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Generalizations of Hermite-Hadamard-Fejer Type Inequalities for Functions Whose Derivatives are s-Convex Via Fractional Integrals

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Abstract In this work, the new results related to right hand side of Hermite-Hadamard-Fejer inequality for s-convex functions in the second sense via fractionals integrals are obtained. This results are generalization of the results obtained by Işcan in [17].

Keywords: *s-Convex Function, Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, Riemann-Liouville fractional integral*

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1. Introduction

A function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be convex function if the inequality

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y) \tag{1.1}$$

holds for all x,y∈[a,b] and t∈[0,1].

One of the most famous inequality for convex functions is so called Hermite-Hadamard's inequality as follows: Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function defined on the interval I of real numbers and a.b∈I with a < b. Then:

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2} \tag{1.2}$$

Fejér [22] gave a generalization of the inequalities (1.2) as the following:

If $f:[a,b]\to\mathbb{R}$ is a convex function, and $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$ then

$$\begin{aligned}f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx&\le\frac{1}{b-a}\int_a^b f(x)g(x)\,dx\\&\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\end{aligned} \tag{1.3}$$

For some results which generalize, improve, and extend the inequalities (1.3), see ([16-21]).

In [23], Hudzik and Maligrada considered among others the class of functions which are s-convex in the second sense.

Definition 1. *A function $f:[0,\infty)\to\mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x+(1-\lambda)y)\le\lambda^s f(x)+(1-\lambda)^s f(y) \tag{1.4}$$

for all x,y∈[0,∞), λ ∈[0,1] and for some fixed s ∈(0,1].

It can be easily seen that s = 1, s-convexity reduces to ordinary convexity of functions defined on [0,∞).

In [24], Dragomir and Fitzpatrick proved Hermite-Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1. *Suppose that f: [0,∞)→[0,∞) is an s-convex functions in the second sense, where s ∈ (0,1), and let a, b∈[0,∞), a < b. If f∈L[a,b], then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{s+1} \tag{1.5}$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throught this paper.

Definition 2. *Let f∈[a,b]. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\ge 0$ are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,x>a \tag{1.6}$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,x<b$$

respectively where $\Gamma(\alpha)=\int_0^{\infty} e^{-t}u^{\alpha-1}du$. Here is $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

In the case of α= 1, the fractional integral reduces to the classical integral.

Let us consider the following special functions:

(1) The Beta function:

$$\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}(1-t)^{b-1}\,dt,$$
$$a,b>0$$

(2) The incomplete Beta function:

$$\beta_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}\,dt, 0<x<1, a,b>0$$

In [13], Sarikaya et. al. represented Hermite-Hadamard's inequality in fractional integral forms as follows.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be positive function with $0\le a<b$ and $f\in L[a,b]$.*

If f is a convex function on [a,b], then the following inequalities for fractional integrals hold

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\le\frac{f(a)+f(b)}{2} \tag{1.7}$$

with $\alpha>0$.

In [13], Sarikaya et. al. proved the following lemma.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with a < b. If $f'\in L[a,b]$ then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]=\frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)\,dt$$

The following Hermite-Hadamard type inequality was proved using the above lemma.

Theorem 3. [13] *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with a < b. If $|f'|$ is convex on [a,b] then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\le\frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(b)|\right].$$

Properties concerning this operator and more and more Hermite-Hadamard type inequalities involving fractional integrals for different classes of functions can be found ([1-15]).

Now, let us give the following lemma which we will use the proof.

Lemma 2. *For $0<\alpha\le 1$ and $0\le a<b$, we have*

$$\left|a^{\alpha}-b^{\alpha}\right|\le(b-a)^{\alpha}$$

Işcan [17] established following lemmas and theorems connected with the right-hand side of Hermite-Hadamard-Fejer type integral inequality for the fractional integrals.

Lemma 3. *If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$ with a < b, then*

$$J_{a+}^{\alpha}g(b)=J_{b-}^{\alpha}g(a)=\frac{1}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]$$

with $\alpha>0$.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be convex function with a < b and $f\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric to (a + b)/2, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\le\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\le\frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]$$

with $\alpha>0$.

Lemma 4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with a < b and $f'\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to (a + b)/2 then the following equality for fractional integral holds*

$$\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]=\frac{1}{\Gamma(\alpha)}\int_a^b\left[\begin{array}{l}\int_a^t(b-s)^{\alpha-1}g(s)ds\\-\int_t^b(s-a)^{\alpha-1}g(s)ds\end{array}\right]f'(t)dt$$

with $\alpha>0$.

