Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 3, 102-109
Available online at http://pubs.sciepub.com/tjant/2/3/9

DOI:10.12691/tjant-2-3-9

On the Moments of the Function $E^*(t)$

Aleksandar Ivić*

Aleksandar Ivić, Katedra Matematike RGF-A, Universitet U Beogradu, Đušina, Beograd, Serbia
*Corresponding author: ivic@rgf.bg.ac.rs, aivic2000@yahoo.com

Received May 19, 2014; Revised July 02, 2014; Accepted July 14, 2014

Abstract Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem, and $E(T)$ the error term in the asymptotic formula for the mean square of $\left|\zeta\left(\frac{1}{2}+it\right)\right|$. If $E^*(t)=E(t)-2\pi\Delta^*(t/2\pi)$ with $\Delta^*(x)=-\Delta(x)+2\Delta(2x)-\frac{1}{2}\Delta(4x)$, then we discuss bounds for third, fourth and fifth power moment of $|E^*(t)|$. We also prove that $E^*(t)$ always changes sign in $\left[T,T+T^{2/3+\varepsilon}\right]$ for $T\ge T_0(\varepsilon)$, and obtain (conditionally) the existence of its large positive, or small negative values.

Keywords: *Riemann zeta-function, moments of the function E*(t), large values*

Cite This Article: Aleksandar Ivić, “On the Moments of the Function $E^*(t)$.” *Turkish Journal of Analysis and Number Theory*, vol. 2, no. 3 (2014): 102-109. doi: 10.12691/tjant-2-3-9.

1. Introduction

This paper is the continuation of the author’s works [6,7,8], where the analogy between the Riemann zeta-function $\zeta(s)$ and the divisor problem was investigated. As usual, let the error term in the classical Dirichlet divisor problem be

$$\Delta(x)=\sum_{n\le x} d(n)-x(\log x+2\gamma-1), \tag{1.1}$$

and

$$E(T)=\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt-T\left(\log\left(\frac{T}{2\pi}\right)+2\gamma-1\right), \tag{1.2}$$

where $d(n)$ is the number of divisors of $n, \zeta(s)$ is the Riemann zeta-function, and $\gamma=-\Gamma'(1)=0.577215\ldots$ is Euler’s constant. In view of F.V. Atkinson’s classical explicit formula for $E(T)$ (see [1], [[4], Chapter 15] and [[5], Chapter 2]) it was known long ago that there are analogies between $\Delta(x)$ and $E(T)$. However, instead of the error-term function $\Delta(x)$ it is more exact to work with the modified function $\Delta^*(x)$ (see M. Jutila [11], [12] and T. Meurman [13]), where

$$\begin{aligned}\Delta^*(x)&:=-\Delta(x)+2\Delta(2x)-\frac{1}{2}\Delta(4x)\\&=\frac{1}{2}\sum_{n\le 4x}(-1)^n d(n)-x(\log x+2\gamma-1),\end{aligned} \tag{1.3}$$

which is a better analogue of $E(T)$ than $\Delta(x)$. M. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t):=E(t)-2\pi\Delta^*\left(\frac{t}{2\pi}\right), \tag{1.4}$$

and this work was continued by the author in [6] and [7]. In [7] he proved the asymptotic formula

$$\int_0^T\left(E^*(t)\right)^2 dt=T^{4/3}p_3(\log T)+O_\varepsilon\left(T^{7/6+\varepsilon}\right), \tag{1.5}$$

where $p_3(y)$ is a polynomial of degree three in y with positive leading coefficient, and all the coefficients may be explicitly evaluated. Here and later, as usual, ε denotes arbitrarily small positive constants, not necessarily the same ones at each occurrence. Moreover $F(x)=O_\varepsilon(G(x))$ (same as $F(x)\ll_\varepsilon G(x)$) means that $|F(x)|\le CG(x)$ for some $C=C(\varepsilon)>0$ and $x\ge x_0(\varepsilon)$. Besides (1.5), the author had proved in [[6], Part IV] that

$$\int_0^T\left|E^*(t)\right|^3 dt\ll_\varepsilon T^{3/2+\varepsilon} \tag{1.6}$$

and in [[6], Part II] that

$$\int_0^T\left|E^*(t)\right|^5 dt\ll_\varepsilon T^{2+\varepsilon}. \tag{1.7}$$

It turns out that the results (1.5)-(1.7) are independent of each other, that is, neither two of them imply the third one. Note that we have odd moments in (1.6) and (1.7), and it seems plausible that the respective moments without absolute value signs are smaller, since a lot of cancellation will probably happen. For example, one expects that the bound

$$\int_0^T \left(E^*(t)\right)^3 dt \ll_\varepsilon T^{3/2-\eta+\varepsilon} \tag{1.8}$$

holds for some constant η such that $0 < \eta < 1/6$, but this seems difficult to prove.

The first aim of this note is to provide unified, simplified and rigorous proofs of (1.6), (1.7) and a result that, by the Cauchy–Schwarz inequality for integrals, follows from (1.6) and (1.7), namely

$$\int_0^T \left|E^*(t)\right|^4 dt \ll_\varepsilon T^{7/4+\varepsilon}. \tag{1.9}$$

Namely in previous work I used a lemma from M. Jutila [[11], Part II]. This was

LEMMA 1. *For $A \in \mathbb{R}$ a constant we have*

$$\cos\left(\sqrt{8\pi nT} + \frac{1}{6}\sqrt{2\pi^3} n^{3/2} T^{-1/2} + A\right) = \int_{-\infty}^{\infty} \alpha(u) \cos\left(\sqrt{8\pi n}\left(\sqrt{T} + u\right) + A\right) du,$$

where $\alpha(u) \ll T^{1/6}$ for $u \neq 0$,

$$\alpha(u) \ll T^{1/6} \exp\left(-bT^{1/4}|u|^{3/2}\right)$$

for $u < 0$, and

$$\alpha(u) = T^{1/8} u^{-1/4} \begin{pmatrix} d \exp\left(ibT^{1/4}u^{3/2}\right) \\ +\bar{d} \exp\left(-ibT^{1/4}u^{3/2}\right) \end{pmatrix} + O\left(T^{-1/8}u^{-7/4}\right)$$

for $u > T^{-1/6}$ and some constants $b\ (> 0)$ and d.

This lemma, useful in its own right, seems insufficient in itself to deal with the case when the exponential integrals that come into play have a saddle point. It can be avoided altogether, and the complete proofs of (1.6)–(1.9) will be given in Sections 4 and 5, while the necessary lemmas are given in Section 2. Actually we shall first prove (1.9), and then use it to derive (1.6) and (1.7).

