Coefficient Estimates for Starlike and Convex Classes of \( m \)-fold Symmetric Bi-univalent Functions

S. Sivasubramanian\(^1\), P. Gurusamy\(^2\*\)

\(^1\)Department of Mathematics, University College of Engineering Tindivanam, Anna University, Tindivanam, India
\(^2\)Department of Mathematics, Velammal Engineering College, Surapet, Chennai, India

*Corresponding author: gurusamy65@gmail.com

Received March 31, 2015; Revised May 31, 2015; Accepted August 02, 2015

Abstract In an article of Pommerenke [10] he remarked that, for an \( m \)-fold symmetric functions in the class \( \mathcal{P} \), the well known lemma stated by Caratheodary for a one fold symmetric functions in \( \mathcal{P} \) still holds good. Exploiting this concept, we introduce certain new subclasses of the bi-univalent function class \( \sigma \) in which both \( f \) and \( f^{-1} \) are \( m \)-fold symmetric analytic with their derivatives in the class \( \mathcal{P} \) of analytic functions. Furthermore, for functions in each of the subclasses introduced in this paper, we obtain the coefficient bounds for \( |a_{n+1}| \) and \( |a_{2m+1}| \). We remark here that the concept of \( m \)-fold symmetric bi-univalent is not in the literature and the authors hope it will make the researchers interested in these type of investigations in the forseeable future. By the working procedure and the difficulty involved in these procedures, one can clearly conclude that there lies an unpredictability in finding the coefficients of a \( m \)-fold symmetric bi-univalent functions.

Keywords: analytic functions, univalent functions, bi-univalent functions, \( m \)-fold symmetric functions, subordination


1. Introduction

Let \( \mathcal{A} \) denote the class of functions of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U},
\]

which are analytic in the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \). Further, by \( \mathcal{S} \), we mean the class of all functions in \( \mathcal{A} \) which are univalent in \( \mathbb{U} \). For more details on univalent functions, see [3]. It is well known that every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), defined by

\[
f^{-1}(f(z)) = z \quad (z \in \mathbb{U})
\]

and

\[
f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); r_0(f) \geq \frac{1}{2} \right).
\]

Indeed, the inverse function may have an analytic continuation to \( \mathbb{U} \), with

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f \) and \( f^{-1} \) are univalent in \( \mathbb{U} \). Let \( \sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \), given by equation (1). An analytic function \( f \) subordinate to an analytic function \( g \), written \( f(z) < g(z) \), provided there is an analytic function \( w \) defined on \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) satisfying \( f(z) = g(w(z)) \). Lewin [8] investigated the class of bi-univalent functions \( \sigma \) and obtained a bound \( |a_2| \leq 1.51 \). Motivated by the work of Lewin [8], Brannan and Clunie [1] conjectured that \( |a_2| \leq \sqrt{2} \). Some examples of bi-univalent functions are \( z, \frac{z}{1-z^2} \) and \( -\log(1-z) \) (see also the work of Srivastava et al. [11]). The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients: \( |a_n| \) \( n \in \mathbb{N}, \ n \geq 3 \) is still open([11]). In recent times, the study of bi-univalent functions gained momentum mainly due to the work of Srivastava et al. [11]. Motivated by this, many researchers (see [4,11,12,13,14,15,17]) recently investigated several interesting subclasses of the class \( \sigma \) and found non-sharp estimates on the first two Taylor-Maclaurin coefficients: \( |a_{n+1}| \) \( (n \in \mathbb{N}, \ n \geq 3) \) in particular on \( a_2 \).

For each function \( f \) in \( \mathcal{S} \), the function \( h(z) = \sqrt{f(z^2)} \) is univalent and maps the unit disk \( \mathbb{U} \) into a region with \( m \)-fold symmetry. A function is \( m \)-fold symmetric (see [10]) if it has the normalized form

\[
f(z) = z + \sum_{n=1}^{\infty} a_{mk+1} z^{mk+1}, \quad z \in \mathbb{U},
\]

and we denote the class of \( m \)-fold symmetric univalent functions by \( \mathcal{S}_m \), which are normalized by the above series expansion. In fact, the functions in the class \( \mathcal{S}_m \) are one fold symmetric. Analogous to the concept of \( m \)-fold symmetric univalent functions, one can think of the concept of \( m \)-fold symmetric bi-univalent functions in a natural way. Each function in the class \( f \) in \( \sigma \), generates an \( m \)-fold symmetric bi-univalent function for each integer \( m \). The normalized form of \( f \) is given as in (5) and \( f^{-1} \) is given as follows.
2. Coefficient Estimates for the Function Class \( ST_{\sigma,m}(\varphi) \)

**Definition 2.1** A function \( f(z) \), given by (5), is said to be in the class \( ST_{\sigma,m}(\varphi) \), if the following conditions are satisfied:

\[
f \in \sigma_m, \quad \frac{zf'(z)}{f(z)} = \varphi(z)
\]

and

\[
\frac{wg'(w)}{g(w)} < \varphi(w), \quad g(w) = f^{-1}(w)
\]

where the function \( g \) is defined by (6).

