} q_1 & \frac{[2]_{q_1, q_2}}{2} & \dots & \frac{[2]_{q_1, q_N}}{2} \\ \frac{[2]_{q_2, q_1}}{2} & q_2 & \dots & \frac{[2]_{q_2, q_N}}{2} \\ \dots & \dots & \dots & \dots \\ \frac{[2]_{q_N, q_1}}{2} & \frac{[2]_{q_N, q_2}}{2} & \dots & q_N \end{pmatrix}$$

$$\begin{aligned}
\left(\mathcal{E}_1(x:q_i,q_j)\right) &= \begin{pmatrix} q_1x - \frac{q_1}{2} & \frac{[2]_{q_1,q_2}}{2}x - \frac{[2]_{q_1,q_2}}{4} & \dots & \frac{[2]_{q_1,q_2}}{2}x - \frac{[2]_{q_1,q_N}}{4} \\ \frac{[2]_{q_1,q_2}}{2}x - \frac{[2]_{q_2,q_1}}{4} & q_2x - \frac{q_2}{2} & \dots & \frac{[2]_{q_1,q_2}}{2}x - \frac{[2]_{q_2,q_N}}{4} \\ \dots & \dots & \dots & \dots \\ \frac{[2]_{q_1,q_2}}{2}x - \frac{[2]_{q_N,q_1}}{4} & \frac{[2]_{q_1,q_2}}{2}x - \frac{[2]_{q_N,q_2}}{4} & \dots & q_Nx - \frac{q_N}{2} \end{pmatrix} \\
\left(\mathcal{E}_2(x:q_i,q_j)\right) &= \begin{pmatrix} q_1x^2 - q_1^2x - \frac{q_1}{2} + \frac{q_1^2}{2} & \dots & \frac{[2]_{q_1,q_N}}{2}x^2 - \frac{([2]_{q_1,q_N})^2}{4}x - \frac{[2]_{q_1,q_N}}{4} + \frac{([2]_{q_1,q_N})^2}{8} \\ \frac{[2]_{q_1,q_2}}{2}x^2 - \frac{([2]_{q_2,q_1})^2}{4}x - \frac{[2]_{q_1,q_2}}{4} + \frac{([2]_{q_1,q_2})^2}{8} & \dots & \frac{[2]_{q_2,q_N}}{2}x^2 - \frac{([2]_{q_2,q_N})^2}{4}x - \frac{[2]_{q_1,q_N}}{4} + \frac{([2]_{q_1,q_N})^2}{8} \\ \dots & \dots & \dots \\ \frac{[2]_{q_N,q_1}}{2}x^2 - \frac{([2]_{q_N,q_1})^2}{4}x - \frac{[2]_{q_N,q_1}}{4} + \frac{([2]_{q_N,q_1})^2}{8} & \dots & q_Nx^2 - q_N^2x - \frac{q_N}{2} + \frac{q_N^2}{2} \end{pmatrix}
\end{aligned}$$

Genocchi-type Numbers and Polynomials:

We begin to compute the first few value of $\mathcal{G}_1(q_i, q_j)$ as follows:

Table 3.

$\mathcal{G}_0(q_i, q_j) = 0$
$\mathcal{G}_1(q_i, q_j) = \frac{[2]_{q_i, q_j}}{2}$
$\mathcal{G}_2(q_i, q_j) = -\frac{([2]_{q_i, q_j})^2}{4}$
$\mathcal{G}_3(q_i, q_j) = \frac{[3]_{q_i, q_j}}{4} \left(-[2]_{q_i, q_j} + \frac{([2]_{q_i, q_j})^2}{2} \right)$
$\mathcal{G}_4(q_i, q_j) = -\frac{[4]_{q_i, q_j} [2]_{q_i, q_j}}{4} + \frac{[4]_{q_i, q_j} [3]_{q_i, q_j} [2]_{q_i, q_j}}{8} - \frac{[4]_{q_i, q_j} [3]_{q_i, q_j}}{8} \left(-[2]_{q_i, q_j} + \frac{([2]_{q_i, q_j})^2}{2} \right)$
$\mathcal{G}_5(q_i, q_j) = -\frac{[5]_{q_i, q_j} [2]_{q_i, q_j}}{4} + \frac{[5]_{q_i, q_j} [4]_{q_i, q_j} [2]_{q_i, q_j}}{8} - \frac{[5]_{q_i, q_j} [4]_{q_i, q_j} [3]_{q_i, q_j}}{8} \left(-1 + \frac{[2]_{q_i, q_j}}{2} \right)$
$-\frac{[5]_{q_i, q_j}}{8} \left(-[4]_{q_i, q_j} [2]_{q_i, q_j} + \frac{[4]_{q_i, q_j} [3]_{q_i, q_j} [2]_{q_i, q_j}}{2} - \frac{[4]_{q_i, q_j} [3]_{q_i, q_j}}{2} \left(-[2]_{q_i, q_j} + \frac{([2]_{q_i, q_j})^2}{2} \right) \right)$

