Symmetric Identities Involving q -Frobenius-Euler Polynomials under Sym (5)

Serkan Araci^{1*}, Ugur Duran², Mehmet Acikgoz²

¹Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep, TURKEY

²University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, Gaziantep, TURKEY

*Corresponding author: mtsrkn@hotmail.com

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Abstract Following the definition of q -Frobenius-Euler polynomials introduced in [3], we derive some new symmetric identities under sym (5), also termed symmetric group of degree five, which are derived from the fermionic p -adic q -integral over the p -adic numbers field.

Keywords: Symmetric identities, q -Frobenius-Euler polynomials, Fermionic p -adic q -integral on $\mathbb{Z}_p$ Invariant under S_5

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1. Introduction

As it is known, the Frobenius-Euler polynomials $H_n(x)$ for $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ are defined by means of the power series expansion at $t = 0$

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \frac{1-\lambda}{e^t - \lambda} e^{xt}. \quad (1.1)$$

Taking $x = 0$ in the Eq. (1.1), we have $H_n(0) := H_n$ that is widely known as n -th Frobenius-Euler number cf. [3,4,5,8,17,18,21].

Let p be chosen as a fixed odd prime number. Throughout this paper, we make use of the following notations: $\mathbb{Z}_p$ denotes topological closure of $\mathbb{Z}$, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes topological closure of $\mathbb{Q}$, and $\mathbb{C}_p$ indicates the field of p -adic completion of an algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

For d an odd positive number with $(p, d) = 1$, let

$$X := X_d = \lim_{\substack{\longrightarrow \\ n}} \mathbb{Z} / dp^N \mathbb{Z} \text{ and } X_1 = \mathbb{Z}_p$$

and

$$a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a \leq dp^N$. See, for details, [1,2,3,4,6-17].

The normalized absolute value according to the theory of p -adic analysis is given by $|p|_p = p^{-1}$. q can be considered as an indeterminate a complex number $q \in \mathbb{C}$ with $|q| < 1$, or a p -adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. It is always clear in the content of the paper.

Throughout this paper, we use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}. \quad (1.2)$$

which is called q -extension of x . It easily follows that $\lim_{q \rightarrow 1} [x]_q = x$ for any x .

Let f be uniformly differentiable function at a point $a \in \mathbb{Z}_p$, which is denoted by $f \in UD(\mathbb{Z}_p)$. Then the p -adic q -integral on $\mathbb{Z}_p$ (or sometimes called q -Volkenborn integral) of a function f is defined by Kim [10]

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \quad (1.3)$$

It follows from the Eq. (1.3) that

$$\begin{aligned} \lim_{q \rightarrow -1} I_q(f) &= I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) \\ &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x. \end{aligned} \quad (1.4)$$

Thus, by the Eq. (1.4), we have

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{s=0}^{n-1} (-1)^{n-s-1} f(s)$$

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$. For the applications of fermionic p -adic integral over the p -adic numbers field, see the references, e. g., [1,2,3,4,6,7,9,11,12,16].

In [3], the q -Frobenius-Euler polynomials are defined by the following p -adic fermionic q -integral on $\mathbb{Z}_p$, with respect to μ_{-1} :

$$\begin{aligned} H_{n,q}(x | -\lambda^{-1}) \\ = \frac{\lambda+1}{2} \int_{\mathbb{Z}_p} \lambda^y [x+y]_q^n d\mu_{-1}(y). \end{aligned} \quad (1.5)$$

Upon setting $x = 0$ into the Eq. (1.5) gives $H_{n,q}(0) := H_{n,q}$ which are called n -th q -Frobenius-Euler number.