The second aim of this paper is to provide some new results on the distribution of values of $E^*(t)$. We have

THEOREM 1. *If $H = T^{2/3+\varepsilon}$, then the function $E^*(t)$ changes sign in every interval [T, T + H] for $T \geq T_0(\varepsilon)$.*

It should be remarked that in [9] the author proved the mean value bound

$$\int_T^{T+H} \left(E^*(t)\right)^2 dt \gg HT^{1/3} \log^3 T \left(T^{2/3+\varepsilon} \leq H \leq T\right). \tag{1.10}$$

From (1.10) one deduces that, under the hypotheses of Theorem 1, the interval [*T*, *T* + *H*] contains a point t_0 such that

$$\left|E^*(t_0)\right| > At_0^{1/6} \log^{3/2} t_0 \quad (A > 0). \tag{1.11}$$

The inequality (1.11) shows the existence of large values of $|E^*(t)|$ in [*T*, *T* + *H*]. Note that $E^*(t)$ is discontinuous at the integers, but *E*(*t*) is continuous and $d(n) \ll_\varepsilon n^{\varepsilon\ \varepsilon}$. Hence by the defining relation (1.4) of $E^*(t)$ and Theorem 1 it follows that every interval [*T*, *T* +*H*] contains a point t_1 such that $E^*(t_1) \ll_\varepsilon t_1^\varepsilon$. It would be interesting to see what is the smallest *H* such that the function $E^*(t)$ changes sign in every interval [*T*, *T* + *H*] for $T > T_0(\varepsilon)$. It would be also interesting to obtain (1.11) without absolute values, namely to find large positive values of $E^*(t)$ and small negative values of $E^*(t)$ in [*T*, *T* + *H*]. This, at present, does not seem possible unconditionally. However, we have

THEOREM 2. Suppose (1.8) holds. Then for any $\varepsilon > 0$ there exist constants $T_0(\varepsilon)$, $A > 0$ such that, for $T \geq T_0(\varepsilon)$, every interval $[T, T + T^{1-\eta+\varepsilon}]$ contains points T1, T2 for which

$$E^*(T_1) > AT_1^{1/6} \log^{3/2} T_1, \quad E^*(T_2) < -AT_2^{1/6} \log^{3/2} T_2. \tag{1.12}$$

The plan of the paper is as follows. In Section 2 the necessary lemmas will be given. Technical preparation is carried out in Section 3, while Section 4 contains the proofs of (1.9). The proofs (1.6) and (1.7) will be given in Section 5. Section 6 contains the proof of Theorem 1, and Section 7 that of Theorem 2.

2. The Necessary Lemmas

In this section we shall present the lemmas needed in the proofs of our results. We begin with the technical

LEMMA 2. *If $T^\varepsilon \leq G \leq T/\log T$ and $C > 0$ is a suitable constant, then we have*

$$\begin{aligned} E^*(T) &\leq \frac{1}{\sqrt{\pi}G} \int_{-\infty}^{\infty} E^*(T + G\log T + u) e^{-(u/G)^2} du \\ &\quad + CGT^\varepsilon, \\ E^*(T) &\geq \frac{1}{\sqrt{\pi}G} \int_{-\infty}^{\infty} E^*(T - G\log T - u) e^{-(u/G)^2} du \\ &\quad - CGT^\varepsilon. \end{aligned} \tag{2.1}$$

Proof. From the defining relations (1.2)−(1.4) one easily obtains

$$E^*(T) \leq E^*(T+u) + O_\varepsilon\left((1+u)T^\varepsilon\right) \quad (0 \leq u \leq T).$$

By integration this gives, for $T^\varepsilon \leq G \leq T/L$, $L := \log T$,

$$E^*(T) \int_0^{2GL} e^{-(u-GL)^2 G^{-2}} du \leq \int_0^{2GL} E^*(T + G\log T + u) e^{-(u/G)^2} du + O_\varepsilon\left(G^2 T^\varepsilon\right),$$

$$E^*(T) \int_{-\infty}^{\infty} e^{-(u-GL)^2 G^{-2}} du \leq \int_{-\infty}^{\infty} E^*(T + G\log T + u) e^{-(u/G)^2} du + O_\varepsilon\left(G^2 T^\varepsilon\right).$$

Since

$$\int_{-\infty}^{\infty} e^{-(u-GL)^2 G^{-2}} du = \int_{-\infty}^{\infty} e^{-(v/G)^2 G^{-2}} dv = \sqrt{\pi}G,$$

we have the first inequality in (2.1), and the second one is derived analogously.

For $E(T)$ we shall use F.V. Atkinson's classical explicit formula (see e.g., his paper [1], or the author's monographs [[4], Chapter 15] or [[5], Chapter 2]). This result is stated here as

LEMMA 3. *Let* $0 < A < A'$ *be any two fixed constants such that* $AT < N < A'T$, *and let* $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$. *Then*

$$E(T) = \sum\nolimits_1(T) + \sum\nolimits_2(T) + O\left(\log^2 T\right), \tag{2.2}$$

where

$$\sum\nolimits_1(T) = 2^{1/2}(T/2\pi)^{1/4} \times \sum_{n\le N}(-1)^n d(n)n^{-3/4}e(T,n)\cos\left(f(T,n)\right), \tag{2.3}$$

$$\sum\nolimits_2(T) = -2\sum_{n\le N'} d(n)n^{-1/2}\left(\log\frac{T}{2\pi n}\right)^{-1}\cos\left(T\log\left(\frac{T}{2\pi n}\right) - T + \frac{1}{4}\pi\right), \tag{2.4}$$

with

$$\begin{aligned} f(T,n) &= 2T\operatorname{arcsinh}\left(\sqrt{\frac{\pi n}{2T}}\right) + \sqrt{2\pi nT + \pi^2 n^2} - \frac{1}{4}\pi \\ &= -\frac{1}{4}\pi + 2\sqrt{2\pi nT} + \frac{1}{6}\sqrt{2\pi^3}n^{3/2}T^{-1/2} \\ &\quad + a_5 n^{5/2}T^{-3/2} + a_7 n^{7/2}T^{-5/2} + \ldots, \end{aligned} \tag{2.5}$$

$$\begin{aligned} e(T,n) &= \left(1 + \pi n/(2T)\right)^{-1/4}\left\{(2T/\pi n)^{1/2}\operatorname{arcsinh}\left(\sqrt{\frac{\pi n}{2T}}\right)\right\}^{-1} \\ &= 1 + O(n/T) \qquad (1 \le n \le T), \end{aligned} \tag{2.6}$$

and $\operatorname{arsinh} x = \log\left(x + \sqrt{1+x^2}\right)$.

The next lemma is the Voronoï-type formula for $\Delta^*(x)$, which is the analogue of the classical truncated Voronoï formula for $\Delta(x)$, only the formula for $\Delta^*(x)$ has the factor $(-1)^n$ in the sum, while that for $\Delta(x)$ does not (see e.g., [[4], Chapter 15]).