As a special case of Table 3, we have

$\mathcal{G}_0(q) = 0$
$\mathcal{G}_1(q) = \frac{[2]_q}{2}$

$\mathcal{G}_2(q) = \frac{([2]_q)^2}{4}$
$\mathcal{G}_3(q) = \frac{[3]_q}{4} \left(-[2]_q + \frac{([2]_q)^2}{2} \right)$
$\mathcal{G}_4(q) = -\frac{[4]_q [2]_q}{4} + \frac{[4]_q [3]_q [2]_q}{8} - \frac{[4]_q [3]_q}{8} \left(-[2]_q + \frac{([2]_q)^2}{2} \right)$
$\mathcal{G}_5(q) = -\frac{[5]_q [2]_q}{4} + \frac{[5]_q [4]_q [2]_q}{8} - \frac{[5]_q [4]_q [3]_q}{8} \left(-1 + \frac{[2]_q}{2} \right)$
$-\frac{[5]_q}{8} \left(-[4]_q [2]_q + \frac{[4]_q [3]_q [2]_q}{2} - \frac{[4]_q [3]_q}{2} \left(-[2]_q + \frac{([2]_q)^2}{2} \right) \right)$

As a special case of Table 3, we have

$G_0 = 0$
$G_1 = 1$
$G_2 = -1$
$G_3 = 0$
$G_4 = 1$
$G_5 = 0$

Moreover the first few Genocchi-type numbers can be shown $N \times N$ matrix with multiple q -numbers elements in the following form

$$\begin{aligned}
 (\mathcal{G}_0(q_i, q_j)) &= \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\
 (\mathcal{G}_1(q_i, q_j)) &= \begin{pmatrix} q_1 & \frac{[2]_{q_1, q_2}}{2} & \dots & \frac{[2]_{q_1, q_N}}{2} \\ \frac{[2]_{q_2, q_1}}{2} & q_2 & \dots & \frac{[2]_{q_2, q_N}}{2} \\ \dots & \dots & \dots & \dots \\ \frac{[2]_{q_N, q_1}}{2} & \frac{[2]_{q_N, q_2}}{2} & \dots & q_N \end{pmatrix} \\
 (\mathcal{G}_2(q_i, q_j)) &= \begin{pmatrix} -\frac{q_1^2}{2} & -\frac{([2]_{q_1, q_2})^2}{4} & \dots & -\frac{([2]_{q_1, q_N})^2}{4} \\ -\frac{([2]_{q_2, q_1})^2}{4} & -\frac{q_2^2}{2} & \dots & -\frac{([2]_{q_2, q_N})^2}{4} \\ \dots & \dots & \dots & \dots \\ -\frac{([2]_{q_N, q_1})^2}{4} & -\frac{([2]_{q_N, q_2})^2}{4} & \dots & -\frac{q_N^2}{2} \end{pmatrix}
 \end{aligned}$$

From Definition 1 and the Table 3, we easily acquire the first few Genocchi-type poly-nomials

Genocchi-type polynomials

$$\mathcal{G}_0(x: q_i, q_j) = 0$$

$$\mathcal{G}_1(x: q_i, q_j) = \frac{[2]_{q_i, q_j}}{2}$$

$$\mathcal{G}_2(x: q_i, q_j) = \frac{([2]_{q_i, q_j})^2}{2} x - \frac{([2]_{q_i, q_j})^2}{4}$$

Usual Genocchi polynomials

$$G_0 = 0$$

$$G_1 = 1$$

$$G_2 = 2x - 1$$

Moreover the first few Genocchi-type polynomials can be shown $N \times N$ matrix with multiple q -numbers elements in the following form

$$\begin{aligned}
 (\mathcal{G}_0(q_i, q_j)) &= \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\
 (\mathcal{G}_1(q_i, q_j)) &= \begin{pmatrix} q_1 & \frac{[2]_{q_1, q_2}}{2} & \dots & \frac{[2]_{q_1, q_N}}{2} \\ \frac{[2]_{q_2, q_1}}{2} & q_2 & \dots & \frac{[2]_{q_2, q_N}}{2} \\ \dots & \dots & \dots & \dots \\ \frac{[2]_{q_N, q_1}}{2} & \frac{[2]_{q_N, q_2}}{2} & \dots & q_N \end{pmatrix}
 \end{aligned}$$