By letting $q \rightarrow 1^-$ in the Eq. (1.5), it yields to

$$\begin{aligned} \lim_{q \rightarrow 1^-} H_{n,q}(x | -\lambda^{-1}) &:= H_n(x | -\lambda^{-1}) \\ &= \frac{\lambda+1}{2} \int_{\mathbb{Z}_p} \lambda^y (x+y)^n d\mu_{-1}(y). \end{aligned}$$

Recently, many mathematicians have studied the symmetric identities on some special polynomials, see, for details, [1,6,7,9,12]. Some of mathematicians also investigated some applications of Frobenius-Euler numbers and polynomials (or its q -analog) cf. [3,4,5,13,14,15,16]. Moreover, Frobenius-Euler numbers at the value $\lambda = -1$ in (1.1) are Euler numbers that is closely related to Bernoulli numbers, Genocchi numbers, etc. For more information about these polynomials, look at [1-21] and the references cited therein.

In the present paper, we obtain not only new but also some interesting identities which are derived from the fermionic p -adic q -integral over the p -adic numbers field. The results derived here is written under Sym (5).

2. Symmetric Identities Involving q -Frobenius-Euler Polynomials

For $w_i \in \mathbb{N}$ with $w_i = 1 \pmod{2}$ with $i \in \{1, 2, 3, 4, 5\}$, by the Eqs. (1.3) and (1.5), we obtain

$$\begin{aligned} \int_{\mathbb{Z}_p} \lambda^{w_1 w_2 w_3 w_4 y} e^{\left[\begin{array}{l} w_1 w_2 w_3 w_4 y \\ + w_1 w_2 w_3 w_4 w_5 x \\ + w_5 w_4 w_2 w_3 i \\ + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k \\ + w_5 w_3 w_1 w_2 h \end{array} \right]_q} d\mu_{-1}(y) \\ = \frac{2}{\lambda+1} \lim_{N \rightarrow \infty} \sum_{y=0}^{p^N-1} (-1)^y \lambda^{w_1 w_2 w_3 w_4 y} \times e^{\left[\begin{array}{l} w_1 w_2 w_3 w_4 y \\ + w_1 w_2 w_3 w_4 w_5 x \\ + w_5 w_4 w_2 w_3 i \\ + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k \\ + w_5 w_3 w_1 w_2 h \end{array} \right]_q} \end{aligned} \quad (2.1)$$

$$\begin{aligned} &= \frac{2}{\lambda+1} \lim_{N \rightarrow \infty} \sum_{l=0}^{p^N-1} \sum_{y=0}^{p^N-1} (-1)^{l+y} \lambda^{w_1 w_2 w_3 w_4 (l+w_5 y)} \\ &\quad \times e^{\left[\begin{array}{l} w_1 w_2 w_3 w_4 (l+w_5 y) + w_1 w_2 w_3 w_4 w_5 x \\ + w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h \end{array} \right]_q} \end{aligned}$$

Taking

$$\frac{\lambda+1}{2} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{h=0}^{w_4-1} \left[\begin{array}{l} (-1)^{i+j+k+h} \\ \times \lambda^{\left(\begin{array}{l} w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h \end{array} \right)} \end{array} \right]$$

on the both sides of Eq. (2.1) gives

$$\begin{aligned} &\frac{\lambda+1}{2} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{h=0}^{w_4-1} \left[\begin{array}{l} (-1)^{i+j+k+h} \\ \times \lambda^{\left(\begin{array}{l} w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h \end{array} \right)} \end{array} \right] \\ &\quad \times \int_{\mathbb{Z}_p} \lambda^{w_1 w_2 w_3 w_4 y} e^{\left[\begin{array}{l} w_1 w_2 w_3 w_4 y \\ + w_1 w_2 w_3 w_4 w_5 x \\ + w_5 w_4 w_2 w_3 i \\ + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k \\ + w_5 w_3 w_1 w_2 h \end{array} \right]_q} d\mu_{-1}(y) \\ &= \lim_{N \rightarrow \infty} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{h=0}^{w_4-1} \sum_{l=0}^{p^N-1} \sum_{y=0}^{p^N-1} (-1)^{i+j+k+h+y+l} \quad (2.2) \\ &\quad \times \lambda^{\left[\begin{array}{l} w_1 w_2 w_3 w_4 (l+w_5 y) + w_5 w_4 w_2 w_3 i \\ + w_5 w_4 w_1 w_3 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h \end{array} \right]} \\ &\quad \times e^{\left[\begin{array}{l} w_1 w_2 w_3 w_4 (l+w_5 y) + w_1 w_2 w_3 w_4 w_5 x \\ + w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h \end{array} \right]_q} \end{aligned}$$

Note that the equation (2.2) is invariant for any permutation $\sigma \in S_5$. Hence, we have the following theorem.