Theorem 5. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with a < b. If $|f'|$ is convex on [a,b] and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to (a + b)/2, then the following inequality for fractional integral holds*

$$\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\le\frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(b)|\right] \tag{1.8}$$

with $\alpha>0$.

Theorem 6. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with a < b: If $|f'|^{q}$, q > 1, is convex on [a,b] and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to (a + b)/2, then the following inequality for fractional integral holds*

$$\left|\begin{array}{l}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \leq \frac{2(b-a)^{\alpha+1}\|g\|_{\infty}}{(b-a)^{1/q}(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{1/q} \tag{1.9}$$

where $\alpha > 0$ and $1/p + 1/q = 1$.

Theorem 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I°$ and $f' \in L[a,b]$ with $a < b$. If $|f'|^q$, $q > 1$, is convex on $[a,b]$ and $g:[a,b] \to \mathbb{R}$ is continuous and symmetric to $(a + b)/2$, then the following inequality for fractional integral holds*

(*i*)

$$\left|\begin{array}{l}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \leq \frac{2^{1/p}\|g\|_{\infty}(b-a)^{\alpha+1}}{(ap+1)^{1/p}\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha p}}\right)\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{1/q} \tag{1.10}$$

with $\alpha > 0$.

(*ii*)

$$\left|\begin{array}{l}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \leq \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{(ap+1)^{1/p}\Gamma(\alpha+1)}\left(\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right)^{1/q} \tag{1.11}$$

for $0 < \alpha \leq 1$ where $1/p + 1/q = 1$.

The main of this paper is to establish some new inequalities related to the left-hand side of the Hermete-Hadamard-Fejer type inequalities for s-convex functions in the second sense via Riemann-Liouville fractional integrals.

2. Main Results

Theorem 8. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on $I°$ and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$, and $g:[a,b] \to \mathbb{R}$ is continious and symmetric to $(a + b)/2$, then the following inequality for fractional integral holds*

$$\left|\begin{array}{l}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \leq \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{(a+s+1)^{1/p}\Gamma(\alpha+1)}\left(|f'|a||+|f'(b)|\right) \times\left(1-(\alpha+s+1)\left[B_{1/2}(s+1,\alpha+1)-B_{1/2}(\alpha+1,s+1)\right]\right) \tag{2.1}$$

with $\alpha > 0$.

Proof. From Lemma 4 we have

$$\left|\begin{array}{l}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \leq \frac{1}{\Gamma(\alpha)}\int_a^b\left|\begin{array}{l}\int_a^t(b-s)^{\alpha-1}g(s)ds\\ -\int_t^b(s-a)^{\alpha-1}g(s)ds\end{array}\right||f'(t)|dt. \tag{2.2}$$

Since $|f'|$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$, we know that for $t \in[a,b]$

$$\begin{aligned}|f'(t)| &= \left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|\\ &\leq \left(\frac{b-t}{b-a}\right)^{s}|f'(a)|+\left(\frac{t-a}{b-a}\right)^{s}|f'(b)|\end{aligned} \tag{2.3}$$

and since $g:[a,b] \to \mathbb{R}$ is symmetric to $(a + b)/2$ we write

$$\begin{aligned}\int_t^b(s-a)^{\alpha-1}g(s)ds &= \int_a^{a+b-t}(b-s)^{\alpha-1}g(a+b-s)ds\\ &= \int_a^{a+b-t}(b-s)^{\alpha-1}g(s)ds,\end{aligned}$$

then we have

$$\begin{aligned}&\left|\int_a^t(b-s)^{\alpha-1}g(s)ds-\int_t^b(s-a)^{\alpha-1}g(s)ds\right|\\ &= \left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)ds\right|\\ &\leq \begin{cases}\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds & t\in\left[a,\frac{a+b}{2}\right]\\ \int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}g(s)\right|ds & t\in\left[\frac{a+b}{2},b\right]\end{cases}\end{aligned} \tag{2.4}$$

