On the Simpson’s Inequality for Convex Functions on the Co-Ordinates

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Abstract In this paper, a new lemma is proved and inequalities of Simpson type are established for convex functions on the co-ordinates and bounded functions.

Keywords: Simpson’s inequality, co-ordinates, convex functions, bounded functions


1. Introduction

The following inequality is well-known in the literature as Simpson’s inequality:

**Theorem 1.** Let \( f: [a,b] \rightarrow \mathbb{R} \) be a four times continuously differentiable mapping on \([a,b]\) and
\[
\left\| f^{(4)} \right\|_{\infty} = \sup_{x \in [a,b]} \left| f^{(4)} (x) \right| < \infty. \text{ Then the following inequality holds:}
\[
\frac{1}{3} \left[ f(a) + f(b) + 2f \left( \frac{a+b}{2} \right) - 2f \left( \frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int_{a}^{b} f(x) dx \leq \frac{1}{2880} \left\| f^{(4)} \right\|_{\infty} (b-a)^4.
\]

For recent results on Simpson’s type inequalities see the papers [11-19].

Convexity on the co-ordinates can be given as following (see [10]):

Let us consider the bidimensional interval \( \Delta = [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). A function \( f: \Delta \rightarrow \mathbb{R} \) will be called convex on the co-ordinates if the partial mappings \( f_y: [a,b] \rightarrow \mathbb{R} \) and \( f_x: [c,d] \rightarrow \mathbb{R} \) are convex where defined for all \( y \in [c,d] \) and \( x \in [a,b] \).

Recall that the mapping \( f: \Delta \rightarrow \mathbb{R} \) is convex on \( \Delta \), if the following inequality holds for all \((x,y),(z,w) \in \Delta \) and \( \lambda \in [0,1] \):
\[
f(\lambda x + (1-\lambda) y, \lambda z + (1-\lambda) w) \leq \lambda f(x,y) + (1-\lambda) f(z,w)
\]

In [10], Dragomir proved the following inequalities:

**Theorem 2.** Suppose that \( f: \Delta = [a,b] \times [c,d] \rightarrow \mathbb{R} \) is convex on the co-ordinates on \( \Delta \). Then one has the inequalities:
\[
\begin{align*}
&f \left( \frac{a+b+c+d}{2} \right) \leq \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x,y) dy dx \\
&\leq f(a,c) + f(b,d) + f(a,d) + f(b,c) - \frac{1}{4}b f(x,y)\mid_{x=a}^{y=b}
\end{align*}
\]

The above inequalities are sharp.

Recently, several papers have been written on the convex functions on the co-ordinates. Similar results can be found in [1-9] and [20,21,22,23].

In this paper, we will give Simpson-type inequalities for convex functions on the co-ordinates and bounded functions on the basis of the following lemma.

2. Main Results

To prove our main result, we need the following lemma.

**Lemma 1.** Let \( f: \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a,b] \times [c,d] \). If \( \frac{\partial^2 f}{\partial x \partial y} \in L(\Delta) \), then the following equality holds:
\[
\begin{align*}
&f \left( \frac{a+c+d}{2} \right) + f \left( \frac{b+c+d}{2} \right) + 4f \left( \frac{a+b+c+d}{4} \right) \\
&+ f \left( \frac{a+b+c}{2} \right) + f \left( \frac{a+b+d}{2} \right)
\end{align*}
\]
\[ + \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{36} \]
\[ - \frac{1}{6(b-a)} \int_a^b \left[ f(x, c) + 4f \left( x, \frac{x+c+d}{2} \right) + f(x, d) \right] dx \]
\[ - \frac{1}{6(d-c)} \int_c^d \left[ f(a, y) + 4f \left( \frac{a+b}{2}, y \right) + f(b, y) \right] dy \]
\[ + \frac{(b-a)(d-c)}{6} \int_a^d f(x, y) dy dx \]
\[ = (b-a)(d-c) \int_0^1 p(t)q(s) \frac{\partial^2 f}{\partial t \partial s} \]
\[ \times (ta+(1-t)b, sc+(1-s)d) dtds \]

where
\[ p(t) = \begin{cases} \left( t - \frac{1}{6}, \ t \in \left[ 0, \frac{1}{2} \right] \right) \\ \left( t - \frac{5}{6}, \ t \in \left[ \frac{1}{2}, 1 \right] \right) \end{cases} \]

and
\[ q(s) = \begin{cases} \left( s - \frac{1}{6}, \ t \in \left[ 0, \frac{1}{2} \right] \right) \\ \left( s - \frac{5}{6}, \ t \in \left[ \frac{1}{2}, 1 \right] \right) \end{cases} \]