Theorem 1. Let $w_i \in \mathbb{N}$ with $w_i = 1 \pmod{2}$ with $i \in \{1, 2, 3, 4, 5\}$. Then the following

$$\begin{aligned} &\frac{\lambda+1}{2} \sum_{i=0}^{w_{\sigma(1)}-1} \sum_{j=0}^{w_{\sigma(2)}-1} \sum_{k=0}^{w_{\sigma(3)}-1} \sum_{h=0}^{w_{\sigma(4)}-1} (-1)^{i+j+k+h} \\ &\quad \times \lambda^{\left[\begin{array}{l} w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(2)} w_{\sigma(3)} i \\ + w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(3)} j \\ + w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(2)} k \\ + w_{\sigma(5)} w_{\sigma(3)} w_{\sigma(1)} w_{\sigma(2)} h \end{array} \right]} \\ &\quad \times \int_{\mathbb{Z}_p} \lambda^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)} (l+w_{\sigma(5)} y)} \\ &\quad \exp([w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)} y \\ &\quad + w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)} w_{\sigma(5)} x \\ &\quad + w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(2)} w_{\sigma(3)} i \\ &\quad + w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(3)} j \\ &\quad + w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(2)} k \\ &\quad + w_{\sigma(5)} w_{\sigma(3)} w_{\sigma(1)} w_{\sigma(2)} h]_q t) d\mu_{-1}(y) \end{aligned}$$

holds true for any $\sigma \in S_5$.

By Eq. (1.2), we easily derive that

$$\begin{aligned} & \left[\begin{array}{l} w_1 w_2 w_3 w_4 y + w_1 w_2 w_3 w_4 w_5 x + w_5 w_4 w_2 w_3 i \\ + w_5 w_4 w_1 w_3 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h \end{array} \right]_q \\ &= [w_1 w_2 w_3 w_4]_q \left[\begin{array}{l} y + w_5 x + \frac{w_5}{w_1} i \\ + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} h \end{array} \right]_{q^{w_1 w_2 w_3 w_4}} \end{aligned} \quad (2.3)$$

From Eq. (2.1) and (2.3), we obtain

$$\begin{aligned} & \int_{\mathbb{Z}_p} \lambda^{w_1 w_2 w_3 w_4 y} e^{\left[\begin{array}{l} w_1 w_2 w_3 w_4 y \\ + w_1 w_2 w_3 w_4 w_5 x \\ + w_5 w_4 w_2 w_3 i \\ + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k \\ + w_5 w_3 w_1 w_2 h \end{array} \right]_q} d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty} [w_1 w_2 w_3 w_4]_q^n \\ & \times \left(\int_{\mathbb{Z}_p} \left\{ \begin{array}{l} \lambda^{w_1 w_2 w_3 w_4 y} \\ \left[y + w_5 x + \frac{w_5}{w_1} i \right]^n \\ + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k \\ + \frac{w_5}{w_4} h \end{array} \right\}_{q^{w_1 w_2 w_3 w_4}} d\mu_{-1}(y) \right) \frac{t^n}{n!}, \end{aligned} \quad (2.4)$$

from which, we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \lambda^{w_1 w_2 w_3 w_4 y} \left[\begin{array}{l} w_1 w_2 w_3 w_4 y \\ + w_1 w_2 w_3 w_4 w_5 x \\ + w_5 w_4 w_2 w_3 i \\ + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k \\ + w_5 w_3 w_1 w_2 h \end{array} \right]_q^n d\mu_{-1}(y) \\ &= \frac{2}{\lambda+1} [w_1 w_2 w_3 w_4]_q^n H_{n,q}^{w_1 w_2 w_3 w_4} \\ & \left(w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} h \mid -\lambda^{-w_1 w_2 w_3 w_4} \right). \end{aligned} \quad (2.5)$$

Thus, by Theorem 1 and (2.5), we procure the following theorem.