By virtue of (2.2), (2.3) and (2.4), we get

$$\left|\begin{array}{l}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\ -\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right| \leq \frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\begin{array}{l}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\\ \left[\left(\frac{b-t}{b-a}\right)^{s}|f'(a)|+\left(\frac{t-a}{b-a}\right)^{s}|f'(b)|\right]dt\end{array}$$

$$+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left[\left(\frac{b-t}{b-a}\right)^{s}\left|f'(a)\right|+\left(\frac{t-a}{b-a}\right)^{s}\left|f'(b)\right|\right]dt$$

$$\leq\frac{\|g\|_{\infty}}{(b-a)^{s}\Gamma(\alpha+1)}\left\{\begin{array}{l}\int_{a}^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]\left[\begin{array}{l}(b-t)^{s}\left|f'(a)\right|\\+(t-a)^{\alpha}\left|f'(b)\right|\end{array}\right]dt\\+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]\left[\begin{array}{l}(b-t)^{s}\left|f'(a)\right|\\+(t-a)^{\alpha}\left|f'(b)\right|\end{array}\right]dt\end{array}\right\}$$

$$=\frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{(\alpha+s+1)\Gamma(\alpha+1)}\left\{\begin{array}{l}\left(1-(a+s+1)\left[\begin{array}{l}B_{1/2}(s+1,\alpha+1)\\-B_{1/2}(\alpha+1,s+1)\end{array}\right]\right)\\\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\end{array}\right\}$$

where

$$\begin{aligned}&\int_{a}^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right](t-a)^{s}dt\\&=\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right](b-t)^{s}dt\\&=\frac{(b-a)^{\alpha+s+1}}{\alpha+s+1}\left(\frac{1}{2^{\alpha+s+1}}+(\alpha+s+1)B_{1/2}(s+1,\alpha+1)\right)\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}&\int_{a}^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right](b-t)^{s}dt\\&=\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right](t-a)^{s}dt\\&=\frac{(b-a)^{\alpha+s+1}}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s+1}}-(\alpha+s+1)B_{1/2}(\alpha+1,s+1)\right)\end{aligned}\tag{2.6}$$

Remark 1. *In Theorem 8, if we take s = 1, then the inequality (2.1) becomes inequality (1.8) of Theorem 5.*

Theorem 9. *Let $f:I\subseteq[0,\infty)\to\mathbb{R}$ be a differentiable mapping on Io and f' ∈L[a,b] with a < b. If $|f'|^q$, q > 1, is s-convex on [a,b] for some fixed s∈(0,1], and $g:[a,b]\to\mathbb{R}$ is continious and symmetric to (a + b)/2, then the following inequality for fractional integral holds*

$$\begin{aligned}&\left|\begin{array}{l}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right|\\&\leq\frac{2\|g\|_{\infty}(b-a)^{\alpha+1}}{(\alpha+1)^{1-1/q}(\alpha+s+1)^{1/q}\Gamma(\alpha+1)}\\&\left(1-\frac{1}{2^{\alpha}}\right)^{1-1/q}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{1/q}\\&\times\left(1-(\alpha+s+1)\left[\begin{array}{l}B_{1/2}(s+1,\alpha+1)\\-B_{1/2}(\alpha+1,s+1)\end{array}\right]\right)^{1/q}\end{aligned}\tag{2.7}$$

where α > 0.

Proof. Using Lemma 4, Hölder's inequality, (2.4) and the s-convexity of $|f'|^q$, it follows that

$$\begin{aligned}&\left|\begin{array}{l}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right|\\&\leq\frac{1}{\Gamma(\alpha)}\left(\int_{a}^{b}\left|\int_{t}^{a+b-t}(b-s)^{\alpha-1}g(s)ds\right|dt\right)^{1-1/q}\\&\times\left(\int_{a}^{b}\left|\int_{t}^{a+b-t}(b-s)^{\alpha-1}g(s)ds\right|\left|f'(t)\right|^{q}dt\right)^{1/q}\\&\leq\frac{1}{\Gamma(\alpha)}\left\{\begin{array}{l}\int_{a}^{\frac{a+b}{2}}\left(\int_{t}^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\\+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\end{array}\right\}^{1-1/q}\\&\times\left\{\begin{array}{l}\int_{a}^{\frac{a+b}{2}}\left(\int_{t}^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left|f'(t)\right|^{q}dt\\+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left|f'(t)\right|^{q}dt\end{array}\right\}^{1/q}\\&\leq\frac{2^{1-1/q}\|g\|_{\infty}}{(b-a)^{s/q}\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{1-1/q}\\&\left\{\begin{array}{l}\int_{0}^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]\left[\begin{array}{l}(b-t)^{s}\left|f'(a)\right|^{q}\\+(t-a)^{s}\left|f'(b)\right|^{q}\end{array}\right]dt\\+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]\left[\begin{array}{l}(b-t)^{s}\left|f'(a)\right|^{q}\\+(t-a)^{s}\left|f'(b)\right|^{q}\end{array}\right]dt\end{array}\right\}^{1/q}\end{aligned}\tag{2.8}$$

where it is easily seen that

$$\begin{aligned}&\int_{a}^{\frac{a+b}{2}}\left(\int_{t}^{a+b-t}\left|(b-s)\right|^{\alpha-1}g(s)ds\right)dt\\&+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)\right|^{\alpha-1}g(s)ds\right)dt\\&=\frac{2(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\end{aligned}$$