LEMMA 4. *We have, for* $1 \ll N \ll x$,

$$\Delta^*(x) = \frac{1}{\pi\sqrt{2}}x^{\frac{1}{4}}\sum_{n\le N}(-1)^n d(n)n^{-\frac{3}{4}}\cos\left(4\pi\sqrt{nx} - \frac{1}{4}\pi\right) + O_\varepsilon\left(x^{\frac{1}{2}+\varepsilon}N^{-\frac{1}{2}}\right). \tag{2.7}$$

We shall also need an arithmetic lemma on the number of small values of four square roots of integers. The was proved, in the general case of k-th roots, by Robert–Sargos [14].

LEMMA 5. *Let* $k \ge 2$ *be a fixed integer and* $\delta > 0$ *be given. Then* $\mathcal{N}$ *, the number of integers* n_1, n_2, n_3, n_4 *such that* $N < n_1, n_2, n_3, n_4 \le 2N$ *and*

$$\left|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}\right| < \delta N^{1/k}$$

satisfies, for any given $\varepsilon > 0$,

$$\mathcal{N} \ll_\varepsilon N^\varepsilon\left(N^4\delta + N^2\right). \tag{2.8}$$

3. The Technical Preparation for the Proofs

It is clear that we may prove the bounds in (1.6)–(1.9) for the integrals over $[T, 2T]$, and then replace T by $T2^{-j}$ and sum the resulting bounds for $j = 1, \ldots$. As is customary in this field, we shall bound the occurrence of large values of $E(t)$ by considering the set of well-spaced points $\{t_r\}_{r=1}^R$ for which

$$E^*(t_r) \ge V(>0)\ \left(\begin{aligned} &T \le t_1 \le \ldots \le t_R \le 2T, \\ &t_{r+1} - t_r \ge G(r = 1, \ldots, R-1) \end{aligned}\right). \tag{3.1}$$

If $E^*(tr) \le -V$ the analysis is similar, and therefore will not be carried out in detail. Namely when $E^*(t_r) > 0$ we shall use the first inequality in (2.1), and in the case when when $E^*(t_r) < 0$ the second one. We use Lemma 3 and Lemma 4 (with $N = T$) to derive the explicit expression for $E^*(t_r + GL + u)$ ($r = 1, \ldots, R$) in the integral ($L = \log T$) on the right-hand side of (2.1). To truncate the sums that occur in these explicit expressions we use Taylor's formula and

$$\int_{-\infty}^{\infty} e^{Au - Bv^2}dv = \sqrt{\frac{\pi}{B}}\exp\left(\frac{A^2}{4B}\right) \quad (\Re e B > 0).$$

This procedure is similar to the one used in [4] in the proof of Lemma 7.2. In this manner it is seen that the terms with $n > TG^{-2}L$ (cf. (3.5)) make a negligible contribution (that is, one which is $\ll T^{-A}$ for any given $A > 0$). The integral can be also truncated at $\pm GL$ with a negligible error. We take then in each of the upper bounds provided by (2.1)

$$G = \frac{V}{2CT^\varepsilon} \tag{3.2}$$

with a suitable $C > 0$. Therefore we obtain

$$\begin{aligned} V &\ll T^\varepsilon V^{-1}\int_{-GL}^{GL}\sum(t_r + GL + u)e^{-(u/G)^2}du + \ldots \\ &\ll T^\varepsilon V^{-1}\int_{t_r}^{t_r+2GL}\left|\sum(t)\right|dt + \ldots \quad (r = 1, \ldots, R). \end{aligned} \tag{3.3}$$

We can suppose that V satisfies the bounds

$$T^{1/6+\varepsilon} \le V \le T^{1/3}. \tag{3.4}$$

Namely if $V \le T^{1/6+\varepsilon}$, then all our results follow trivially from the mean square formula (1.5), while the upper bound in (3.4) follows from the trivial bound $E^*(T) \ll T^{1/3}$ (see e.g., [4] and [5]). Furthermore by picking a suitable subsequence $\{t'_r\}$ of $\{t_r\}$ and calling it again $\{t_r\}$, with a slight abuse of notation, we may assume that all the intervals $[t_r,\, t_r + 2GL]$, for the t_r's satisfying (3.3), are non-overlapping. In (3.3) we have set

$$\sum(t) = \frac{1}{\pi\sqrt{2}} T^{\frac{1}{4}} \sum_{n\le M} (-1)^n d(n) n^{-\frac{3}{4}} \begin{Bmatrix} \cos\left(\sqrt{8\pi n t} - \frac{1}{4}\pi\right) \\ -\cos\left(f(t,n)\right) \end{Bmatrix} \quad (3.5)$$

with $M = T^{1+\varepsilon}V^{-2}$, where we have simplified the function $e(T,\, n)$ (see (2.6)) by Taylor's formula. Here and later +. . . will mean that in the relevant formula there are more expressions of a similar nature (structure) present, but they are of a lower order of magnitude than the expressions that are explicitly stated.

4. The Bound for the Fourth Moment

There are two natural ways to bound the quantity R appearing in (3.3): by the mean square or the mean fourth power of the function $\Sigma(t)$ in (3.5). In this section we want to prove the bound (1.9), which is equivalent (by (2.1)) to the bound

$$R \ll_\varepsilon T^{7/4+\varepsilon} V^{-5}, \quad (4.1)$$

provided that (3.1), (3.2) and (3.4) hold. In Section 5 we shall use this result to derive (1.6) and (1.7). From (3.3) and (3.5), on squaring and using the Cauchy-Schwarz inequality for integrals, we have

$$\begin{aligned} R &\ll_\varepsilon T^\varepsilon V^{-3} \int_T^{3T} \left(\sum(t)\right)^2 dt \\ &\ll_\varepsilon T^\varepsilon V^{-3} \int_T^{3T} \left(|S_1(t)|^2 + |S_2(t)|^2\right) dt, \end{aligned} \quad (4.2)$$

say, where

$$\begin{aligned} S_1(t) &:= t^{\frac{1}{4}} \sum_{n\le M} (-1)^n d(n) n^{-\frac{3}{4}} \exp\left(i\sqrt{8\pi n t} - i\pi/4\right), \\ S_2(t) &:= t^{\frac{1}{4}} \sum_{n\le M} (-1)^n d(n) n^{-\frac{3}{4}} \exp\left(if(t,n)\right). \end{aligned} \quad (4.3)$$

