The Rogers-Ramanujan Identities

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Abstract In 1894, Rogers found the two identities for the first time. In 1913, Ramanujan found the two identities later and then the two identities are known as The Rogers-Ramanujan Identities. In 1982, Baxter used the two identities in solving the Hard Hexagon Model in Statistical Mechanics. In 1829 Jacobi proved his triple product identity; it is used in proving The Rogers-Ramanujan Identities. In 1921, Ramanujan used Jacobi’s triple product identity in proving his famous partition congruences. This paper shows how to generate the generating function for \( C'_n \), \( C''_n \), and \( C'''_n \), and shows how to prove the Corollaries 1 and 2 with the help of Jacobi’s triple product identity. This paper shows how to prove the Remark 3 with the help of various auxiliary functions and shows how to prove The Rogers-Ramanujan Identities with help of Ramanujan’s device of the introduction of a second parameter \( a \).

Keywords: at most, auxiliary function, convenient, expansion, minimal difference, operator, Ramanujan’s device


1. Introduction

In this article, we give some related definitions of \( P(n) \), \( C'(n) \), \( C''(n) \), \( C'''(n) \), \( P_m(n-m(m+1)) \) and \( C'_1(n) \). We describe the generating functions for \( C'(n) \), \( C''(n) \), \( C'''(n) \), \( P_m(n-m(m+1)) \), and \( C'_1(n) \), and establish the Remarks 1 and 2 with numerical examples and also prove the Corollaries 1 and 2 with the help of Jacobi’s triple product identity [3]. We transfer the auxiliary function into another auxiliary function with the help of Ramanujan’s device of the introduction of a second parameter \( a \) [5];

\[
G_k(a,x) = \prod_{n=0}^{\infty} \left( 1 + \sum_{m=0}^{\infty} (-1)^m a^{2kn} x^{(n(5n+1)-2kn)/2} \right) C_n
\]

to

\[
G_1(x) = \sum_{m=0}^{\infty} \prod_{k=1}^{\infty} \left( 1 - x^{5m+1} \right) \left( 1 - x^{5m+4} \right) \left( 1 - x^{5m+5} \right)
\]

where \( k = 1 \), and \( a = x \), it is used in proving The Rogers-Ramanujan Identity 1. We prove The Rogers-Ramanujan Identities with the help of auxiliary functions.

2. Some Related Definitions

\( P(n) \) [7]: The number of partitions of \( n \) like: 4, 3+1, 2+2, 2+1+1, 1+1+1+1 ... \( P(4) = 5 \).

\( C'(n) \) [6]: The number of partitions of \( n \) into parts of which is of one of the forms \( 5m + 1 \) and \( 5m + 4 \).

\( C''(n) \) : The number of partitions of \( n \) into \( m \) parts at most.

\( C'''(n) \) : The number of partitions of \( n \) into parts without repetitions or parts whose minimal difference is 2.

\( C''''(n) \) : The number of partitions of \( n \) into \( m \) parts at most.

\( C'''''(n) \) : The number of partitions of \( n \) into parts not less than 2 and with minimal difference 2.

3. Generating Functions for \( C'(n) \) and \( C''(n) \)

In this section we describe the generating functions for \( C'(n) \) and \( C''(n) \) respectively. The generating function for \( C'(n) \) is of the form [5];
\[
\sum_{m=0}^{\infty} \frac{1}{\left(1-x^m\right)\left(1-x^{m+1}\right)} = \frac{1}{(1-x)(1-x^2) \cdots (1-x^m)} \quad (1)
\]

\[
= 1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + \cdots \infty
\]

where the coefficient \(C'(n)\) of \(x^n\) is the number of partitions of \(n\) into parts each of which is of one of these forms \(5m+1\) and \(5m+4\).

Now we consider a special function, which is given by

\[
\sum_{n=0}^{\infty} \frac{1}{\left(1-x^m\right)\left(1-x^{m+2}\right) \cdots (1-x^m)} = \frac{1}{1-x^2} \sum_{n=1}^{\infty} \frac{x^m}{(1-x)(1-x^2) \cdots (1-x^m)} \quad (2)
\]

where the coefficient \(C'_1(n)\) of \(x^n\) is the number of partitions of \(n\) into parts without repetitions or parts, whose minimal difference is 2.

From (1) and (2) we can establish the following Remark:

**Remark 1:**

\[
C'_1(11) = C'(11) \quad (3)
\]

i.e., the number of partitions of \(n\) with minimal difference 2 is equal to the number of partitions of \(n\) into parts of the forms \(5m+1\) and \(5m+4\).