$$\left(\mathcal{G}_2(x: q_i, q_j) \right) = \begin{pmatrix} 2q_1^2x - \frac{q_1^2}{2} & \frac{([2]_{q_1, q_2})^2}{2} - \frac{([2]_{q_1, q_2})^2}{4} & \dots & \frac{([2]_{q_1, q_N})^2}{2} - \frac{([2]_{q_1, q_N})^2}{4} \\ \frac{([2]_{q_2, q_1})^2}{2} - \frac{([2]_{q_2, q_1})^2}{4} & 2q_2^2x - \frac{q_2^2}{2} & \dots & \frac{([2]_{q_2, q_N})^2}{2} - \frac{([2]_{q_2, q_N})^2}{4} \\ \dots & \dots & \dots & \dots \\ \frac{([2]_{q_N, q_1})^2}{2} - \frac{([2]_{q_N, q_1})^2}{4} & \frac{([2]_{q_N, q_2})^2}{2} - \frac{([2]_{q_N, q_2})^2}{4} & \dots & 2q_N^2x - \frac{q_N^2}{2} \end{pmatrix}$$

References

- [1] G. M. Philips, *Bernstein polynomials based on the q -integers*, Annals of Numerical Mathematics, vol. 4, no. 1-4, pp. 511-518, 1997.
- [2] C. S. Ryoo, *A note on q -Bernoulli numbers and polynomials*, Appl. Math. Lett. 20 (2007) 524-531.
- [3] V. Kac and P. Cheung, *Quantum Calculus*, New York: Springer, 2002.
- [4] G. S. Cheon, *A note on the Bernoulli and Euler polynomials*, Appl. Math. Lett. 16 (2003) 365-368.
- [5] Q. M. Luo, *Some results for the q -Bernoulli and q -Euler polynomials*, J. Math. Anal. Appl. 363 (2010) 7-18.
- [6] H. M. Srivastava, *Some generalizations and basic (or q -) extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inform. Sci. 5 (2011), 390-444.
- [7] H. M. Srivastava, *Some formulas for the Bernoulli and Euler polynomials at rational arguments*, Math. Proc. Cambridge Philos. Soc. 129 (2000), 77-84.
- [8] S. Nalci and O. K. Pashaev, *q -Bernoulli Numbers and Zeros of q -Sine Function*, arXiv: 1202.2265v12012.
- [9] S. Nalci and O.K. Pashaev, *Exactly Solvable q -extended Nonlinear Classical and Quantum Models*, Lambert Academic Publishing, 2014.
- [10] M. Acikgoz, S. Araci, U. Duran, *Research on Some New Theorems and Some Elementary Functions Under Multiple Quantum Calculus*, preprint.
- [11] U. Duran, M. Acikgoz, A. Esi, S. Araci, *Some new symmetric identities involving q -Genocchi polynomials under S_4* , Journal of Mathematical Analysis, Vol 6, Issue 4 (2015).
- [12] U. Duran, M. Acikgoz, S. Araci, *Symmetric identities involving weighted q -Genocchi polynomials under S_4* , Proceedings of the Jangjeon Mathematical Society, 18 (2015), No. 4, pp 455-465.
- [13] S. Araci, E. Ağyüz and M. Acikgoz, *On a q -analog of some numbers and polynomials*, J. Ineq. Appl. (2015) 2015:19.
- [14] S. Araci, M. Acikgoz, A. Bagdasaryan, E. Sen, *The Legendre Polynomials Associated with Bernoulli, Euler, Hermite and Bernstein Polynomials*, Turkish J. Anal. Number Theory, Vol. 1, No. 1, (2013) 1-3.
- [15] S. Araci, M. Acikgoz, E. Sen, *Some New Formulae for Genocchi Numbers and Polynomials Involving Bernoulli and Euler Polynomials*, Int. J. Math. Math. Sci., vol. 2014(2014), Art. ID: 760613, 7pp.
- [16] S. Araci, U. Duran, M. Acikgoz, *Symmetric Identities Involving q -Frobenius-Euler Polynomials under $Sym(5)$* , Turkish J. Anal. Number Theory, Vol. 3, No. 3, (2015) pp 90-93.
- [17] T. Kim, *q -Volkenborn integration*, Russ. J. Math. Phys. Vol. 9, no.3, pp. 288-299, 2002.
- [18] T. Kim, *q -Generalized Euler numbers and polynomials*, Russ. J. Math. Phys., Vol. 13, No. 3 (2006), pp. 293-298.
- [19] T. Kim, *Symmetry p -adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Differ. Equ. Appl. 14 (2008), 1267-1277.
- [20] B. A. Kupersmidt, *Reflection Symmetries of q -Bernoulli Polynomials*, J. Nonlinear Math. Phys. 12 (Suppl. 1), 412-422 (2005).
- [21] N. I. Mahmudov, *On a class of q -Bernoulli and Euler polynomials*, Advances in Difference Equations 2013, 2013:108.