The mean square of both $S_1(t)$ and $S_2(t)$ is estimated similarly (see e.g., [[4], Chapter 15]). The former is technically a little simpler. As usual, we split $S_j(t)$ $(j = 1,2)$ into $O(\log T)$ subsums $S_j(t,K)$ where the range of summation for n is

$$K < n \le K' \le 2K, \quad 1 \ll K \ll M = T^{1+\varepsilon}V^{-2}.$$

We obtain

$$\begin{aligned} &\int_T^{3T} |S_1(t,K)|^2 dt \\ &\ll T^{1/2} \sum_{K<m,n\le K'} \frac{d(m)d(n)}{(mn)^{3/4}} \left| \int_T^{3T} \exp\left(i\sqrt{8\pi} \begin{pmatrix} \sqrt{m} \\ -\sqrt{n} \end{pmatrix} \sqrt{t} \right) dt \right| \\ &= O_\varepsilon\left(T^{\frac{3}{2}} K^{\varepsilon - \frac{1}{2}} \right) \\ &\quad + T^{\frac{1}{2}} \sum_{K<m\ne n\le K'} \frac{d(m)d(n)}{(mn)^{3/4}} \left| \int_T^{3T} \exp\left(i\sqrt{8\pi} \begin{pmatrix} \sqrt{m} \\ -\sqrt{n} \end{pmatrix} \sqrt{t} \right) dt \right| \\ &\ll_\varepsilon T^{3/2}K^{\varepsilon-1/2} + T \sum_{K<m<n\le K'} \frac{d(m)d(n)}{(mn)^{3/4}\left(\sqrt{n}-\sqrt{m}\right)} \\ &\ll_\varepsilon T^{3/2}K^{\varepsilon-1/2} + TK^{\varepsilon-1} \sum_{K<m<n\le K'} \frac{1}{n-m} \\ &\ll_\varepsilon T^{3/2}K^{\varepsilon-1/2} + TK^\varepsilon \ll_\varepsilon T^{3/2}K^{\varepsilon-1/2}, \end{aligned}$$

since $K < T$. Here we estimated trivially the "diagonal" terms $m = n$ (using $d(n) \ll_\varepsilon n^\varepsilon$) and used the first derivative test (i.e., Lemma 2.1 of [4]) for the terms $m \ne n$. The same bound holds for the mean square of $|S_2(t,K)|$. If RK is the number of t_r's in (3.3) pertaining to $S_j(t,K)$ $(j = 1,2)$, then we have shown that

$$R_K \ll_\varepsilon T^{3/2+\varepsilon} V^{-3} K^{-1/2} \quad \left(1 \ll K \ll M = T^{1+\varepsilon}V^{-2}\right). \quad (4.4)$$

There is a possibility for another type of large values estimate involving the technique developed by the author in [3] and [[4], Chapter 13] to estimate the occurrence of large values of $\Delta(x)$. The present problem is similar, with $f(t,\, n)$ standing in place of $\sqrt{8\pi n t}$ in S_2, which is not problematic in the present situation. We can use (13.65) of [4] with $k = 2, K \equiv N$ and R_0 the number of points t_r counted by R_K which lie in interval of length $T_0 (V \ll T_0 \le T)$. Then we obtain

$$R_0 V^2 \ll_\varepsilon T^{1+\varepsilon} V^{-1} + R_0 T_0^\kappa T^{(1-\kappa)/2+\varepsilon} K^{(2\lambda-1-\kappa)/2}.$$

Hence with

$$T_0 = cV^{2/\kappa} T^{(\kappa-1)/(2\kappa)-\varepsilon} K^{(1+\kappa-2\lambda)/(2\kappa)}$$

and suitable $c > 0$, we obtain

$$\begin{aligned} R_K &\ll R_0\left(1 + T/T_0\right) \\ &\ll_\varepsilon T^{1+\varepsilon}V^{-3} \\ &\quad + T^{(1+3\kappa)/(2\kappa)+\varepsilon} V^{-(3\kappa+2)/\kappa} K^{(2\lambda-1-\kappa)/(2\kappa)}, \end{aligned} \quad (4.5)$$

where $(\kappa,\, \lambda)$ is an one-dimensional exponent pair. For definitions and properties of exponent pairs see Graham-Kolesnik [2] and [[4], Chapter 2]. The condition $T_0 \gg V$ becomes

$$V \gg T^{(1-\kappa)/(4-2\kappa)} K^{(2\lambda-1-\kappa)/(4-2\kappa)}. \quad (4.6)$$

If we use the standard exponent pair $(\kappa,\, \lambda) = (1/2, 1/2)$, then from (4.5) and (4.6) we obtain

$$R_K \ll_\varepsilon T^{1+\varepsilon}V^{-3} + T^{5/2+\varepsilon}V^{-7}K^{-1/2} \quad \left(V \ge T^{1/6}K^{-1/6}\right),$$

valid also for the whole range $1 \ll K \ll M = T^{1+\varepsilon}V^{-2}$. The condition on V may be dropped, since we certainly assume $V \ge T^{1/6}$ to hold. Also note that $T^{1+\varepsilon}V^{-3} \le T^{5/2+\varepsilon}V^{-7}K^{-1/2}$ since $V < T^{1/3}$, hence we obtain

$$R_K \ll_\varepsilon T^{5/2+\varepsilon}V^{-7}K^{-1/2}. \tag{4.7}$$

Multiplying (4.4) and (4.7) and then taking square roots we infer that

$$R_K \ll_\varepsilon T^{2+\varepsilon}V^{-5}K^{-1/2}. \tag{4.8}$$

If $K \ge T^{1/2-\varepsilon}$ we see that (4.8) gives the desired bound

$$R_K \ll_\varepsilon T^{7/4+\varepsilon}V^{-5}. \tag{4.9}$$

Suppose now that $K \ge T^{1/2-\varepsilon}$ fails, namely that

$$1 \ll K < T^{1/2-\varepsilon}. \tag{4.10}$$

Analogously to (4.2) we shall obtain, by raising (3.3) to the fourth power,

$$R \ll_\varepsilon T^\varepsilon V^{-5}\int_T^{3T}\left(|S_3(t)|^4 + |S_4(t)|^4 + |S_5(t)|^4\right)dt, \tag{4.11}$$

where we have set

$$\begin{aligned} S_3(t) &:= t^{1/4}\sum_{n \le T^{1/3-\varepsilon_1}} (-1)^n d(n) n^{-\frac{3}{4}} \times \begin{Bmatrix} \exp\left(i\sqrt{8\pi n t} - \pi i/4\right) \\ -\exp\left(if(t,n)\right) \end{Bmatrix}, \\ S_4(t) &:= t^{1/4}\sum_{T^{1/3-\varepsilon_1} < n < T^{1/2-\varepsilon}} (-1)^n d(n) n^{-\frac{3}{4}} \times \left\{\exp\begin{pmatrix} i\sqrt{8\pi n t} \\ -\pi i/4 \end{pmatrix} - \frac{1}{4}\pi i\right\} \\ S_5(t) &:= t^{1/4}\sum_{T^{1/3-\varepsilon_1} < n < T^{1/2-\varepsilon}} (-1)^n d(n) n^{-\frac{3}{4}} \times \exp\left(if(t,n)\right). \end{aligned} \tag{4.12}$$

