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Some Integral Inequalities of Hermite-Hadamard Type for Functions Whose n-times Derivatives are (α,m)-Convex

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Abstract In the paper, the authors find some new integral inequalities of Hermite-Hadamard type for functions whose derivatives of the n-th order are (α,m)-convex and deduce some known results. As applications of the newly-established results, the authors also derive some inequalities involving special means of two positive real numbers.

Keywords: *Hermite-Hadamard integral inequality, convex function, (α,m)-convex function, differentiable function; application; mean*

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1. Introduction

It is common knowledge in mathematical analysis that a function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval $I \neq \varnothing$ if

$$f(\lambda x+(1-\lambda)y) \le f(x)+(1-\lambda)f(y) \qquad (1.1)$$

for all x,y∈I and $\lambda \in[0,1]$, if the inequality (1.1) reverses, then f is said to be concave on I.

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I and a,b∈I with a < b. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \qquad (1.2)$$

This inequality is well known in the literature as Hermite-Hadamard integral inequality for convex functions. See [4,12] and closely related references therein.

The concept of usually used convexity has been generalized by a number of mathematicians. Some of them can be recited as follows.

Definition 1.1. *([20]). Let $f:[0,b] \to \mathbb{R}$ be a function and m∈[0,1]. If*

$$f(\lambda x+m(1-\lambda)y) \le \lambda f(x)+m(1-\lambda)f(y) \qquad (1.3)$$

holds for all $f:[0,b] \to \mathbb{R}$ and $\lambda \in[0,1]$, then we say that f(x) is m-convex on [0,b].

Definition 1.2. *([11]). Let $f:[0,b] \to \mathbb{R}$ R be a function and (α,m)∈[0,1]×[0,1]. If*

$$f(\lambda x+m(1-\lambda)y) \le \lambda^{\alpha} f(x)+m(1-\lambda^{\alpha})f(y) \qquad (1.4)$$

is valid for all x,y∈[0,b] and α∈(0,1], then we say that f(x) is (α,m)-convex on [0,b].

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It is not difficult to see that when (α,m)∈f(α,0), (1,0), (1,m), (1,1), (α,1)g the (α,m)-convex function becomes the α-star-shaped, star-shaped, m-convex, convex, and α-convex functions respectively.

The famous Hermite-Hadamard inequality (1.2) has been refined or generalized by many mathematicians. Some of them can be reformulated as follows.

Theorem 1.1. *([[14], Theorem 3]). Let $f: I^{\circ} \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L([a,b])$ for a,b∈I with a < b. If $|f''(x)|^q$ is m-convex on [a,b] for some fixed q > 1 and m∈[0,1], then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{8}$$

$$\times\left[\frac{\Gamma(1+p)}{\Gamma(3/2+p)}\right]^{1/p}\left[\frac{|f''(a)|^q+m|f''(b/m)|^q}{2}\right]^{1/q}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and Γ is the classical Euler gamma function which may be defined for R(z) > 0 by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt. \quad (1.5)$$

Theorem 1.2. *([[17], Theorem 4]). Let $f: I\subseteq\mathbb{R}$ be an open interval and $a,b\in I$ with $a<b$, and let $f: I\to\mathbb{R}$ be a twice differentiable mapping such that $f''(x)$ is integrable. If $0\le\lambda\le 1$ and $|f''(x)|$ is convex on [a,b], then*

$$\left|(\lambda-1)f\left(\frac{a+b}{2}\right)-\lambda\frac{f(a)+f(b)}{2}+\int_a^b f(x)dx\right|$$
$$\le\begin{cases}\frac{(b-a)^2}{24}\left\{\left[\lambda^4+(1+\lambda)(1-\lambda)^3+\frac{5\lambda-3}{4}\right]|f''(a)|\right.\\ \left.+\left[\lambda^4+(2-\lambda)\lambda^3+\frac{1-3\lambda}{4}\right]|f''(b)|\right\}, 0\le\lambda\le\frac{1}{2};\\ \frac{(b-a)^2}{48}(3\lambda-1)\left(|f''(a)|+|f''(b)|\right), \frac{1}{2}\le\lambda\le 1.\end{cases}$$