**Example 1:** For \(n = 11\), there are 7 partitions of 11 that are enumerated by \(C'_1(n)\) of above statement, which are given bellow [6]:

11, 10+1, 9+2+8+3, 7+4+7+3+1, 6+4+1+

\[C'_1(11) = 7.\]

There are 7 partitions of 11 are enumerated by \(C'_1(n)\) of above statement, which are given bellow: 11, 9+1+1+6+4+1, 6+1+1+1+1+1+1+1, 4+4+1+1+1+4+1+1+1+1+1+1+1,

\[C'(11) = 7.\]

Hence, \(C'_1(11) = C'(11)\).

We can conclude that, \(C'_1(11) = C'(11)\).

\[
1 + \sum_{n=1}^{\infty} C'(n) x^n = 1 + \sum_{n=1}^{\infty} C'_1(n) x^n.
\]

\[
1 + \sum_{n=1}^{\infty} \frac{x^m}{\left(1-x^m\right)(1-x^2) \cdots (1-x^m)} = \frac{1}{1-x^5} \sum_{m=0}^{\infty} \frac{1}{\left(1-x^{5m+1}\right)} \left(1-x^{5m+4}\right).
\]

which will be proved later as identity 1, it is known as The Rogers-Ramanujan identity 1.

The generating function for \(C^*(n)\) is of the form [1]:

\[
\sum_{m=0}^{\infty} \frac{1}{\left(1-x^{5m+2}\right)(1-x^{5m+3})} = \frac{1}{(1-x^2)(1-x^3)(1-x^7)(1-x^8) \cdots \infty} \quad (4)
\]

\[
= 1 + 0.x + x^2 + x^3 + x^4 + x^5 + 2x^6 + 2x^7 + \cdots \infty
\]

\[
= 1 + \sum_{n=1}^{\infty} C^*(n) x^n
\]

where the coefficient \(C^*(n)\) is the number of partitions of \(n\) into parts of the forms \(5m+2\) and \(5m+3\).

Now we consider a special function, which is of the form [1]:

\[
\frac{x^{m(m+1)}}{(1-x)(1-x^2) \cdots (1-x^m)} = x^{m(m+1)} \sum_{n=m(m+1)}^{\infty} P_m \left(n - m(m+1)\right) x^{n-m(m+1)}
\]

where the coefficient \(P_m\) of \(x^n\) in the above expansion is the number of partitions of \(n-m(m+1)\) into \(m\) parts at most.

Another special function, which is defined as;
where the coefficient $C'_n(n)$ is the number of partitions of $n$ into parts not less than 2 and with minimal difference 2.

From (4) and (5) we can establish the following Remark:

Remarks 2:

$$C'_n(n) = C''(n),$$  
(6)

i.e., the number of partitions of $n$ into parts not less than 2 and with minimal difference 2 is equal to the number of partitions of $n$ into parts of the forms $5m + 2$ and $5m + 3$.

Example 2: If $n = 11$, the four partitions of 11 into parts not less than 2 and with minimal difference 2 are given below:

$$11, 9 + 2, 8 + 3, 7 + 4.$$  

Hence, $C'_n(11) = 4$.

Again the four partitions of 11 into parts of the form $5m + 2$ and $5m + 3$ are given as;

$$8 + 3, 7 + 2 + 2, 3 + 3 + 3 + 2, 3 + 2 + 2 + 2 + 2.$$  

Hence, $C''(11) = 4$.

$\therefore C'_n(11) = C''(11)$.

We can conclude that,

$$C'_n(n) = C''(n).$$

i.e.,

$$1 + \sum_{m=1}^{\infty} C'_n(n) x^n = 1 + \sum_{m=1}^{\infty} C''(n) x^n$$

$$= 1 + \sum_{m=1}^{\infty} x^m (1-x^2)(1-x^3) \cdots (1-x^n)$$

$$= \prod_{m=0}^{\infty} \frac{1}{1-x^{5m+2}} \prod_{m=0}^{\infty} \frac{1}{1-x^{5m+3}},$$  

which will be proved later as identity 2, it is known as The Rogers-Ramanujan identity 2.

Now we give two Corollaries, which are related to the Jacobi’s triple product identity [3].

**Corollary 1:**

$$\prod_{n=0}^{\infty} \left(1-x^{5n+1}\right) \left(1-x^{5n+4}\right) \left(1-x^{5n+5}\right)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n x^\frac{n(5n+3)}{2}.$$

**Proof:** From Jacobi’s Theorem [2] we have;

$$\prod_{n=0}^{\infty} \left(1-x^{2n}\right) \left(1+x^{2n+1}\right) \left(1+x^{2n-1}\right)$$

$$= \sum_{n=-\infty}^{\infty} x^n z^n,$$

for all $z$ except $z = 0$, if $|x| < 1$.