For the special choices of the function \( \varphi(z) \) and for the choice of \( m = 1 \), our class reduces to the following.

1. For \( m = 1 \) and \( \varphi(z) = \frac{1+e^z}{1-z} \), \( 0 < \gamma \leq 1 \), \( ST_{\sigma,1}(1, \left( \frac{1+e^z}{1-z} \right)^\gamma) \equiv ST_\gamma(\varphi) \), \( 0 < \gamma \leq 1 \), the class of strongly bi-starlike functions of order \( \gamma \) studied by Brannan and Taha [2].

2. For \( m = 1 \) and \( \varphi(z) = \frac{1+(1-2\gamma)z}{1-z} \), \( 0 \leq \gamma < 1 \), \( ST_{\sigma,1}(1, \left( \frac{1+(1-2\gamma)z}{1-z} \right)^\gamma) \equiv ST_\gamma(\varphi) \), \( 0 \leq \gamma < 1 \), the class of bi-starlike functions of order \( \alpha \) studied by Brannan and Taha [2].

We first state and prove the following theorem.

**Theorem 2.1** Let \( f(z) \), given by (5), be in the class \( ST_{\sigma,m}(\varphi) \).

Then

\[
|a_{m+1}| \leq \frac{B_1\sqrt{B_1}}{m\sqrt{B_1^2 - B_2^2}}
\]

and

\[
|a_{2m+1}| \leq \begin{cases} \frac{(1+m)B_1}{2m^2} & \text{if } |B_2| \leq B_1 \\ \frac{(1+m)(B_2^2 - B_1^2 + 2|B_2|B_1)}{2m^2(B_1^2 - B_2^2 + B_1^2)} & \text{if } |B_2| > B_1 \end{cases}
\]

**Proof.** Let \( f \in ST_{\sigma,m}(\varphi) \) and \( g = f^{-1} \). Then there are analytic functions \( u, v : \mathbb{U} \rightarrow \mathbb{U} \), with \( u(0) = v(0) = 0 \) satisfying

\[
\frac{zf'(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} = \varphi(v(w)).
\]

Since

\[
\frac{zf'(z)}{f(z)} = 1 + ma_{m+1}z^m + (2ma_{2m+1} - ma_{m+1}^2)z^{2m} + \cdots
\]

and

\[
\frac{wg'(w)}{g(w)} = 1 - ma_{m+1}w^m + (m(2m+1)a_{m+1}^2 - 2ma_{2m+1})w^{2m} + \cdots
\]

it follows from (12), (13) and (16) that

\[
m\sigma_{m+1} = B_1b_m, \quad 2ma_{2m+1} - ma_{m+1}^2 = B_1b_{2m} + B_2b_{m+1}^2, \quad -ma_{m+1} = B_1c_m,
\]

and

\[
m(2m+1)a_{m+1}^2 - 2ma_{2m+1} = B_1c_2m + B_2b_{m+1}^2.
\]

From (17) and (19), we get

\[
c_m = -b_m.
\]

By adding (18) and (20) and in view of the computations using (17) and (21), we get

\[
2m^2(B_1^2 - B_2^2)a_{m+1}^2 = B_1^2(b_{2m} + c_{2m+1}).
\]

Further, (21), (22), together with (11), gives

\[
|m^2(B_1^2 - B_2^2)a_{m+1}^2| \leq B_1^2(1 - |b_m|^2).
\]

Now from (17) and (23), we get
\[ |a_{m+1}| \leq \frac{b_1 \sqrt{b_1}}{m |b_1 + |b_1 - b_2|} \]
as asserted in (14).