Theorem 1.3. *([[13], Theorem 3]). Let $b^*>0$ and $f:[0,b^*]\to\mathbb{R}$ be a twice differentiable function such that $f''\in L([a,b])$ for $a,b\in[0,b^*]$ with $a<b$. If $|f''(x)|^q$ is (α,m)-convex on [a,b] for $(\alpha,m)\in[0,1]\times[0,1]$ and $q\ge 1$, then*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|$$
$$\le\frac{(mb-a)^2}{2}\left(\frac{1}{6}\right)^{1-1/q} \quad (1.6)$$
$$\left\{\frac{|f''(a)|^q}{(\alpha+2)(\alpha+3)}+m\right\}^{1/q}$$

In recent years, some other kinds of Hermite-Hadamard type inequalities were generated in [1,2,3,15,16,19,26,29,30], for example. For more systematic information, please refer to monographs [4,12] and related references therein.

In this paper, we will establish some new inequalities of Hermite-Hadamard type for functions whose derivatives of n-th order are (α,m)-convex and deduce some known results in the form of corollaries.

2. A Lemma

For establishing new integral inequalities of Hermite-Hadamard type for functions whose derivatives of n-th order are (α,m)-convex, we need the following lemma.

Lemma 2.1. *Let $0<m\le 1$ and $b>a>0$ satisfying $a<mb$. If $f^{(n)}(x)$ for $n\in\{0\}\cup\mathbb{N}$ exists and is integrable on the closed interval [0,b], then*

$$\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx$$
$$-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{(k+1)!}f^{(k)}(a)=\frac{1}{2}\frac{(mb-a)^n}{n!} \quad (2.1)$$
$$\times\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+m(1-t)b)dt,$$

where the sum above takes 0 when n = 1 and n = 2.

Proof. When n = 1, it is easy to deduce the identity (2.1) by performing an integration by parts in the integrals from the right side and changing the variable.

When n = 2, we have

$$\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx \quad (2.2)$$
$$=\frac{(mb-a)^n}{2}\int_0^1 t(1-t)f''(ta+m(1-t)b)dt.$$

This result is same as [[13], Lemma 2].

When n = 3, the identity (2.1) is equivalent to

$$\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx$$
$$-\frac{(mb-a)^2}{12}f''(a)=\frac{(mb-a)^3}{12} \quad (2.3)$$
$$\times\int_0^1 t^2(n-2t)f^{(3)}(ta+m(1-t)b)dt,$$

which may be derived from integrating the integral in the second line of (2.3) and utilizing the identity (2.2).

When n ≥ 4, computing the second line in (2.1) by integration by parts yields

$$\frac{(mb-a)^n}{n!}\int_0^1 t^2(n-2t)f^{(3)}(ta+m(1-t)b)dt$$
$$=-\frac{(n-2)(mb-a)^{n-1}}{n!}f^{(n-1)}(a)+\frac{(mb-a)^{n-1}}{(n-1)!}$$
$$\times\int_0^1 t^{n-2}(n-1-2t)f^{(n-1)}(ta+m(1-t)b)dt,$$

which is a recurrent formula

$$S_{a,mb}(n)=-T_{a,mb}(n-1)+S_{a,mb}(n-1)$$

on n where

$$S_{a,mb}(n)=\frac{1}{2}\frac{(mb-a)^n}{n!}\times\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+m(1-t)b)dt$$

and

$$T_{a,mb}(n-1)=\frac{1}{2}\frac{(n-2)(mb-a)^{n-1}}{n!}f^{(n-1)}(a)$$

for n≥4. By mathematical induction, the proof of Lemma 2.1 is complete.