Proof: Integrating by parts, we can write
\[ \int_0^1 p(t)q(s) \frac{\partial^2 f}{\partial t \partial s} \]
\[ = \int_0^1 q(s) \left[ \left( t - \frac{1}{6} \right) \frac{\partial f}{\partial s} \right] dt \]
\[ + \int_0^1 \left[ \left( t - \frac{5}{6} \right) \frac{\partial^2 f}{\partial t \partial s} \right] dt \]

By integrating the right hand side of equality, we get
\[ \int_0^1 q(s) \left[ \left( t - \frac{1}{6} \right) \frac{1}{a-b} \frac{\partial f}{\partial s} \right] dt \]
\[ + \int_0^1 \left[ \left( t - \frac{5}{6} \right) \frac{1}{a-b} \frac{\partial f}{\partial s} \right] dt \]
\[ = \frac{1}{a-b} \int_0^1 \left( t - \frac{1}{6} \right) \frac{\partial f}{\partial s} \left( ta+(1-t)b, sc+(1-s)d \right) dt \]
\[ + \frac{1}{a-b} \int_0^1 \left( t - \frac{5}{6} \right) \frac{\partial f}{\partial s} \left( ta+(1-t)b, sc+(1-s)d \right) dt \]

Computing these integrals and using the change of the variable \( x = ta+(1-t)b \) and \( y = sc+(1-s)d \) for \( t, s \in [0, 1]^2 \), then multiplying both sides with \( (b-a)(d-c) \), we get the desired result.

**Theorem 3.** Let \( f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping \( \Delta = [a, b] \times [c, d] \). If \( \frac{\partial^2 f}{\partial t \partial s} \) is a convex function on the co-ordinates on \( \Delta \) and \( \frac{\partial^2 f}{\partial t \partial s} \in L(\Delta) \), then the following inequality holds:
\[ \int f(a, c) + f(b, c) + f(a, d) + f(b, d) \]
\[ + f\left( a, \frac{c+d}{2} \right) + f\left( b, \frac{c+d}{2} \right) + 4f\left( \frac{a+b}{2}, \frac{c+d}{2} \right) \]
\[ + f\left( \frac{a+b}{2}, c \right) + f\left( a, \frac{c+d}{2} \right) \]
\[ + f\left( a, \frac{c+d}{2} \right) + f\left( b, \frac{c+d}{2} \right) \]
\[ + \frac{1}{36} \int_0^1 \left[ \left( s - \frac{1}{6} \right) \frac{\partial f}{\partial s} \left( ta+(1-t)b, sc+(1-s)d \right) \right] ds \]
\[ + \frac{1}{36} \int_0^1 \left[ \left( s - \frac{5}{6} \right) \frac{\partial f}{\partial s} \left( ta+(1-t)b, sc+(1-s)d \right) \right] ds \]
\[ + \frac{1}{36} \int_0^1 \left[ \left( s - \frac{5}{6} \right) \frac{\partial f}{\partial s} \left( ta+(1-t)b, sc+(1-s)d \right) \right] ds \]
\[ + \frac{1}{36} \int_0^1 \left[ \left( s - \frac{5}{6} \right) \frac{\partial f}{\partial s} \left( ta+(1-t)b, sc+(1-s)d \right) \right] ds \]
\[ \leq \frac{25(b-a)(d-c)}{72} \]
where
\[
A = \frac{1}{6(b-a)} \int_a^b \left[ f(x, c) + 4f \left( x, \frac{c + d}{2} \right) + f(x, d) \right] dx \\
+ \frac{1}{6(d-c)} \int_c^d \left[ f(a, y) + 4f \left( \frac{a + b + c + d}{2}, y \right) + f(b, y) \right] dy.
\]

Proof. By using Lemma 1, we can write
\[
\begin{bmatrix}
\frac{\partial^2 f}{\partial t \partial s}(a, sc + (1-s)d) \\
\frac{\partial^2 f}{\partial t \partial s}(a, sc + (1-s)d)
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 f}{\partial t \partial s}(b, sc + (1-s)d) \\
\frac{\partial^2 f}{\partial t \partial s}(b, sc + (1-s)d)
\end{bmatrix}
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\begin{bmatrix}
t \left( \frac{t^2}{\partial t \partial s}(a, sc + (1-s)d) \\
t \left( \frac{t^2}{\partial t \partial s}(b, sc + (1-s)d)
\end{bmatrix}
\]

