Generalizations of Hermite-Hadamard-Fejer Type Inequalities for Functions Whose Derivatives are s-Convex Via Fractional Integrals

ERHAN SET1, IMDAT ISCAN2, ILKER MUMCU1

1Department of Mathematics, Faculty of Arts and Sciences, Ordu University, Ordu, Turkey  
2Department of Mathematics, Faculty of Arts and Sciences, Giresun University, Giresun, Turkey  
*Corresponding author: erhanset@yahoo.com

Received September 03, 2014; Revised October 06, 2014; Accepted October 13, 2014

Abstract In this work, the new results related to right hand side of Hermite-Hadamard-Fejer inequality for s-convex functions in the second sense via fractional integrals are obtained. This results are generalization of the results obtained by İşcan in [17].

Keywords: s-Convex Function, Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, Riemann-Liouville fractional integral


1. Introduction

A function $f:[a,b] \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1.1)$$

holds for all $x,y \in [a,b]$ and $t \in [0,1]$.

One of the most famous inequality for convex functions is so called Hermite-Hadamard’s inequality as follows: Let $f:I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then:

$$\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \quad (1.2)$$

Fejér [22] gave a generalization of the inequalities (1.2) as the following:

If $f:[a,b] \to \mathbb{R}$ is a convex function, and $g:[a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $a+b$ then:

$$\int_a^b \left( \frac{a+b}{2} \right) g(x) \, dx \leq \frac{1}{b-a} \int_a^b f(x) g(x) \, dx \leq \frac{f(a) + f(b)}{2} \quad (1.3)$$

For some results which generalize, improve, and extend the inequalities (1.3), see ([16-21]).

In [23], Hudzik and Maligrada considered among others the class of functions which are s-convex in the second sense.

Definition 1. A function $f:[0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \quad (1.4)$$

for all $x,y \in [0,\infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

It can be easily seen that $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [24], Dragomir and Fitzpatrick proved Hermite-Hadamard’s inequality which holds for s-convex functions in the second sense.

Theorem 1. Suppose that $f: [0,\infty) \to [0,\infty)$ is an s-convex functions in the second sense, where $s \in (0,1)$, and let $a, b \in [0,\infty)$, $a < b$. If $f \in L[a,b]$, then the following inequalities hold:

$$2^{s-1} \int_a^b \left( \frac{a+b}{2} \right) \, dx \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \quad (1.5)$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used through this paper.

Definition 2. Let $f \in [a,b]$. The Riemann-Liouville integrals $J_a^\alpha f$ and $J_b^- f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, x > a \quad (1.6)$$

and

$$J_b^- f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt, x < b$$

for some results which generalize, improve, and extend the inequalities (1.3), see ([16-21]).
respectively where \( \Gamma(a) = \int_0^\infty e^{-u}u^{a-1}du \). Here is

\[ J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x). \]

In the case of \( a=1 \), the fractional integral reduces to the
classical integral.

Let us consider the following special functions:

(1) The Beta function:

\[ \beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \]

\( a,b > 0 \)

(2) The incomplete Beta function:

\[ \beta_a(t,a,b) = \int_t^1 t^{a-1}(1-t)^{b-1}dt, 0 < t < 1, a,b > 0 \]

In [13], Sarıkaya et al. represented Hermite-Hadamard’s inequality in fractional integral forms as follows.

**Theorem 2.** Let \( f :[a,b] \to \mathbb{R} \) be positive function

with \( 0 \leq a < b \) and \( f \in L[a,b] \).

If \( f \) is a convex function on \([a,b]\), then the following inequalities for fractional integrals hold:

\[ f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(a+1)}{2(b-a)^a} \left[ J_{a+}^a f(b) + J_{b-}^a f(a) \right] \]

\[ \leq \frac{f(a) + f(b)}{2} \]

with \( a > 0 \).

In [13], Sarıkaya et al. proved the following lemma.

**Lemma 1.** Let \( f :[a,b] \to \mathbb{R} \) be a differentiable

mapping on \((a,b)\) with \( a < b \). If \( f' \in L[a,b] \) then the following equality for fractional integrals holds:

\[ \frac{f(a) + f(b)}{2} - \frac{\Gamma(a+1)}{2(b-a)^a} \left[ J_{a+}^a f(b) + J_{b-}^a f(a) \right] \]

\[ = \frac{b-a}{2} \left[ (1-t)^{a-1} - t^{a-1} \right] f'(ta + (1-t)b)dt \]

The following Hermite-Hadamard type inequality was proved using the above lemma.