Remark 2.1. *Similar integral identities to (2.1), produced by replacing $f^{(k)}(a)$ in (2.1) by $f^{(k)}(b)$ or by $f^{(k)}\left(\frac{a+b}{2}\right)$, and corresponding integral inequalities of Hermite-Hadamard type have been established in [10,22,23].*

Remark 2.2. *When m = 1, our Lemma 2.1 becomes [[7], Lemma 2.1].*

3. Inequalities of Hermite-Hadamard Type

Now we are in a position to establish some integral inequalities of Hermite-Hadamard type for functions whose derivatives of n-th order are (α,m)-convex.

Theorem 3.1. *Let $(\alpha,m)\in[0,1]\times(0,1]$ and $b > a > 0$ with $a < mb$. If $f(x)$ is n-time differentiable on $[0,b]$ such that $\left|f^{(n)}(x)\right|\in L([0.mb])$ and $\left|f^{(n)}(x)\right|^p$ is (α,m)-convex on $[0,mb]$ for $n\geq 2$ and $p\geq 1$, then*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{(k+1)!}f^{(k)}(a)\right| \leq \frac{1}{2}\frac{(mb-a)^n}{n!}\times\left(\frac{n-1}{n+1}\right)^{1-1/p} \left\{\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\left|f^{(n)}(a)\right|^p + m\left[\frac{n-1}{n+1}-\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\left|f^{(n)}(b)\right|^p\right]\right\}^{1/p} \tag{3.1}$$

where the sum above takes 0 when n = 2.

Proof. It follows from Lemma 2.1 that

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{(k+1)!}f^{(k)}(a)\right| \leq \frac{1}{2}\frac{(mb-a)^n}{n!} \times\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+m(1-t)b)dt, \tag{3.2}$$

When p = 1, since $\left|f^{(n)}(x)\right|$ is (α,m)-convex, we have

$$\left|f^{(n)}(ta+m(1-t)b)\right|\leq t^{\alpha}\left|f^{(n)}(a)\right|+m\left(1-t^{a}\right)\left|f^{(n)}(b)\right|.$$

Multiplying by the factor $t^{n-1}(n-2t)$ on both sides of the above inequality and integrating with respect to $t\in[0,1]$ lead to

$$\begin{aligned}&\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+m(1-t)b)dt\\ &\leq\int_0^1 t^{n-1}(n-2t)\left[t^{\alpha}\left|f^{(n)}(a)\right|+m\left(1-t^{\alpha}\right)\left|f^{(n)}(b)\right|\right]dt\\ &=\left|f^{(n)}(a)\right|\int_0^1 t^{n+\alpha-1}(n-2t)dt\\ &+m\left|f^{(n)}(b)\right|\int_0^1 t^{n-1}(n-2t)\left(1-t^{\alpha}\right)dt\\ &=\left(\frac{n}{n+\alpha}-\frac{2}{n+\alpha+1}\right)\left|f^{(n)}(a)\right|\\ &+m\left|f^{(n)}(b)\right|\left(\frac{n-1}{n+1}-\frac{n}{n+\alpha}+\frac{2}{n+\alpha+1}\right)\\ &=\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\left|f^{(n)}(a)\right|\\ &+m\left[\frac{n-1}{n+1}-\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\right]\left|f^{(n)}(b)\right|.\end{aligned}$$

The proof for the case p = 1 is complete.