If we write $x^{5/2}$ for $x$, $-x^{5/2}$ for $z$ and replace $n$ by $n + 1$ on the left hand side we obtain;

$$\prod_{n=0}^{\infty} \left(1-x^{5n+1}\right) \left(1-x^{5n+4}\right) \left(1-x^{5n+5}\right)$$

$$= \prod_{n=-\infty}^{\infty} (-1)^n x^\frac{n(5n+3)}{2}.$$

Hence, the Corollary.

**Corollary 2:**

$$\prod_{n=0}^{\infty} \left(1-x^{5n+2}\right) \left(1-x^{5n+3}\right) \left(1-x^{5n+5}\right)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n x^\frac{n(5n+3)}{2}.$$  

**Proof:** From Jacobi’s Theorem we have;

$$\prod_{n=0}^{\infty} \left(1-x^{2n}\right) \left(1+x^{2n+1}\right) \left(1+x^{2n-1}\right)$$

$$= \sum_{n=-\infty}^{\infty} x^n z^n,$$

for all $z$ except $z = 0$, when $|x| < 1$.

If we write $x^{5/2}$ for $x$, $-x^{5/2}$ for $z$ and replace $n$ by $n + 1$ on the left hand side we obtain;

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$$= \sum_{n=-\infty}^{\infty} (-1)^n x^\frac{n(5n+3)}{2}.$$

Hence the Corollary.

4. The Rogers-Ramanujan Identities

First we transfer the following auxiliary function into another auxiliary function. Let us consider the auxiliary function $[1, 2]$ with $|a| < 1$ and $|b| < 1$.

$$G_k(a, x) = \prod_{n=0}^{\infty} \left(1-x^{2kn}\right) \left(1-a^2 x^{2kn}\right) \left(1-a x^{2kn}\right)$$

it is known as Ramanujan’s device of the introduction of a second parameter $a$, where $k = 0, 1$ or 2 and $C_0 = 1$, by
\[ C_n = \frac{(1-a)(1-ax) \cdots (1-ax^{n-1})}{(1-x)(1-x^2) \cdots (1-x^n)}. \]

Hence,
\[
G_k(a,x) = \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{n(5n+1)-2kn} \frac{n(5n+1)-2kn}{2} \left(1-a^k x^{2kn}\right) \left(1-a^k x^{2kn}\right).
\]

\[
G_k(a,x) = \frac{1}{(1-a)(1-ax) \cdots (1-ax^{n-1})} \times \frac{1}{(1-x)(1-x^2) \cdots (1-x^n)} \cdot \frac{1}{(1-a^k x^{2kn})} \left(1-a^k x^{2kn}\right). \]

Again for \( k = 2 \) and \( a = x \), we get;
\[
H_2(x,x) = \frac{G_2(x,x)}{(1-x)(1-x^2)(1-x^3) \cdots \infty} = \prod_{m=0}^{\infty} \left(1-x^{5m+1}\right)\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)\left(1-x^{5m+5}\right) \cdots \infty
\]
\[
= \prod_{m=0}^{\infty} \left(1-x^{5m+1}\right)\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)\left(1-x^{5m+4}\right).
\]

Now we can consider the following Remark [2].

Remark 3: \( H_k - H_{k-1} = a^{k-1} \eta H_{3-k} \), where the operator \( \eta \) is defined by \( \eta f(a) = f(ax) \), and \( k = 1 \) or 2.

Proof: From (8) we have;
\[
H_k = H_k(a,x) = \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{n(5n+1)-2kn} \frac{n(5n+1)-2kn}{2} \left(1-a^k x^{2kn}\right)P_nQ_n(a),
\]
where \( P_n = \prod_{r=1}^{n} \frac{1}{1-ax^r} \), and \( Q_n(a) = \prod_{r=1}^{n} \frac{1}{1-ax^r} \).