**Theorem 3.** [13] Let \( f :[a,b] \to \mathbb{R} \) be a differentiable

mapping on \((a,b)\) with \( a < b \). If \( f' \) is convex on \([a,b]\) then the following inequality for fractional integrals holds:

\[ \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(a+1)}{2(b-a)^a} \left[ J_{a+}^a f(b) + J_{b-}^a f(a) \right] \right| \]

\[ \leq \frac{b-a}{2} \left[ 1 - \frac{1}{2^a} \right] \left[ |f'(a)| + |f'(b)| \right]. \]

Properties concerning this operator and more and more Hermite-Hadamard type inequalities involving fractional integrals for different classes of functions can be found ([1-15]).

Now, let us give the following lemma which we will use the proof.

**Lemma 2.** For \( 0 < \alpha \leq 1 \) and \( 0 \leq a < b \), we have

\[ |a^\alpha - b^\alpha| \leq (b-a)^\alpha \]

Işcan [17] established following lemmas and theorems connected with the right-hand side of Hermite-Hadamard-Fejer type integral inequality for the fractional integrals.

**Lemma 3.** If \( g :[a,b] \to \mathbb{R} \) is integrable and

symmetric to \( \frac{a+b}{2} \) with \( a < b \), then

\[ J_{a+}^a g(b) = J_{b-}^a g(a) = \frac{1}{2} \left[ J_{a+}^a g(b) + J_{b-}^a g(a) \right] \]

with \( \alpha > 0 \).

**Theorem 4.** Let \( f :[a,b] \to \mathbb{R} \) be convex function with

\( a < b \) and \( f \in L[a,b] \). If \( g :[a,b] \to \mathbb{R} \) is nonnegative, integrable and symmetric to \( (a+b)/2 \), then the following inequalities for fractional integrals hold:

\[ f \left( \frac{a+b}{2} \right) \left[ J_{a+}^a g(b) + J_{b-}^a g(a) \right] \]

\[ \leq \left[ J_{a+}^a (fg)(b) + J_{b-}^a (fg)(a) \right] \]

\[ \leq \frac{f(a) + f(b)}{2} \left[ J_{a+}^a g(b) + J_{b-}^a g(a) \right] \]

with \( \alpha > 0 \).

**Lemma 4.** Let \( f :[a,b] \to \mathbb{R} \) be a differentiable mapping on \([a,b]\) with \( a < b \) and \( f' \in L[a,b] \). If \( g :[a,b] \to \mathbb{R} \) is integrable and symmetric to \( (a+b)/2 \) then the following equality for fractional integrals holds

\[ \left( \frac{f(a) + f(b)}{2} \right) \left[ J_{a+}^a g(b) + J_{b-}^a g(a) \right] \]

\[ - \left[ J_{a+}^a (fg)(b) + J_{b-}^a (fg)(a) \right] \]

\[ = \frac{1}{\Gamma(a)} \int_a^b \left[ (b-s)^{a-1} g(s)ds - \int_a^b (s-a)^{a-1} g(s)ds \right] f'(t)dt \]

with \( \alpha > 0 \).

**Theorem 5.** Let \( f :I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \) and \( f' \in L[a,b] \) with \( a < b \). If \( f' \) is convex on \([a,b]\) and \( g :[a,b] \to \mathbb{R} \) is continuous and symmetric to \( (a+b)/2 \), then the following inequality for fractional integral holds

\[ \left( \frac{f(a) + f(b)}{2} \right) \left[ J_{a+}^a g(b) + J_{b-}^a g(a) \right] \]

\[ - \left[ J_{a+}^a (fg)(b) + J_{b-}^a (fg)(a) \right] \]

\[ \leq \frac{1}{\Gamma(a+1)} \left[ \left| f'(a) \right| + \left| f'(b) \right| \right] \]

with \( \alpha > 0 \).

**Theorem 6.** Let \( f :I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \) and \( f' \in L[a,b] \) with \( a < b \). If \( f' \in L[a,b] \) and \( g :[a,b] \to \mathbb{R} \) is continuous and symmetric to \( (a+b)/2 \), then the following inequality for fractional integral holds
be a differentiable mapping on \( I^\circ \) and \( f' \in L[a, b] \) with \( a < b \). If \( |f'|^q \) is \( s \)-convex on \( [a, b] \) and symmetric to \((a + b)/2\), then the following inequality for fractional integral holds

\[
\left( \frac{f(a) + f(b)}{2} \right) - \left( J_a^a g(b) + J_a^b g(a) \right) - \left[ J_a^a (fg)(b) + J_a^b (fg)(a) \right] \leq \frac{2(b - a)^{\alpha+1}}{(b - a)^{\alpha+1} \Gamma(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) \left( \int f'(a)^q + f'(b)^q \right)^{1/q} \tag{1.9}
\]

where \( \alpha > 0 \) and \( 1/p + 1/q = 1 \).