In other words, we consider separately the cases $n \le T^{1/3-\varepsilon_1}$ and $T^{1/3-\varepsilon_1} < n < T^{1/2-\varepsilon}$. It is in the former case that we can take advantage of the "closeness" of the functions $E(t)$ and $\Delta^*(t/(2\pi))$. In the other case we treat the sums coming from $E(t)$ and $\Delta^*(t/(2\pi))$ separately, or even "trivially". In $S_3(t)$ we use (2.5) to replace exp($if(t, n)$) by

$$\left(1 + cn^{3/2}t^{-1/2}\right)\exp\left(i\sqrt{8\pi n t} - \frac{1}{4}\pi i\right)$$

plus terms of a lower order of magnitude (in view of (4.10)). Hence we obtain

$$\int_T^{3T} |S_3(t)|^4 dt \ll \frac{\log T}{T}\max_{K \le T^{1/3-\varepsilon_1}} \sum_{K < m,n,k,\ell \le K'} d(m)d(n)d(k)d(\ell)(mnk\ell)^{3/4} J + \ldots \tag{4.13}$$

with

$$J = J(T;m,n,k,\ell) := \int_{T/4}^{4T}\varphi(t)\exp\left(i\sqrt{8\pi t}\left(\sqrt{m}+\sqrt{n}-\sqrt{k}-\sqrt{\ell}\right)\right)dt.$$

Here $\varphi(t)$ (≥ 0) is a smooth function supported in $[T/4, 4T]$ that equals unity in $[T, 3T]$, which implies that $\varphi^{(r)}(t) \ll_r T^{-r}$ for $r = 0, 1, 2, \ldots$. We set

$$\Delta = \Delta(m,n,k,\ell) := \sqrt{8\pi}\left(\sqrt{m}+\sqrt{n}-\sqrt{k}-\sqrt{\ell}\right) \quad \left(K < m,n,k,\ell \le K' \le 2K\right) \tag{4.14}$$

and note that integration by parts gives

$$J = \frac{2i}{\Delta}\int_{T/4}^{4T}\left(\frac{\varphi(t)}{2\sqrt{t}} + \sqrt{t}\varphi'(t)\right)\exp\left(i\Delta\sqrt{t}\right)dt \quad (\Delta \ne 0).$$

This is the same type of exponential integral as the original one, only its integrandis decreased by a factor which is $\ll 1/(\Delta\sqrt{T})$. Thus, after sufficiently manyintegrations by parts, it is seen that the contribution of J is negligible if

$$|\Delta| > T^{\varepsilon-1/2}.$$

If $|\Delta| \le T^{\varepsilon-1/2}$, then by Lemma 5 (with $k = 2$, $\delta = T^{\varepsilon-1/2}K^{-1/2}$) and trivial estimation we obtain

$$\begin{aligned} &\int_T^{3T}|S_3(t)|^4 dt \\ &\ll \frac{\log T}{T}\max_{K \le T^{1/3-\varepsilon_1}} K^3 T^{1+\varepsilon}\left(K^4K^{-1/2}T^{-1/2} + K^2\right) \\ &\ll_\varepsilon T^\varepsilon \max_{K \le T^{1/3-\varepsilon_1}}\left(K^{13/2}T^{-1/2} + K^5\right) \ll_\varepsilon T^{5/3+\varepsilon}. \end{aligned} \tag{4.15}$$

The contribution to (4.1) is then $\ll_\varepsilon T^{5/3+\varepsilon}V^{-5}$, which is more than sufficient.

The integrals in (4.11) with S_4 and S_5 are estimated analogously, only the latter is more difficult and thus will be considered in detail. In $S_5(t)$ we replace exp($if(t, n)$) by $\exp\left(i\sqrt{8\pi n t} - i\pi/4 + ic_3 n^{3/2}t^{-1/2}\right)$ plus terms of a lower order of magnitude. We obtain

$$\int_T^{3T}|S_5(t)|^4 dt \ll T\log T \max_{T^{1/3-\varepsilon_1} < K < T^{1/2-\varepsilon}}\int_T^{3T}|S_5(t;K)|^4 dt + \ldots,$$

$$S_5(t;K) := \sum_{K < n \le K' \le 2K} (-1)^n d(n) n^{-3/4} \times \exp\begin{pmatrix} i\sqrt{8\pi n t} - \frac{1}{4}\pi i \\ + ic_3 n^{3/2}t^{-1/2} \end{pmatrix}.$$

Thus with Δ given by (4.14) and

$$E = E(m,n,k,\ell) := c_3\left(m\sqrt{m} + n\sqrt{n} - k\sqrt{k} - \ell\sqrt{\ell}\right)$$

we see that the integral of $|S_5(t;K)|^4$ is equal to

$$\sum_{K<m,n,k,\ell\le K'\le 2K} \frac{(-1)^{m+n+k+\ell}\, d(m)d(n)d(k)d(\ell)}{(mnk\ell)^{3/4}} \int_T^{3T} \exp\left(i\Delta\sqrt{t}+iEt^{-1/2}\right)dt. \tag{4.16}$$

If $|\Delta|\le T^{\varepsilon-1/2}$, we apply again Lemma 5 as in the case of $S_3(t)$. We obtain that in this case

$$\begin{aligned} R_K &\ll_\varepsilon T^2K^{\varepsilon-3}\left(K^4(TK)^{-1/2}+K^2\right)V^{-5} \\ &\ll_\varepsilon \left(T^{3/2+\varepsilon}K^{1/2}+T^{2+\varepsilon}K^{-1}\right)V^{-5} \\ &\ll_\varepsilon T^{7/4+\varepsilon}V^{-5}, \end{aligned} \tag{4.17}$$

since $T^{1/3-\varepsilon_1} \ll K < T^{1/2-\varepsilon}$. Again, this makes a contribution of $T^{7/4+\varepsilon}V^{-5}$ to R in (4.1).

If $|\Delta|\ge T^{\varepsilon-1/2}$, suppose that $\Delta>0$ (the case $\Delta<0$ is analogous). We may also suppose that $E>0$, for if $E\le 0$, then all the derivatives of $F(t):=\Delta\sqrt{t}+E^{t-1/2}$ are of the same sign. Thus by repeated integration by parts the integral in (4.16) makes again a negligible contribution in view of $|\Delta|\ge T^{\varepsilon-1/2}$.