When p > 1, by the well-known Hölder integral inequality, we obtain

$$\begin{aligned}&\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(ta+m(1-t)b)\right|dt\\ &\leq\left[\int_0^1 t^{n-1}(n-2t)dt\right]^{1-1/p}\\ &\left[\int_0^1 t^{n-1}(n-2t)\times\left|f^{(n)}(ta+m(1-t)b)\right|dt\right]^{1/p}\end{aligned} \tag{3.3}$$

Using the (α,m)-convexity of $\left|f^{(n)}(x)\right|^p$ produces

$$\begin{aligned}&\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(ta+m(1-t)b)\right|^p dt\\ &\leq\int_0^1 t^{n-1}(n-2t)\left[t^{\alpha}\left|f^{(n)}(a)\right|^p+m\left(1-t^{\alpha}\right)\left|f^{(n)}(b)\right|^p\right]dt\\ &=\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\left|f^{(n)}(a)\right|^p\\ &+m\left[\frac{n-1}{n+1}-\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\right]\left|f^{(n)}(b)\right|^p.\end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) yields the inequality (3.1). This completes the proof of Theorem 3.1.

Corollary 3.1. *Under conditions of Theorem 3.1,*

1. when m = 1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^{b}f(x)dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{(k+1)!}f^{(k)}(a)\right| \leq \frac{1}{2}\frac{(b-a)^n}{n!}\times\left(\frac{n-1}{n+1}\right)^{1-1/p} \left\{\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\left|f^{(n)}(a)\right|^p + \left[\frac{n-1}{n+1}-\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\left|f^{(n)}(b)\right|^p\right]\right\}^{1/p}$$

2. when n = 2, we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right| \leq \frac{(mb-a)^2}{4}\left(\frac{1}{3}\right)^{1-1/p} \left\{\frac{2}{(\alpha+2)(\alpha+3)}\left|f^{n}(a)\right|^p + m\left[\frac{1}{3}-\frac{2}{(\alpha+2)(\alpha+3)}\right]\left|f^{n}(b)\right|^p\right\};$$

3. when $m=\alpha=p=1$ and n = 2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^{b}f(x)dx\right| \leq \frac{(b-a)^2}{24}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

4. when m =α= 1 and p = n = 2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{12}\left[\frac{|f''(a)|^2+|f''(b)|^2}{2}\right]^{1/2}.$$

Remark 3.1. *Under conditions of Theorem 3.1,*

1. when n = 2, the inequality (3.1) becomes the one (1.6) in [[13], Theorem 3];

2. when α = m = 1, Theorem 3.1 becomes [[7], Theorem 3.1].

Theorem 3.2. *Let (α,m)∈[0,1]×(0,1] and b >a > 0 with a < mb. If f(x) is n-time differentiable on [0,b] such that* $\left|f^{(n)}(x)\right| \in L([0,mb])$ *and* $\left|f^{(n)}(x)\right|^p$ *is (α,m)-convex on [0,mb] for n≥2 and p > 1, then*

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{(k+1)!}f^{(k)}(a)\right| \le \frac{1}{2}\frac{(mb-a)^n}{n!}\left[\frac{n^{q+1}-(n-2)^{q+1}}{2(q+1)}\right]^{1/q} \times\left\{\frac{1}{p(n-1)+\alpha+1}\left|f^{(n)}(a)\right|^p+\frac{m\alpha\left|f^{(n)}(b)^p\right|}{\left[p(n-1)+1\right]\left[p(n-1)+\alpha+1\right]}\right\}^{1/p}, \tag{3.5}$$

where the sum above takes 0 when n = 2 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. It follows from Lemma 2.1 that

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{(k+1)!}f^{(k)}(a)\right| \le \frac{1}{2}\frac{(mb-a)^n}{n!} \times\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(ta+m(1-t)b)\right|dt. \tag{3.6}$$

By the well-known Hölder integral inequality, we obtain

$$\begin{aligned}&\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+m(1-t)b)dt\\ &\le\left[\int_0^1 (n-2t)^q dt\right]^{1/q}\left[\int_0^1 t^{p(n-1)}\left|f^{(n)}(ta+m(1-t)b)\right|^p dt\right]^{1/p}\\ &=\left[\frac{n^{q+1}-(n-2)^{q+1}}{2(q+1)}\right]^{1/q}\\ &\times\left[\int_0^1 t^{p(n-1)}\left|f^{(n)}(ta+m(1-t)b)\right|^p dt\right]^{1/p}.\end{aligned} \tag{3.7}$$