It is convenient to define \( P_0 = 1 \), \( H_0 = 1 \). We have;
\[
H_k - H_{k-1} = \sum_{n=0}^{\infty} \left\{ (-1)^n a^{2n} x^{n(5n+1)-2kn} \frac{n(5n+1)-2kn}{2} \left(1-a^k x^{2kn}\right)P_nQ_n(a) \right\}
\]
\[
= \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{n(5n+1)} \frac{n(5n+1)}{2} \left[ a^{k-1} x^{(k-1)n} \left(1-a^k \right) \right] P_nQ_n.
\]
\[
= \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{n(5n+1)-2kn} \frac{n(5n+1)-2kn}{2} P_nQ_n.
\]

Now we have, \( (1-ax^n)Q_n = Q_{n+1} \) and \( (1-x^n)P_n = P_{n-1} \), hence,
\[
H_k - H_{k-1} = \sum_{n=0}^{\infty} (-1)^n a^{2n+k-1} x^{n(5n+1)+2n(k-1)} P_{n+1}Q_{n+1}
\]
\[
+ \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{n(5n+1)-2kn} P_{n+2}Q_{n+1}.
\]

In the second sum on the right hand side of the Identity we change \( n \) into \( n + 1 \). Thus,
\[
H_k - H_{k-1} = \sum_{n=0}^{\infty} (-1)^n a^{2n+k-1} x^{n(5n+1)+2n(k-1)} P_{n+1}Q_{n+1}
\]
\[
- \sum_{n=0}^{\infty} (-1)^n a^{2(n+1)} x^{(n+1)(5(n+1)+2k(n-1))} P_{n+1}Q_{n+1}.
\]
We have $Q_{n+1} = \eta Q_n$ and so,

$$H_k - H_{k-1} = a^{k-1} \eta \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{2n(3-k)} P_n Q_n$$

$$= a^{k-1} \eta H_{3-k}.$$

Hence, the Remark.

The Rogers-Ramanujan Identities

Identity 1 [4]:

$$1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)(1-x^2)\cdots(1-x^m)} = \prod_{m=0}^{\infty} \frac{1}{1-x^{5m+1}} \left(1-x^{5m+3}\right).$$

Identity 2 [4]:

$$1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)(1-x^2)\cdots(1-x^m)} = \prod_{m=0}^{\infty} \frac{1}{1-x^{5m+2}} \left(1-x^{5m+3}\right).$$

\textbf{Proof:} From (8) we have;

$$H_k (a,x) = \frac{G_k (a,x)}{(1-a)(1-ax)\cdots} \quad (13)$$

where $H_0 = 0$.

From above Remark we have;

$$H_k - H_{k-1} = a^{k-1} \eta H_{3-k}$$

where the operator $\eta$ is defined by $\eta f(a) = f(ax)$, and $k = 1$ or 2. In particular

$$H_1 = \eta H_2,$$

$$H_2 - H_1 = a \eta H_1. \quad (14)$$

So we have,

$$H_2 = \eta H_2 + a \eta^2 H_1. \quad (15)$$

Suppose now that;

$$H_2 = 1 + c_1 a + c_2 a^2 + \cdots.$$

where the coefficients depend on $x$ only. Substituting this into (15), we obtain;

$$1 + c_1 a + c_2 a^2 + \cdots = 1 + c_1 ax + c_2 a^2 x^2 + \cdots + a \left(1 + c_1 ax^2 + c_2 a^2 x^4 + \cdots\right).$$

Hence, equating the coefficients of various powers of $a$ from both sides we get;

$$c_1 = \frac{1}{1-x}, c_2 = \frac{x^2}{1-x^2} c_1, c_3 = \frac{x^4}{1-x^3} c_2, \cdots,$$

$$c_n = \frac{x^{n(n-1)}}{(1-x)(1-x^2)\cdots(1-x^n)}.$$
\[
1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \cdots = G_1(x, x).
\]

Therefore,

\[
1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)(1-x^2)\cdots(1-x^m)} = \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})}.
\]

Hence the Identity 2.

5. Conclusion

In this study, we have shown \(C'(n) = C^*(n)\) with the help of a numerical example when \(n=11\), and also have shown \(C'(n) = C^*(n)\) with the help of a numerical example when \(n = 11\). We have transferred the auxiliary function into another auxiliary function with the help of Ramanujan’s device of the introduction of a second parameter \(a\), i.e.,

\[
G_k(a, x) = \prod_{n=0}^{\infty} (-1)^n a^{2^n} x^{\frac{n(5n+1)-2kn}{2}} \left(1-a^k x^{2kn}\right) C_n
\]

to

\[
G_2(x, x) = \sum_{m=0}^{\infty} \left(1-x^{5m+2}\right) \left(1-x^{5m+3}\right) \left(1-x^{5m+5}\right),
\]

where \(k = 2\), and \(a = x\), it is used in proving The Rogers-Ramanujan Identity 2. Finally we have proved The Rogers-Ramanujan Identities with the help of auxiliary function,

\[
H_k(a, x) = \frac{G_k(a, x)}{(1-a) (1-ax) \cdots}.
\]

where \(H_0 = 0\).

References