**Theorem 7.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^\circ \) and \( f' \in L[a, b] \) with \( a < b \). If \( |f'|^q \), \( q > 1 \), is convex on \([a, b]\) and \( g : [a, b] \to \mathbb{R} \) is continuous and symmetric to \((a + b)/2\), then the following inequality for fractional integral holds

\[
\left( \frac{f(a) + f(b)}{2} \right) - \left( J_a^a g(b) + J_a^b g(a) \right) - \left[ J_a^a (fg)(b) + J_a^b (fg)(a) \right] \leq \frac{\|f'\|_p (b - a)^{\alpha+1}}{(a + s)^{\alpha+1} \Gamma(\alpha + 1)} \left( (f'(a)^q + f'(b)^q) \right)^{1/q} \times \left( 1 - (a + s)^{\alpha+1} \right) \left[ B_{1/2} (s + 1, \alpha + 1) - B_{1/2} (\alpha + 1, s + 1) \right] \tag{2.1}
\]

with \( \alpha > 0 \).

**Proof.** From Lemma 4 we have

\[
\left( \frac{f(a) + f(b)}{2} \right) - \left( J_a^a g(b) + J_a^b g(a) \right) = \frac{f(t)ds}{\Gamma(\alpha + 1)} \cdot \int f'(a)^q + f'(b)^q \right)^{1/q} \frac{\|f'\|^q}{(a + s)^{\alpha+1} \Gamma(\alpha + 1)} \left( (f'(a)^q + f'(b)^q) \right)^{1/q} \times \left( 1 - (a + s)^{\alpha+1} \right) \left[ B_{1/2} (s + 1, \alpha + 1) - B_{1/2} (\alpha + 1, s + 1) \right] \tag{2.2}
\]

Since \( |f'| \) is \( s \)-convex on \([a, b]\) for some fixed \( s \in (0, 1]\), we know that for \( t \in [a, b] \)

\[
|f'(t)| = \left| \frac{b - t}{b - a} f(t) + \frac{t - a}{b - a} f'(b) \right| \tag{2.3}
\]

and since \( g : [a, b] \to \mathbb{R} \) is symmetric to \((a + b)/2\) we write

\[
\int_{a}^{b} (s - a)^{\alpha-1} g(s) ds = \int_{a}^{a+b-t} (b - s)^{\alpha-1} g(a + b - s) ds = \int_{a}^{a+b-t} (b - s)^{\alpha-1} g(s) ds,
\]

then we have

\[
\int_{a}^{a+b-t} (b - s)^{\alpha-1} g(s) ds = \int_{a}^{a+b-t} (b - s)^{\alpha-1} g(s) ds \tag{2.4}
\]

By virtue of (2.2), (2.3) and (2.4), we get

\[
\left( \frac{f(a) + f(b)}{2} \right) - \left( J_a^a g(b) + J_a^b g(a) \right) - \left[ J_a^a (fg)(b) + J_a^b (fg)(a) \right] \leq \frac{\|f'\|^q}{(a + s)^{\alpha+1} \Gamma(\alpha + 1)} \left( (f'(a)^q + f'(b)^q) \right)^{1/q} \times \left( 1 - (a + s)^{\alpha+1} \right) \left[ B_{1/2} (s + 1, \alpha + 1) - B_{1/2} (\alpha + 1, s + 1) \right] \tag{2.5}
\]
then the following inequality for fractional integral holds

\[
\left\| \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{b} \frac{d}{dt} \left( (t-a)^\alpha - (t-a)^\alpha \right) \right\|_{a}^{\alpha+1} \leq \frac{1}{(b-a)^\alpha} \Gamma(\alpha+1) \left\{ \frac{1}{(\alpha+1)} \right\}
\]

where

\[
\int_{\alpha}^{b} \frac{d}{dt} \left( (t-a)^\alpha - (t-a)^\alpha \right) dt = \left\{ \frac{1}{(\alpha+1)} \right\}
\]

and

\[
\int_{\alpha}^{b} \frac{d}{dt} \left( (t-a)^\alpha - (t-a)^\alpha \right) \left\{ \frac{1}{(\alpha+1)} \right\}
\]

**Remark 1.** In Theorem 8, if we take \(s = 1\), then the inequality (2.1) becomes inequality (1.8) of Theorem 5.

**Theorem 9.** Let \( f : I \subseteq [0, \infty) \rightarrow \mathbb{R} \) be a differentiable mapping on \( I \) and \( f' \in L([a,b]) \) with \( a < b \). If \( |f'|^q \), \( q > 1 \), is \( s \)-convex on \([a,b]\) for some fixed \( s \), then the following inequality for fractional integrals holds:

\[
\left\| \left[ \frac{f(a) + f(b)}{2} \right] \right\|_{a}^{\alpha+1} \leq \frac{2 \left\| f' \right\|_{a}^{\alpha+1}}{(\alpha+1)^{1-q} \cdot (\alpha+1)^{1/q}} \left\{ \frac{1}{2} \right\}^{1-q} \times \left( \frac{1}{2^\alpha} \right)^{1-q} \left( \frac{1}{\Gamma(\alpha+1)} \right)^{1-q} \left\{ \frac{1}{(\alpha+1)} \right\}^{1-q}
\]

where \( \alpha > 0 \).