Making use of the (α,m)-convexity of $\left|f^{(n)}(x)\right|^p$ reveals

$$\begin{aligned}&\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+m(1-t)b)dt\\ &\le\int_0^1 t^{p(n-1)}\left[t^\alpha\left|f^{(n)}(a)\right|^p+m(1-t)^\alpha\left|f^{(n)}(b)\right|^p\right]dt\\ &=\left|f^{(n)}(a)\right|^p\int_0^1 t^{p(n-1)+\alpha}dt\\ &+m\left|f^{(n)}(b)\right|^p\int_0^1 t^{p(n-1)}\left(1-t^\alpha\right)dt\\ &=\frac{\left|f^{(n)}(a)\right|^p}{p(n-1)+\alpha+1}\\ &+\frac{m\alpha}{\left[p(n-1)+1\right]\left[p(n-1)+\alpha+1\right]}\left|f^{(n)}(b)^p\right|.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) with (3.6) results in the inequality (3.5). This completes the proof of Theorem 3.2.

Corollary 3.2. *Under conditions of Theorem 3.2,*

1. when m = 1, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{(k+1)!}f^{(k)}(a)\right| \le \frac{1}{2}\frac{(b-a)^n}{n!}\left[\frac{n^{q+1}-(n-2)^{q+1}}{2(q+1)}\right]^{1/q} \times\left\{\frac{1}{p(n-1)+\alpha+1}\left|f^{(n)}(a)\right|^p+\frac{\alpha\left|f^{(n)}(b)^p\right|}{\left[p(n-1)+1\right]\left[p(n-1)+\alpha+1\right]}\right\}^{1/p};$$

2. when n = 2, we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right| \le \frac{(mb-a)^2}{2}\left(\frac{1}{q+1}\right)^{1/q} \left[\frac{1}{p+\alpha+1}\left|f''(a)\right|^p+\frac{m\alpha}{(p+1)(p+\alpha+1)}\left|f''(b)\right|^p\right]^{1/p};$$

3. when m =α= p = 1 and n = 2, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2(p+1)^{1/p}(q+2)^{1/q}} \times\left[\frac{(q+1)\left|f''(a)\right|^q+\left|f''(b)\right|^q}{q+1}\right]^{1/q}, \tag{3.9}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 3.3. *Let $(\alpha,m)\in[0,1]\times(0,1]$ and $b>a>0$ with $a<mb$. If $f(x)$ is n-time differentiable on $[0,b]$ such that $\left|f^{(n)}(x)\right|\in L([0,mb])$ and $\left|f^{(n)}(x)\right|^p$ is (α,m)-convex on $[0,mb]$ for $n\geq 2$ and $p>1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{(k+1)!}f^{(k)}(a)\right|\\&\leq\frac{(n-1)^{1-1/p}}{2}\frac{(mb-a)^n}{n!}\\&\times\left\{\begin{array}{l}\frac{(n-2)(pn-p+\alpha)+2(n-1)}{(pn-p+\alpha+1)(pn-p+\alpha+2)}\left|f^{(n)}(a)\right|^p\\-m\left[\begin{array}{l}\frac{(n-2)(pn-p+\alpha)+2(n-1)}{(pn-p+\alpha+1)(pn-p+\alpha+2)}\\-\frac{(n-1)(pn-2p+\alpha)}{(pn-p+\alpha+1)(pn-p+\alpha+2)}\end{array}\right]\\\left|f^{(n)}(b)\right|^p\end{array}\right\}^{1/p},\end{aligned}\tag{3.10}$$

where the sum above takes 0 when n = 2.