If we have $\Delta> C_1E/T$ with a sufficiently large $C_1>0$, then $F'(t)\gg \Delta/\sqrt{T}$ in[T, $3T$]. Hence by the first derivative test and Lemma 5 (with $k=2$, $\delta=\Delta K^{-1/2}$) we have, supposing that $\Delta \asymp 2^jT^{\varepsilon-1/2}$,

$$\begin{aligned} R_K &\ll_\varepsilon \sum_j \begin{matrix} T^{1+\varepsilon}K^{-3}T^{1/2}2^{-j}T^{1/2-\varepsilon} \\ \times\left(K^4 2^jT^{\varepsilon-1/2}+K^2\right)V^{-5} \end{matrix} \\ &\ll_\varepsilon \left(T^{3/2+\varepsilon}K^{1/2}+T^{2+\varepsilon}K^{-1}\right)V^{-5} \\ &\ll_\varepsilon T^{7/4+\varepsilon}V^{-5}, \end{aligned} \tag{4.18}$$

since $T^{1/3-\varepsilon_1} \ll K < T^{1/2-\varepsilon}$ and there are $<<\log T$ possible values of j. Here we used the notation $f(x)\asymp g(x)$ which means that $f(x)<<g(x)<<f(x)$.

If $\Delta\ge T^{\varepsilon-1/2}$ and $\Delta< C_2E/T$ with a sufficiently small $C_2>0$, then $F'(t)>>ET^{-3/2}$. Hence

$$\left|F'(t)\right|^{-1} \ll T^{3/2}E^{-1} \ll T^{1/2}\Delta^{-1}$$

and by the preceding argument we have again

$$R_K \ll_\varepsilon T^{7/4+\varepsilon}V^{-5}$$

Finally if $\Delta\ge \mathrm{T}^{\varepsilon-1/2}$ and $\Delta\asymp \mathrm{E}/\mathrm{T}$, then there may exist a saddle point $\mathrm{t}_0=\mathrm{E}/\Delta$ (root of $\mathrm{F}'(\mathrm{t}_0)=0$) in [T, 3T] if $\Delta\mathrm{T}\asymp E$. Hence by the saddle-point method (see [[4], Chapter 2] or by the use the second derivative test, making first the change of variable $\sqrt{t}=u$ we obtain

$$R_K \ll TV^{-5}\left|\sum{}^{*} \frac{(-1)^{m+n+k+\ell}\,\dfrac{d(m)d(n)d(k)d(\ell)}{(mnk\ell)^{3/4}}}{\left(F''(t_0)\right)^{-1/2}e^{2i\sqrt{\Delta E}}}\right| \tag{4.19}$$

plus the contribution of the error terms which is of a lower order of magnitude. Here Σ^* denotes summation with the conditions

$$\begin{aligned} &K<m,n,k,\ell\le K'\le 2K, \Delta>0, \\ &E>0, \Delta\ge T^{\varepsilon-1/2}, \Delta\asymp\frac{E}{T} \end{aligned} \tag{4.20}$$

However, the estimation of the sum in (4.19) with the summation condition (4.20) is complicated, and we shall use another approach which avoids the use of higher dimensional exponential sums. Suppose now that

$$T^{\varepsilon-1/2}\le\Delta\le\Delta_0$$

where Δ_0, which will be determined later, does not depend on $(m,\ n,\ k,\ \ell)$. Further suppose that

$$2^{-j}\Delta_0<\Delta\le 2^{1-j}\Delta_0\left(1\le j\le J\left(\asymp\log\left(\Delta_0T^{1/2-\varepsilon}\right)\right)\right).$$

Since $F''(t_0)=\frac{1}{2}\Delta^{5/2}E^{-3/2}\asymp\Delta T^{-3/2}$, it follows from (4.19) by trivial estimation and Lemma 5 that

$$\begin{aligned} R_K &\ll_\varepsilon \sum_{j\le J} \begin{matrix} T^{1+\varepsilon}K^{-3}T^{3/4}2^{j/2}\Delta_0^{-1/2} \\ \left(K^4 2^{-1}\Delta_0K^{-1/2}+K^2\right)V^{-5} \end{matrix} \\ &\ll_\varepsilon T^{7/4+\varepsilon}V^{-5}\left((K\Delta_0)^{1/2}+2^{j/2}\Delta_0^{-1/2}K^{-1}\right) \\ &\ll_\varepsilon T^{7/4+\varepsilon}V^{-5} \end{aligned} \tag{4.21}$$

if $\Delta_0=1/K$, since $2^{-J}\Delta_0\asymp T^{\varepsilon-1/2}, K>>T^{1/3-\varepsilon 1}$. There remains the case when $\Delta\ge 1/\mathrm{K}$, but we shall show that this is impossible, which will complete the proof of (4.1)

To this end note that by the elementary identity

$$x^3+y^3+z^3=(x+y+z)^3-3(x+y)(x+z)(y+z)$$

With $x=\sqrt{m}, y=\sqrt{n}, z=-\sqrt{k}$ we obtain

$$\begin{aligned} E/c_3 &= \left(\sqrt{m}+\sqrt{n}-\sqrt{k}\right)^3-\left(\sqrt{\ell}\right)^3 \\ &-3\left(\sqrt{m}+\sqrt{n}\right)\left(\sqrt{m}-\sqrt{k}\right)\left(\sqrt{n}-\sqrt{k}\right) \\ &= O(\Delta K)-3\left(\sqrt{m}+\sqrt{n}\right)\left(\sqrt{m}-\sqrt{k}\right)\left(\sqrt{n}-\sqrt{k}\right) \end{aligned}$$

In the critical case when $\Delta\asymp E/T$ holds this gives

$$\Delta\asymp\frac{\sqrt{K}\left(\sqrt{m}-\sqrt{k}\right)\left(\sqrt{k}-\sqrt{n}\right)}{T} \tag{4.22}$$

Thus if $\Delta\ge 1/K$, (4.22) gives

$$\begin{aligned} &\left(\sqrt{m}-\sqrt{k}\right)\left(\sqrt{k}-\sqrt{n}\right)\gg T\Delta K^{-1/2} \\ &\gg TK^{-3/2}\gg T^{1/2-(3\varepsilon_1)/2} \end{aligned}$$

since $K>>T^{1/3-\varepsilon 1}$ holds in S_4 and S_5. But trivially

$$\left(\sqrt{m}-\sqrt{k}\right)\left(\sqrt{k}-\sqrt{n}\right)\ll K,$$

which yields $K>>T^{1/2-(3\varepsilon 1)/2}$, and this is a contradiction by (4.10), if $\varepsilon_1=\varepsilon/2$, say.