By simple calculations from (18) and (20) using with the equations (17) and (21), we get
\[ 4m^2a_{m+1} = (1 + 2m)B_1B_{2m} + B_1\gamma \Delta m + 2(1 + m)B_2 b_2^m \] (24)
Then using the equation (11) in (24), we get
\[ |a_{m+1}| \leq \frac{(1+m)B_1}{2m^2} + \frac{(1+m)(\Delta m - B_1)|b_m|^2}{2m^2}. \] (25)
Since
\[ |b_m|^2 \leq \frac{b_1}{|b_1 - b_2|} \] (26)
substituting (26) in (25), we get
\[ |a_{m+1}| \leq \begin{cases} \frac{(1+m)B_1}{2m^2} & \text{if } |B_2| \leq B_1 \\ \frac{(1+m)}{2m^2} \left| B_2 - B_1 \right| & \text{if } |B_2| > B_1 \end{cases} \]
as asserted in (15). This completes the proof of Theorem 2.1.

For the case of one fold symmetric functions, Theorem 2.1 reduces to the coefficient estimates for Ma-Minda bi-starlike functions in Srivastava et al [11].

**Corollary 2.1** Let \( f(z) \), given by (5), be in the class \( ST_\sigma(\phi) \). Then
\[ |a_2| \leq \frac{b_1 \sqrt{b_1}}{|b_1 + (b_1 - b_2)|} \] (27)
and
\[ |a_3| \leq B_1 + |B_2 - B_1|. \] (28)

For the case of one fold symmetric functions and for the class of strongly starlike functions, the function \( \phi \) is given by
\[ \phi(z) = \left( \frac{1 + z}{1 - z} \right)^{\gamma} = 1 + 2\gamma z + z^2 + \cdots, \quad (0 < \gamma \leq 1) \] (29)
which gives \( B_1 = 2\gamma \) and \( B_2 = 2\gamma^2 \). Hence Theorem 2.1 reduce to the result in Brannan and Taha [2].

**Corollary 2.2** [2] Let \( f(z) \), given by (5), be in the class \( ST_\sigma(\phi) \). Then
\[ |a_2| \leq \frac{2\gamma}{\sqrt{1 + \gamma}} \] (30)
and
\[ |a_3| \leq 2\gamma^2. \] (31)

For the case of one fold symmetric functions and for the class of strongly starlike functions, the function \( \phi \) is given by
\[ \phi(z) = 1 + 2(1 - \gamma)z + 2(1 - \gamma)^2z^2 + \cdots \]
then \( B_1 = B_2 = 2(1 - \gamma) \), and the Theorem 2.1 reduce to the result in Brannan and Taha [2].

**Corollary 2.3** [2] Let \( f(z) \), given by (5), be in the class \( ST_\sigma(\phi) \). Then
\[ |a_2| \leq \sqrt{2(1 - \gamma)}, \] (32)
and
\[ |a_3| \leq \sqrt{2(1 - \gamma)}. \] (33)

### 3. Coefficient Bound for the Function Class \( CV_{\sigma,m}(\lambda, \varphi) \)

**Definition 3.1** A function \( f(z) \), given by (5), is said to be in the class \( CV_{\sigma,m}(\lambda, \varphi) \), if the following conditions are satisfied:
\[ f \in \sigma_m, 1 + \frac{zf''(z)}{f'(z)} < \varphi(z) \]
and
\[ 1 + \frac{wg''(w)}{\varphi'(w)} < \varphi(w), \]
where the function \( g \) is defined by (6).

For one fold symmetric, a function in the class \( CV_{\sigma,m}(\lambda, \varphi) \) is called bi-Mocanu-convex function of Minda type. For the special choices of the function \( \varphi(z) \), and for the choice of \( m = 1 \), our class reduces to the following.

1. For \( m = 1 \) and \( \varphi(z) = \frac{1 + (1-2\gamma)z}{1-z}, (0 \leq \gamma < 1) \), \( CV_{\sigma,1} \left( 1, \left( \frac{1 + z}{1 - z} \right)^{\gamma} \right) \equiv CV_{\sigma}(\gamma) \), the class of strongly bi-convex functions of order \( \gamma \) studied by Brannan and Taha [2].
2. For \( m = 1 \) and \( \varphi(z) = \frac{1 + (1-2\gamma)z}{1-z}, (0 \leq \gamma < 1) \), \( CV_{\sigma,1} \left( 1, \left( \frac{1 + z}{1 - z} \right)^{\gamma} \right) \equiv C_{\sigma}(\gamma), (0 \leq \gamma < 1) \), the class of bi-convex functions of order \( \gamma \) studied by Brannan and Taha [2].