By a similar argument for the above integral, we have
\[
\int_0^1 \left[ \frac{\partial^2 f}{\partial t \partial s}(a, sc + (1-s)d) + \frac{\partial^2 f}{\partial t \partial s}(b, sc + (1-s)d) \right] dt
\]

Computing the integral in the right hand side of above inequality, we have
\[
\int_0^1 \left[ \frac{\partial^2 f}{\partial t \partial s}(a, sc + (1-s)d) + \frac{\partial^2 f}{\partial t \partial s}(b, sc + (1-s)d) \right] dt
\]

Since \( f : \Delta \to \mathbb{R} \) is co-ordinated convex on \( \Delta \), we get
\[
\begin{align*}
&\left[ f \left( a, \frac{c + d}{2} \right) + f \left( b, \frac{c + d}{2} \right) + 4f \left( \frac{a + b + c + d}{2}, \frac{c + d}{2} \right) \right] \\
+ &\frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{9}
\end{align*}
\]

By similar argument for the above integral, we have
\[
\int_0^1 \left[ \frac{\partial^2 f}{\partial t \partial s}(a, sc + (1-s)d) + \frac{\partial^2 f}{\partial t \partial s}(b, sc + (1-s)d) \right] dt
\]

We obtain
\[
\int_0^1 \left[ \frac{\partial^2 f}{\partial t \partial s}(a, sc + (1-s)d) + \frac{\partial^2 f}{\partial t \partial s}(b, sc + (1-s)d) \right] dt
\]

By a similar argument for the above integral, we have
\[
\int_0^1 \left[ \frac{\partial^2 f}{\partial t \partial s}(a, sc + (1-s)d) + \frac{\partial^2 f}{\partial t \partial s}(b, sc + (1-s)d) \right] dt
\]

We obtain
If we use (2.3) in (2.2), we get the required result.

**Theorem 4.** Let \( f: \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \). If \( \frac{\partial^2 f}{\partial t \partial s} \) is bounded, i.e.,

\[
\left\| \frac{\partial^2 f}{\partial t \partial s} ((ta + (1-t)b), sc + (1-s)d) \right\|_\infty = \sup_{(t,s) \in [0,1]^2} \left\| \frac{\partial^2 f}{\partial t \partial s} ((ta + (1-t)b), sc + (1-s)d) \right\| < \infty
\]

for all \((t,s) \in [0,1]^2\) and \( \frac{\partial^2 f}{\partial t \partial s} \in L(\Delta) \). Then the following inequality holds:

\[
f \left( a, \frac{c + d}{2} \right) + f \left( b, \frac{c + d}{2} \right) + 4f \left( \frac{a + b + c + d}{2}, \frac{a + b + c + d}{2} \right)
+ f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, \frac{a + b}{2} \right)
+ f \left( a, c + d \right) + f \left( b, c + d \right) + f \left( a, d \right) + f \left( b, d \right)
\]

\[
\leq \frac{25(b-a)(d-c)}{1296}
\times \left\| \frac{\partial^2 f}{\partial t \partial s} ((ta + (1-t)b), sc + (1-s)d) \right\|_\infty
\]

where \( A \) is as in Theorem 3.

**Proof.** From Lemma 1 and using the property of modulus, we have

\[
\left[ f \left( a, \frac{c + d}{2} \right) + f \left( b, \frac{c + d}{2} \right) + 4f \left( \frac{a + b + c + d}{2}, \frac{a + b + c + d}{2} \right)
+ f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, \frac{a + b}{2} \right) \right]
\]

\[
= \frac{25(b-a)(d-c)}{1296}
\times \left\| \frac{\partial^2 f}{\partial t \partial s} ((ta + (1-t)b), sc + (1-s)d) \right\|_\infty
\]

By a simple calculation,

\[
\int_0^1 \int_0^1 |p(t)q(s)|dsdt = \frac{25}{1296}
\]

If we use (2.5) in (2.4), we have

\[
f \left( a, \frac{c + d}{2} \right) + f \left( b, \frac{c + d}{2} \right) + 4f \left( \frac{a + b + c + d}{2}, \frac{a + b + c + d}{2} \right)
+ f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) + f \left( \frac{a + b}{2}, \frac{a + b}{2} \right)
+ f \left( a, c + d \right) + f \left( b, c + d \right) + f \left( a, d \right) + f \left( b, d \right)
\]

\[
\leq \frac{25(b-a)(d-c)}{1296}
\times \left\| \frac{\partial^2 f}{\partial t \partial s} ((ta + (1-t)b), sc + (1-s)d) \right\|_\infty
\]

This completes the proof.
References