**Proof.** Using Lemma 4, Hölder’s inequality, (2.4) and the s-convexity of \( |f'|^q \), it follows that

\[
\left[ \frac{f(a) + f(b)}{2} \right] \left[ J_{a^\alpha}^b \left( g(s) \right) + J_{b^\alpha}^a \left( g(s) \right) \right] \leq \frac{1}{\Gamma(\alpha)} \left\{ \frac{1}{2} \right\}^{1-q} \times \left( \frac{1}{2^\alpha} \right)^{1-q} \left\{ \frac{1}{\Gamma(\alpha+1)} \right\}^{1-q} \left\{ \frac{1}{(\alpha+1)} \right\}^{1-q}
\]

Hence, if we use (2.5) and (2.6) in (2.8), we have

\[
\left[ \frac{f(a) + f(b)}{2} \right] \left[ J_{a^\alpha}^b \left( g(s) \right) + J_{b^\alpha}^a \left( g(s) \right) \right] \leq \frac{2 \left\| f' \right\|_{a}^{\alpha+1}}{(\alpha+1)^{1-q} \cdot (\alpha+1)^{1/q}} \left\{ \frac{1}{2^\alpha} \right\}^{1-q} \times \left( \frac{1}{(\alpha+1)} \right)^{1-q} \left\{ \frac{1}{\Gamma(\alpha+1)} \right\}^{1-q} \left\{ \frac{1}{(\alpha+1)} \right\}^{1-q}
\]
Remark 2. In Theorem 9, if we take \( s = 1 \), then the inequality (2.7) becomes inequality (1.9) of Theorem 6.

Theorem 10. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^n \) and \( f' \in L_1[a,b] \) with \( a < b \). If \( |f'|^q, q > 1 \), is \( s \)-convex on \([a, b]\) for some fixed \( s \in (0, 1] \), and \( g : [a, b] \to \mathbb{R} \) is continuous and symmetric to \((a + b)/2\), then the following inequality for fractional integral holds

\[
\left( \frac{f(a) + f(b)}{2} \right) \left[ \int_a^b f'(t)^q \, dt \right]^{1/q} \leq \left( \frac{f'(a)^q + f'(b)^q}{2} \right)^{1/q} + \left( \frac{f''(a) + f''(b)}{2} \right)^{1/q}
\]

for \( 0 < \alpha < 1 \), where \( 1/p + q/1 = 1 \).

Proof. (i) Using Lemma 4, Hölder’s inequality, the inequality (2.4) and the \( s \)-convexity of \( |f'|^q \), it follows that

\[
\left[ \frac{f(a) + f(b)}{2} \right] \left[ \int_a^b f'(t)^q \, dt \right]^{1/q} \leq \left( \frac{f'(a)^q + f'(b)^q}{2} \right)^{1/q} + \left( \frac{f''(a) + f''(b)}{2} \right)^{1/q}
\]

and (2.8) becomes inequalities (1.9) of Theorem 6.

Remark 2. In Theorem 9, if we take \( s = 1 \), then the inequality (2.7) becomes inequality (1.9) of Theorem 6.

Theorem 10. Let \( f : I \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable mapping on \( I^n \) and \( f' \in L_1[a,b] \) with \( a < b \). If \( |f'|^q, q > 1 \), is \( s \)-convex on \([a, b]\) for some fixed \( s \in (0, 1] \), and \( g : [a, b] \to \mathbb{R} \) is continuous and symmetric to \((a + b)/2\), then the following inequality for fractional integral holds

\[
\left( \frac{f(a) + f(b)}{2} \right) \left[ \int_a^b f'(t)^q \, dt \right]^{1/q} \leq \left( \frac{f'(a)^q + f'(b)^q}{2} \right)^{1/q} + \left( \frac{f''(a) + f''(b)}{2} \right)^{1/q}
\]

for \( 0 < \alpha < 1 \), where \( 1/p + q/1 = 1 \).

Proof. (i) Using Lemma 4, Hölder’s inequality, the inequality (2.4) and the \( s \)-convexity of \( |f'|^q \), it follows that

\[
\left( \frac{f(a) + f(b)}{2} \right) \left[ \int_a^b f'(t)^q \, dt \right]^{1/q} \leq \left( \frac{f'(a)^q + f'(b)^q}{2} \right)^{1/q} + \left( \frac{f''(a) + f''(b)}{2} \right)^{1/q}
\]

for \( 0 < \alpha < 1 \), where \( 1/p + q/1 = 1 \).

References