Hence, if we use (2.5) and (2.6) in (2.8), we have

$$\begin{aligned}&\left|\begin{array}{l}\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\\-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\end{array}\right|\\&\leq\frac{2\|g\|_{\infty}(b-a)^{\alpha+1}}{(\alpha+1)^{1-1/q}(\alpha+s+1)^{1/q}\Gamma(\alpha+1)}\\&\left(1-\frac{1}{2^{\alpha}}\right)^{1-1/q}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{1/q}\\&\times\left(1-(\alpha+s+1)\left[\begin{array}{l}B_{1/2}(s+1,\alpha+1)\\-B_{1/2}(\alpha+1,s+1)\end{array}\right]\right)^{1/q}\end{aligned}$$

Remark 2. *In Theorem 9, if we take s = 1, then the inequality (2.7) becomes inequality (1.9) of Theorem 6.*

Theorem 10. *Let* $f: I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *and* $f' \in L[a,b]$ *with* $a < b$. *If* $|f'|^q$, $q > 1$, *is s-convex on [a,b] for some fixed* $s \in (0,1]$, *and* $g: [a,b] \to \mathbb{R}$ *is continious and symmetric to (a + b)/2, then the following inequality for fractional integral holds*

(i)

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] - \left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \le \frac{2^{1/p}\|g\|_\infty (b-a)^{\alpha+1}}{(\alpha p+1)^{1/p}\Gamma(\alpha+1)} \left(\left[1-\frac{1}{2^{\alpha p}}\right]\right)^{1/p}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{1/q} \tag{2.9}$$

with $\alpha>0$.

(ii)

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] - \left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \le \frac{\|g\|_\infty (b-a)^{\alpha+1}}{(\alpha p+1)^{1/p}\Gamma(\alpha+1)}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{1/q} \tag{2.10}$$

for $0 < \alpha < 1$, *where* $1/p + 1/q = 1$.

Proof. (i) Using Lemma 4, Hölder's inequality, the inequality (2.4) and the s-convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] - \left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)ds\right|^p dt\right)^{1/p}\left(\int_a^b|f'(t)|^q\right)^{1/q} \\
&\le \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(\int_a^{\frac{a+b}{2}}\left[(b-t)^\alpha-(t-a)^\alpha\right]^p dt + \int_{\frac{a+b}{2}}^b\left[(t-a)^\alpha-(b-t)^\alpha\right]^p dt\right)^{1/p} \\
&\times\left(\int_a^b\left[\left(\frac{b-t}{b-a}\right)^s|f'(a)|^q+\left(\frac{t-a}{b-a}\right)^s|f'(b)|^q\right]dt\right)^{1/q} \\
&= \frac{\|g\|_\infty (b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^\alpha-t^\alpha\right]^p dt + \int_{\frac{1}{2}}^1\left[t^\alpha-(1-t)^\alpha\right]^p dt\right)^{1/p} \\
&\times\left(\frac{|f'(a)^q+f'(b)^q|}{s+1}\right)^{1/q} \\
&\le \frac{\|g\|_\infty (b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha p}-t^{\alpha p}\right]dt + \int_{\frac{1}{2}}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right]dt\right)^{1/p} \\
&\times\left(\frac{|f'(a)^q+f'(b)^q|}{s+1}\right)^{1/q} \\
&\le \frac{\|g\|_\infty (b-a)^{\alpha+1}}{\Gamma(\alpha+1)} \\
&\left(\frac{2}{ap+1}\left[1-\frac{1}{2^{\alpha p}}\right]\right)^{1/p}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{1/q}.
\end{aligned}$$

Here we use

$$[(1-t)^\alpha - t^\alpha]^p \le (1-t)^{\alpha p} - t^{\alpha p}$$

for $t \in [0,1/2]$ and

$$[t^\alpha - (1-t)^\alpha]^p \le t^{\alpha p} - (1-t)^{\alpha p}$$

for $t \in [1/2,1]$, which follows from

$$(A-B)^q \le A^q - B^q,$$

for any $A \ge B \ge 0$ and $q \ge 1$. Hence the inequality (2.9) is proved.

(ii) The inequality (2.10) is easily proved using (2.11) and Lemma 2.

Remark 3. *In Theorem 10, if we take s = 1, then inequalities (2.9) and (2.10) becomes inequalities (1.10) and (1.11) of Theorem 7.*

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