5. The Fifth and the Third Moment

Having at our disposal (4.1) we shall use it to obtain (1.7) and then (1.6). The bound (1.7) follows from the large values estimate

$$R \ll_\varepsilon T^{2+\varepsilon}V^{-6} \tag{5.1}$$

under the spacing conditions (3.1), (3.2) and (3.4). For $V \le T^{1/4+\varepsilon}$ we have, from the fourth moment bound (4.1),

$$R \ll_\varepsilon T^{7/4+\varepsilon}V^{-5} = T^{7/4+\varepsilon}V\cdot V^{-6} \ll_\varepsilon T^{2+2\varepsilon}V^{-6} \tag{5.2}$$

In the case when $V > T^{1/4+\varepsilon}$ we have (see the discussion after (4.3))

$$\mathrm{K} \ll_\varepsilon \mathrm{T}^{1+\varepsilon}\mathrm{V}^{-2} \ll_\varepsilon \mathrm{T}^{1/2-\varepsilon} \tag{5.3}$$

In this case we follow the discussion that was made in (4.11)–(4.18). The contribution of $S_3(t)$ will be, like in (4.13), $<<_\varepsilon T^{5/3+\varepsilon}V^{-5} \ll_\varepsilon T^{2+\varepsilon}V^{-6}$. The contribution of $S_4(t)$ and $S_5(t)$ will be (by the first bound in (4.17) and (4.18)),

$$R \ll_\varepsilon T^{3/2+\varepsilon}K^{1/2}V^{-5} + T^{2+\varepsilon}K^{-1}V^{-6} \ll_\varepsilon T^{2+\varepsilon}V^{-6} \tag{5.4}$$

since $\mathrm{K} <<_\varepsilon T^{1+\varepsilon}V^{-2}$ by (5.3) and $K > T^{1/3-\varepsilon 1}$. We remark that (5.4) also incorporates the bound $\mathrm{O}(\mathrm{K}^2)$ corresponding to the cases m = n and k = ℓ in Section 4. Hence we have proved (5.1).

Likewise the bound (1.6) follows from

$$R \ll_\varepsilon T^{3/2+\varepsilon}V^{-4} \tag{5.5}$$

We have, by (5.1),

$$\mathrm{R} \ll \varepsilon \mathrm{T}^{2+\varepsilon}\mathrm{V}^{-6} \ll T^{3/2+3\varepsilon}V^{-4}$$

for $\mathrm{V} \ge \mathrm{T}^{1/4-\varepsilon}$. If $\mathrm{V} < \mathrm{T}^{1/4-\varepsilon}$, then by (4.4)

$$\begin{aligned}\mathrm{R}_\mathrm{K} &\ll_\varepsilon \mathrm{T}^{3/2+\varepsilon}\mathrm{V}^{-3}\mathrm{K}^{-1/2}\\ &= T^{3/2+\varepsilon}V^{-4}\cdot \mathrm{V}\,\mathrm{K}^{-1/2} \ll_\varepsilon \mathrm{T}^{3/2+\varepsilon}\mathrm{V}^{-4}\end{aligned}$$

for $V \le K^{1/2}$. If $K > T^{1/2-\varepsilon}$, then we have $V \le T^{1/4-\varepsilon/2} \le K^{1/2}$, hence (5.5) holds for R_K.

It remains to deal with the case

$$\mathrm{T}^{1/6} \le \mathrm{V} \le \mathrm{T}^{1/4-\varepsilon},\ \mathrm{K} \le \mathrm{T}^{1/2-\varepsilon}. \tag{5.6}$$

Namely for $V \le T^{1/6}$ the bound (5.5) easily follows from (1.5). As in the case of the proof of (5.1) we note that the contribution of $S_3(t)$ is

$$\ll \varepsilon \mathrm{T}^{5/3+\varepsilon}\mathrm{V}^{-5} \le T^{3/2+\varepsilon}V^{-4}$$

in view of $\mathrm{T}^{1/6} \le \mathrm{V}$. But in the remaining case of $\mathrm{S}_4(\mathrm{t})$ and $\mathrm{S}_5(\mathrm{t})$ we have the bound in (5.4). Thus if $\mathrm{T}^{2+\varepsilon}\mathrm{K}^{-1}\mathrm{V}^{-5}$ dominates in size then, since $\mathrm{K} > \mathrm{T}^{1/3-\varepsilon 1}$,

$$\mathrm{R}_\mathrm{K} \ll_\varepsilon \mathrm{T}^{2+\varepsilon}\mathrm{K}^{-1}\mathrm{V}^{-5} \ll_\varepsilon \mathrm{T}^{5/3+\varepsilon}\mathrm{V}^{-5} \ll_\varepsilon \mathrm{T}^{3/2+\varepsilon}\mathrm{V}^{-4}.$$

If $\mathrm{T}^{3/2+\varepsilon}\mathrm{K}^{1/2}\mathrm{V}^{-5}$ dominates in (5.4), then by (4.4) and (5.)

$$\mathrm{R}_\mathrm{K} = \mathrm{R}_K^{1/2}\mathrm{R}_K^{1/2} \ll \varepsilon \begin{Bmatrix}\mathrm{T}^{3/2+\varepsilon}\mathrm{K}^{-1/2}\mathrm{V}^{-3}\\ \cdot\mathrm{T}^{3/2+\varepsilon}\mathrm{K}^{1/2}\mathrm{V}^{-5}\end{Bmatrix}^{1/2} = \mathrm{T}^{3/2+\varepsilon}\mathrm{V}^{-4},$$

and the proof of (5.5) is finished. Thus the proof of the bounds (1.6)–(1.9) is complete.

6. Proof of Theorem 1

Suppose, contrary to the assertion of Theorem 1, that the function $\mathrm{E}^*(\mathrm{t})$ does not change sign in some interval [T, T+H] for some $\mathrm{T} > \mathrm{T}_0(\varepsilon)$ and $\mathrm{H} = \mathrm{T}^{2/3+\varepsilon}$.