Theorem 2. For $w_i \in \mathbb{N}$ with $w_i \equiv 1 \pmod{2}$ with $i \in \{1, 2, 3, 4, 5\}$, the following

$$\begin{aligned} & \left[w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)} \right]_q^n \\ & \times \sum_{i=0}^{w_{\sigma(1)}-1} \sum_{j=0}^{w_{\sigma(2)}-1} \sum_{k=0}^{w_{\sigma(3)}-1} \sum_{h=0}^{w_{\sigma(4)}-1} (-1)^{i+j+k+h} \\ & \times \lambda^{\left(\begin{array}{l} w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(2)} w_{\sigma(3)} i + w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(3)} j \\ + w_{\sigma(5)} w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(2)} k + w_{\sigma(5)} w_{\sigma(3)} w_{\sigma(1)} w_{\sigma(2)} h \end{array} \right)} \end{aligned}$$

$$\times H_{n,q}^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)}}$$

$$\left(w_{\sigma(5)} x + \frac{w_{\sigma(5)}}{w_{\sigma(1)}} i + \frac{w_{\sigma(5)}}{w_{\sigma(2)}} j + \frac{w_{\sigma(5)}}{w_{\sigma(3)}} k \right. \\ \left. + \frac{w_{\sigma(5)}}{w_{\sigma(4)}} h \mid -\lambda^{-w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)}} \right)$$

holds true for any $\sigma \in S_5$.

It is shown by using the definition of $[x]_q$ that

$$\begin{aligned} & \left[y + w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} h \right]_{q^{w_1 w_2 w_3 w_4}}^n \\ &= \sum_{m=0}^n \binom{n}{m} \left(\frac{[w_5]_q}{[w_1 w_2 w_3 w_4]_q} \right)^{n-m} \\ & \times \left[\begin{array}{l} w_4 w_2 w_3 i + w_4 w_1 w_3 j \\ + w_4 w_1 w_2 k + w_3 w_1 w_2 h \end{array} \right]_{q^{w_5}}^{n-m} \\ & \times q^{m \left(\begin{array}{l} w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j \\ + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h \end{array} \right)} [y + w_5 x]_{q^{w_1 w_2 w_3 w_4}}^m. \end{aligned} \quad (2.6)$$

Taking $\int_{\mathbb{Z}_p} \lambda^{w_1 w_2 w_3 w_4 y} d\mu_{-1}(y)$ on the both sides of Eq.(2.6) gives

$$\begin{aligned} & \int_{\mathbb{Z}_p} \lambda^{w_1 w_2 w_3 w_4 y} \left[\begin{array}{l} y + w_5 x + \frac{w_5}{w_1} i \\ + \frac{w_5}{w_2} j \\ + \frac{w_5}{w_3} k + \frac{w_5}{w_4} h \end{array} \right]_{q^{w_1 w_2 w_3 w_4}}^n d\mu_{-1}(y) \\ &= \sum_{m=0}^n \binom{n}{m} \left(\frac{[w_5]_q}{[w_1 w_2 w_3 w_4]_q} \right)^{n-m} \\ & \times \left[\begin{array}{l} w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k + w_3 w_1 w_2 h \end{array} \right]_{q^{w_5}}^{n-m} \\ & \times q^{m \left(\begin{array}{l} w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h \end{array} \right)} \\ & \times \int_{\mathbb{Z}_p} \lambda^{w_1 w_2 w_3 w_4 y} [y + w_5 x]_{q^{w_1 w_2 w_3 w_4}}^m d\mu_{-1}(y) \quad (2.7) \\ &= \frac{2}{1+\lambda} \sum_{m=0}^n \binom{n}{m} \left(\frac{[w_5]_q}{[w_1 w_2 w_3 w_4]_q} \right)^{n-m} \\ & \times \left[\begin{array}{l} w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k + w_3 w_1 w_2 h \end{array} \right]_{q^{w_5}}^{n-m} \\ & \times q^{m \left(\begin{array}{l} w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h \end{array} \right)} \\ & \times H_{n,q}^{w_1 w_2 w_3 w_4} \left(w_5 x \mid -\lambda^{-w_1 w_2 w_3 w_4} \right). \end{aligned}$$