**Theorem 3.1** Let \( f(z) \), given by (5), be in the class \( CV_{\sigma,m}(\lambda, \varphi), \lambda \geq 0 \). Then
\[ |a_{m+1}| \leq \frac{b_1 \sqrt{b_1}}{m(1 + m)B_1 + (1 + m)|B_1^2 - (1 + m)B_2^2|} \] (34)
and
\[ |a_{m+1}| \leq \begin{cases} \frac{B_1}{2m^2} & \text{if } |B_2| \leq B_1 \\ \frac{B_1^2 - (1 + m)B_2}{2m^2} & \text{if } |B_2| > B_1 \end{cases} \] (35)

**Proof.** Let \( f \in CV_{\sigma,m}(\lambda, \varphi) \). Then there are analytic functions \( u, v: \mathbb{U} \rightarrow \mathbb{U} \), with \( u(0) = v(0) = 0 \) satisfying
\[ 1 + \frac{zf'(z)}{f(z)} = \varphi \left( u(z) \right) \text{ and } 1 + \frac{wg''(w)}{\varphi'(w)} = \varphi \left( v(w) \right). \] (36)

Since
\[ 1 + \frac{zf'(z)}{f(z)} = 1 + m(1 + m)a_{m+1}z^m + \cdots \]
and
\[ 1 + \frac{wg''(w)}{\varphi'(w)} = 1 + m\left( 1 + m \right)a_{m+1}z^m + \cdots \]
(40) using with the equations (37) and (41), we get
\[ 2m^2 \left( 1 + m \right) \left( B_1^2 - \left( 1 + m \right) B_2 \right) \underline{a}_{m+1}^2 \]
\[ = B_1^3 \left( b_{2m} + c_{2m} \right). \]

Further, from the equations (41), (42), together with (11), we have
\[ \left| \left( 1 + m \right) m^2 \left( B_1^2 - (1 + m) B_2 \right) \underline{a}_{m+1}^2 \right| \leq B_1^3 \left( 1 - |b_m|^2 \right). \tag{43} \]

Now from (37) and (43), we get
\[ |a_{m+1}| \leq \frac{B_1 \sqrt{m}}{m \left( 1 + m |b_1|^2 + (1 + m) |b_2|^2 \right)} \]
as asserted in (34). By simple calculations from (38) and (40) using with the equations (37) and (41), we get
\[ 4m^2 \left( 1 + 2m \right) a_{2m+1}^2 \]
\[ = (1 + 3m) B_1 b_{2m} + (1 + m) B_1 c_{2m} + 2(1 + 2m) B_2 b_{2m}^2. \tag{44} \]

Then using the equation (11) in (44), we get
\[ |a_{2m+1}| \leq \frac{|B_1|}{2m^2} + \frac{|B_1 - B_2| |b_m|^2}{2m^2}. \tag{45} \]

Since
\[ |b_m|^2 \leq \frac{(1 + m) B_1}{|B_1^2 - (1 + m) B_2^2| + (1 + m) |B_1|^2}, \tag{46} \]

substituting (46) in (45), we get
\[ |a_{2m+1}| \leq \begin{cases} \frac{B_1}{2m^2} & \text{if } |B_2| \leq B_1 \\ \frac{B_1^2 - (1 + m) B_2 |B_1| (1 + m) |B_2|}{2m^2} & \text{if } |B_2| > B_1 \end{cases} \]

as asserted in (35).

For one fold symmetric functions then, Theorem 3.1 gives the coefficient for Ma-Minda bi-convex functions in Brannan and Taha [2]

\begin{corollary} \[ \text{Let } f(z), \text{ given by (5), be in the class } \mathcal{CT}_c(\varphi). \text{ Then} \]
\[ |a_2| \leq \frac{B_1 \sqrt{2m}}{\sqrt{2|B_1^2 + 4(B_1 - B_2)|}} \]
and
\[ |a_3| \leq \frac{B_1 + |B_2 - B_1|}{2}. \]

For the case of one fold symmetric functions and for the class of strongly starlike functions, the function \( \varphi \) is given by
\[ \varphi(z) = 1 + 2(1 - \gamma)z + 2(1 - \gamma)^2 z^2 + \ldots \]
then \( B_1 = B_2 = 2(1 - \gamma) \), and the Theorem 3.1 reduce to the result in Brannan and Taha [2].

\begin{corollary} \[ \text{Let } f(z), \text{ given by (5), be in the class } \mathcal{CT}_c(\varphi). \text{ Then} \]
\[ |a_2| \leq 2(1 - \gamma) \]
and
\[ |a_3| \leq 1 - \gamma. \]

\end{corollary}

\begin{thebibliography}{10}
\bibitem{3} P. L. Duren, Univalent functions, Springer-Verlag, New York, Berlin, Hiedelberg and Tokyo, 1983.
\end{thebibliography}