Proof. Utilizing Lemma 2.1, Hölder integral inequality, and the (α,m)-convexity of $\left|f^{(n)}(x)\right|^p$ yields

$$\begin{aligned}&\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(mb-a)^k}{(k+1)!}f^{(k)}(a)\right|\\&\leq\frac{1}{2}\frac{(mb-a)^n}{n!}\times\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(ta+m(1-t)b)\right|dt\\&\leq\frac{1}{2}\frac{(mb-a)^n}{n!}\left[\int_0^1(n-2t)^q\,dt\right]^{1-1/q}\\&\times\left\{\int_0^1 t^{p(n-1)}(n-2t)\left[\begin{array}{l}t^{\alpha}\left|f^{(n)}(a)\right|^p\\+m\left(1-t^{\alpha}\right)\left|f^{(n)}(b)\right|^p\end{array}\right]dt\right\}^{1/p}\\&=\frac{(n-1)^{1-1/p}}{2}\frac{(mb-a)^n}{n!}\\&\times\left\{\begin{array}{l}\frac{(n-2)(pn-p+\alpha)+2(n-1)}{(pn-p+\alpha+1)(pn-p+\alpha+2)}\left|f^{(n)}(a)\right|^p\\-m\left[\begin{array}{l}\frac{(n-1)(pn-2p+\alpha)}{(pn-p+\alpha+1)(pn-p+\alpha+2)}\\-\frac{(n-2)(pn-p+\alpha)+2(n-1)}{(pn-p+\alpha+1)(pn-p+\alpha+2)}\end{array}\right]\left|f^{(n)}(x)\right|^p\end{array}\right\}^{1/p}.\end{aligned}$$

This completes the proof of Theorem 3.3.

Corollary 3.3. *Under conditions of Theorem 3.3,*

1. when m = 1, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^{b}f(x)dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{(k+1)!}f^{(k)}(a)\right|\\&\leq\frac{(n-1)^{1-1/p}}{2}\frac{(b-a)^n}{n!}\\&\times\left\{\begin{array}{l}\frac{(n-2)(pn-p+\alpha)+2(n-1)}{(pn-p+\alpha+1)(pn-p+\alpha+2)}\left|f^{(n)}(a)\right|^p\\+\\-m\left[\begin{array}{l}\frac{(n-1)(pn-2p+2)}{(pn-p+\alpha+1)(pn-p+\alpha+2)}\\-\frac{(n-2)(pn-p+\alpha)+2(n-1)}{(pn-p+\alpha+1)(pn-p+\alpha+2)}\end{array}\right]\left|f^{(n)}(b)\right|^p\end{array}\right\}^{1/p};\end{aligned}$$

2. when n = 2, we have

$$\begin{aligned}&\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb}f(x)dx\right|\\&\leq\frac{(mb-a)^2}{4}\left\{\begin{array}{l}\frac{2}{(p+\alpha+1)(p+\alpha+2)}\left|f''(a)\right|^p\\+m\left[\begin{array}{l}\frac{2}{(p+1)(p+2)}\\-\frac{2}{(p+\alpha+1)(p+\alpha+2)}\end{array}\right]\left|f''(b)\right|^p\end{array}\right\}\end{aligned}$$

3. when $m=\alpha=1$ and n = 2, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^{b}f(x)dx\right|\\&\leq\frac{(b-a)^2}{2^{2-1/p}}\times\left[\frac{(p+1)\left|f''(a)\right|^p+2\left|f''(b)\right|^p}{(p+1)(p+2)(p+3)}\right]^{1/p}.\end{aligned}$$

4. Applications to Special Means

It is well known that, for positive real numbers α and β with α≠β, the quantities

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2},G(\alpha,\beta)=\sqrt{\alpha\beta},$$

$$H(\alpha,\beta)=\frac{2}{1/\alpha+1/\beta},I(\alpha,\beta)=\frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right)^{1/(\beta-\alpha)},$$

$$L(\alpha,\beta)=\frac{\alpha-\beta}{\ln\alpha-\ln\beta},L_r(\alpha,\beta)=\left[\frac{\beta^{r+1}-\alpha^{r+1}}{(r+1)(\beta-\alpha)}\right]^{1/r}$$

for $r\neq 0$, -1 are respectively called the arithmetic, geometric, harmonic, exponential, logarithmic, and generalized logarithmic means.