By the Eq. (2.7), we have

$$[w_1 w_2 w_3 w_4]_q^n \frac{\lambda+1}{2} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{h=0}^{w_4-1} (-1)^{i+j+k+h}$$

$$\begin{aligned}
 & \times \lambda^{(w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h)} \\
 & \times \int_{\mathbb{Z}_p} \lambda^{w_1 w_2 w_3 w_4 y} \left[y + w_5 x + \frac{w_5}{w_1} i + \frac{w_5}{w_2} j + \frac{w_5}{w_3} k + \frac{w_5}{w_4} h \right]_q^{n-1} d\mu_{-1}(y) \\
 & = \sum_{m=0}^n \binom{n}{m} [w_1 w_2 w_3 w_4]_q^m [w_5]_q^{n-m} \\
 & \times H_{n,q}^{w_1 w_2 w_3 w_4} \left(w_5 x \mid -\lambda^{-w_1 w_2 w_3 w_4} \right) \\
 & \times \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{h=0}^{w_4-1} (-1)^{i+j+k+h} \\
 & \times \lambda^{(w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h)} \\
 & \times q^{m(w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k + w_3 w_1 w_2 h)} \\
 & \times [w_2 w_4 w_3 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k + w_1 w_2 w_3 h]_q^{n-m} \\
 & = \sum_{m=0}^n \binom{n}{m} [w_1 w_2 w_3 w_4]_q^m [w_5]_q^{n-m} \\
 & \times H_{n,q}^{w_1 w_2 w_3 w_4} \left(w_5 x \mid -\lambda^{-w_1 w_2 w_3 w_4} \right) \\
 & \times C_{n,q}^{w_5} (w_1, w_2, w_3, w_4 \mid m), \tag{2.8}
 \end{aligned}$$

where

$$\begin{aligned}
 & C_{n,q}^{w_5} (w_1, w_2, w_3, w_4 \mid m) \\
 & = \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_3-1} \sum_{h=0}^{w_4-1} (-1)^{i+j+k+h} \\
 & \times \lambda^{(w_5 w_4 w_2 w_3 i + w_5 w_4 w_1 w_3 j + w_5 w_4 w_1 w_2 k + w_5 w_3 w_1 w_2 h)} \\
 & \times q^{m(w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k + w_3 w_1 w_2 h)} \\
 & \times [w_2 w_4 w_3 i + w_1 w_3 w_4 j + w_1 w_2 w_4 k + w_1 w_2 w_3 h]_q^{n-m}. \tag{2.9}
 \end{aligned}$$

Consequently, by (2.9), we get the following theorem.

Theorem 3. Let $w_i \in \mathbb{N}$ with $w_i \equiv 1 \pmod{2}$ with $i \in \{1, 2, 3, 4, 5\}$. Then the following expression

$$\begin{aligned}
 & \sum_{m=0}^n \binom{n}{m} [w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)}]_q^m [w_5]_q^{n-m} \\
 & \times H_{n,q}^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)}} \left(w_{\sigma(5)} x \mid -\lambda^{\left[\begin{smallmatrix} -w_{\sigma(1)} w_{\sigma(2)} \\ \times w_{\sigma(3)} w_{\sigma(4)} \end{smallmatrix} \right]} \right) \\
 & \times C_{n,q}^{w_{\sigma(5)}} (w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)}, w_{\sigma(4)} \mid m)
 \end{aligned}$$

holds true for some $\sigma \in S_5$.

3. Conclusion

We have derived some new interesting identities of q -Frobenius-Euler polynomials. We also showed that these symmetric identities are written by symmetric group of degree five.

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