The cases $\mathrm{E}^*(\mathrm{t}) > 0$ and $\mathrm{E}^*(\mathrm{t}) < 0$ are analogous, so only the former will be considered. On one hand we have the lower bound (1.10). On the other hand, since $\mathrm{E}^*(\mathrm{t}) > 0$ in [T, T+H],

$$\begin{aligned}\int_\mathrm{T}^{\mathrm{T+H}} (\mathrm{E}^*(t))^2\,\mathrm{d}t &\le \sup_{T\le t\le T+H}\int_\mathrm{T}^{\mathrm{T+H}} \mathrm{E}^*(t)\mathrm{d}t\\ &= \sup_{T\le t\le T+H} \mathrm{E}*(t)(R(T+H)-R(T) + \frac{3\pi}{4}).\end{aligned} \tag{6.1}$$

The function $R(T)$ is defined (for some results on $R(T)$ see Part III of [6,9] and [10]) by the relation

$$\int_0^\mathrm{T} \mathrm{E}^*(t)\mathrm{d}t = \frac{3\pi}{4}T + R(T)$$

In [10] it was proved that, if $R(T) \ll \varepsilon T\alpha + \varepsilon$, then $E^*(\mathrm{T}) <<_\varepsilon T^{\alpha/2+\varepsilon}$ holds. It was conjectured that $\alpha = 1/2$ is permissible, but unconditionally it was proved that $\alpha 6\ 593/912 = 0.6502129 < 2/3$ holds. Hence (6.1) yields

$$HT^{1/3}\log^3 T \ll \sup_{T\le t\le T+H} \mathrm{E}^*(t)(\frac{3\pi}{4}\mathrm{H} + \mathrm{T}^{\alpha+\varepsilon}).$$

It follows, since $\alpha < 2/3$ and $E^*(T) << \varepsilon T^{\alpha/2+\varepsilon}$,

$$\mathrm{HT}^{1/3}\log^3\mathrm{T} \ll \mathrm{T}^{3\alpha/2+\varepsilon} \ll \mathrm{T},$$

which contradicts our choice $H = T^{2/3+\varepsilon}$. Therefore Theorem 1 is proved.

7. Proof of Theorem 2

We shall use a technique similar to the one employed by K.-M. Tsang in his work on the $\Omega\pm-$results in the sphere problem (see [[15], Lemma 1]). For any function f define

$$f_+ = \frac{1}{2}(|\mathrm{f}|+\mathrm{f}),\ f_- = \frac{1}{2}(|\mathrm{f}|-\mathrm{f}),$$

so that f_+ is the positive, and f_- is the negative part of f. Then $\mathrm{f}_+\mathrm{f}_- \equiv 0$, and therefore the binomial theorem yields

$$f^k = f_+^k + (-1)^\mathrm{k}\mathrm{f}_-^k\ (\mathrm{k} \in \mathbb{N}). \tag{7.1}$$

From (1.8) and (7.1) it follows that

$$\int_T^{T+H} E_+^*(t)^3\,dt - \int_T^{T+H} E_+^*(t)^3\,dt \ll_\varepsilon T^{3/2-\eta+\varepsilon}\ (1 \le H \le T). \tag{7.2}$$

On the other hand, from the mean square formula (1.5) we obtain, for $T^{5/6+\varepsilon} \le H \le T$, $T \to \infty$ (this is why we had the restriction $0 < \eta < 1/6$ in the formulation of Theorem 1),

$$HT^{1/3}\log^3 T \sim \int_T^{T+H} E^*(t)^2\,dt$$
$$= \int_T^{T+H} E_+^*(t)^2\,dt + \int_T^{T+H} E_-^*(t)^2\,dt$$
$$\le H^{1/3}\left(\int_T^{T+H} E_+^*(t)^3\,dt\right) + H^{1/3}\left(\int_T^{T+H} E_-^*(t)^3\,dt\right)$$

by Hölder's inequality for integrals. By raising both sides to the power 3/2 one obtains

$$\int_T^{T+H} E_+^*(t)^3\,dt + \int_T^{T+H} E_-^*(t)^3\,dt \gg HT^{1/2}\log^{9/2} T(T^{5/6+\varepsilon} \le H \le T). \tag{7.3}$$

In deriving (7.3), instead of using (1.5), we could have used the lower bound in (1.10).

Take now $H = T^{1-\eta+2\varepsilon}$, so that for any constant $C > 0$ and $T \ge T_1(C, \varepsilon)$ we have

$$HT^{1/2}\log^{9/2}T > CT^{3/2-\eta+\varepsilon}.$$

Then (7.2) and (7.3) imply that

$$HT^{1/2}\log^{9/2}T \ll \int_T^{T+H} E_\pm^*(t)^3\,dt \le H \sup_{T \le t \le T+H} E_\pm^*(t)^3,$$

and the assertion of Theorem 2 follows. It would be interesting to ascertain what is the lower bound for the integral in (1.8).

Acknowledgement

The author wishes to thank Prof. Kai-Man Tsang for valuable remarks.

References

[1] F.V. Atkinson, The mean value of the Riemann zeta-function, Acta Math. 81(1949), 353-376.

[2] S.W. Graham and G. Kolesnik, Van der Corput's method of exponential sums, London Mathematical Society Lecture Note Series 126, Cambridge University Press, Cambridge, 1991. vi+120 pp.

[3] A. Ivić, Large values of the error term in the divisor problem, Inventiones Math. 71(1983), 513-520.

[4] A. Ivić, The Riemann zeta-function, John Wiley & Sons, New York, 1985 (2nd ed. Dover, Mineola, New York, 2003).

[5] A. Ivić, The mean values of the Riemann zeta-function, LNs 82, Tata Inst. of Fundamental Research, Bombay (distr. by Springer Verlag, Berlin etc.), 1991.

[6] A. Ivić, On the Riemann zeta-function and the divisor problem, Central European J. Math. (2)(4) (2004), 1-15; II, ibid. (3)(2) (2005), 203-214; III, Annales Univ. Sci. Budapest, Sect. Comp. 29(2008), 3-23.; IV, Uniform Distribution Theory 1(2006), 125-135.

[7] A. Ivić, On the mean square of the zeta-function and the divisor problem, Annales Acad. Scien. Fennicae Mathematica 23(2007), 1-9.

[8] A. Ivić, Some remarks on the moments of |(ζ(1/2+ *it*)| in short intervals, Acta Math. Hung. 119(2008), 15-24.

[9] A. Ivić, On some mean square estimates for the zeta-function in short intervals, Annales Univ. Sci. Budapest., Sect. Comp. 40(2013), 321-335.

[10] A. Ivić, On some mean value results for the zeta-function in short intervals, Acta Arith. 162.2(2014), 141-158.

[11] M. Jutila, Riemann's zeta-function and the divisor problem, Arkiv Mat. 21(1983), 75-96 and II, ibid. 31(1993), 61-70.

[12] M. Jutila, On a formula of Atkinson, in "Coll. Math. Sci. J´anos Bolyai 34, Topics in classical Number Theory, Budapest 1981", North-Holland, Amsterdam, 1984, pp. 807-823.

[13] T. Meurman, A generalization of Atkinson's formula to *L*-functions, Acta Arith. 47(1986), 351-370.

[14] O. Robert and P. Sargos, Three-dimensional exponential sums with monomials, J. reine angew. Math. 591(2006), 1-20.

[15] K.-M. Tsang, Counting lattice points in the sphere, Bull. London Math. Soc. 32(2000), 679-688.