Basing on inequalities of Hermite-Hadamard type in the above section, we shall derive some inequalities of the above defined means as follows.

Theorem 4.1. *Let $r\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $b>a>0$. Then, for $p,q>1$,*

$$\left|A\left(a^r,b^r\right)-\left[L_r\left(a,b\right)\right]^r\right| \le \frac{\left(b-a\right)^2 r\left(r-1\right)}{2\left(p+1\right)^{1/p}\left(q+2\right)^{1/q}}\left[a^{(r-2)q}+\frac{b^{(r-2)q}}{q+1}\right]^{1/q}, \quad (4.1)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. This follows from applying the inequality (3.9) to the function $f(x) = x^r$.

Theorem 4.2. *Let $r\in(-\infty, 0)\cup[1,\infty)\backslash\{-1\}$ and $b > a > 0$. Then, for $p \ge 1$,*

$$\left|A\left(a^r,b^r\right)-\left[L_r\left(a,b\right)\right]^r\right| \le \frac{\left(b-a\right)^2 r\left(r-1\right)}{2^{2-1/p}}\left[\frac{\left(p+1\right)a^{(r-2)p}+2b^{(r-2)p}}{\left(p+1\right)\left(p+2\right)\left(p+3\right)}\right]^{1/p}.$$

Proof. This follows from applying the inequality (3.11) to the function $f(x) = x^r$.

Theorem 4.3. *Let $r\in(-\infty, 0)\cup[1,\infty)\backslash\{-1\}$ and $b > a > 0$. Then*

$$\left|A\left(a^r,b^r\right)-\left[L_r\left(a,b\right)\right]^r\right| \le \frac{\left(b-a\right)^2 r\left(r-1\right)}{24}A\left(a^{r-2},b^{r-2}\right).$$

Proof. This follows from applying the inequality (3.11) for $p = 1$ to the function $f(x) = x^r$.

Theorem 4.4. *Let $b > a > 0$. Then for $p,q > 1$ we have*

$$\left|\frac{1}{H\left(a,b\right)}-\frac{1}{L\left(a,b\right)}\right| \le \frac{\left(b-a\right)^2}{\left(p+1\right)^{1/p}\left(q+2\right)^{1/q}}\left[\frac{1}{a^{3q}}+\frac{1}{\left(q+1\right)b^{3q}}\right]^{1/q}, \quad (4.2)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. This follows from applying the inequality (3.9) to the function $f\left(x\right)=\frac{1}{x}$.

Theorem 4.5. *Let $b > a > 0$. Then for $p\ge 1$ we have*

$$\left|\frac{1}{H\left(a,b\right)}-\frac{1}{L\left(a,b\right)}\right| \le \frac{\left(b-a\right)^2 r\left(r-1\right)}{2^{1-1/p}\left[\left(p+2\right)\left(p+3\right)\right]^{1/p}}\times\left[\frac{1}{a^{3p}}+\frac{1}{\left(p+1\right)b^{3q}}\right]^{1/p}. \quad (4.3)$$

Proof. This follows from the inequality (3.11) to the function $f(x) = x^r$.

Theorem 4.6. *Let $b > a > 0$. Then we have*

$$\ln\frac{I\left(a,b\right)}{G\left(a,b\right)} \le \frac{\left(b-a\right)^2}{24}A\left(\frac{1}{a^2},\frac{1}{b^2}\right). \quad (4.4)$$

Proof. This follows from applying the inequality (3.11) for $p = 1$ to the function $f(x) = -\ln x$.

Remark 4.1. *This paper is a combined version of the preprints [8,9].*

Remark 4.2. *Finally, we would like to recommend some newly published articles [5,6,18,21,24,25,27,28,31,32,33] which have something to do with this topic.*

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