Hermite-Hadamard and Simpson Type Inequalities for Differentiable Quasi-Geometrically Convex Functions

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Abstract In this paper, the authors define a new identity for differentiable functions. By using of this identity, authors obtain new estimates on generalization of Hadamard and Simpson type inequalities for quasi-geometrically convex functions.

Keywords: quasi-geometrically convex functions, hermite–hadamard type inequalities, simpson type inequality


1. Introduction

Let real function \( f \) be defined on some nonempty interval \( I \) of real line \( \mathbb{R} \). The function \( f \) is said to be convex on \( I \) if inequality

\[ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \]

holds for all \( x, y \in I \) and \( t \in [0,1] \).

Following inequalities are well known in the literature as Hermite-Hadamard inequality and Simpson inequality respectively:

**Theorem 1.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following double inequality holds

\[ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \]

**Theorem 2.** Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping on \( (a, b) \) and

\[ \|f^{(4)}\|_{\infty} = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty. \]

Then the following inequality holds:

\[ \left| \frac{1}{3} \int_a^b \left( f(a) + f(b) \right) dx + 2 \int \left( \frac{a+b}{2} \right) dx - \frac{1}{b-a} \int_b^a f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4. \]

In recent years, many athors have studied errors estimations for Hermite-Hadamard, Ostrowski and Simpson inequalities; for refinements, counterparts, generalization see [2,9,10].

The following definitions are well known in the literature.

**Definition 1 ([7,8]).** A function \( f : I \subseteq (0, \infty) \to \mathbb{R} \) is said to be GA-convex (geometric-arithmetically convex) if

\[ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \]

for all \( x, y \in I \) and \( t \in [0,1] \).

**Definition 2 ([7,8]).** A function \( f : I \subseteq (0, \infty) \to (0, \infty) \) is said to be GG-convex (called in [13] geometrically convex function) if

\[ f(tx^{1-t}) \leq f(x)^{1-t} f(y)^{1-t} \]

for all \( x, y \in I \) and \( t \in [0,1] \).

In [3], İşcan gave definition of quasi-geometrically convexity as follows:

**Definition 3.** A function \( f : I \subseteq (0, \infty) \to \mathbb{R} \) is said to be quasi-geometrically convex on \( I \) if

\[ f\left(\frac{x'}{y'}\right) \leq \sup \{ f(x), f(y) \}, \]

for any \( x, y \in I \) and \( t \in [0,1] \).

Clearly, any GA-convex and geometrically convex functions are quasi-geometrically convex functions. Furthermore, there exist quasi-geometrically convex functions which are neither GA-convex nor GG-convex [3].

For some recent results concerning Hermite-Hadamard type inequalities for GA-convex, GG-convex, quasi-geometrically convex functions we refer interested reader to [1,3,4,5,6,11,12,14].

The goal of this article is to establish some new general integral inequalities of Hermite-Hadamard and Simpson type for quasi-geometrically convex functions by using a new integral identity.
2. Main Results

Let $f: I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^*$, the interior of $I$, throughout this section we will take
\[
I_f(\lambda, \mu, a, b) = (\lambda - \mu) f(\sqrt{ab}) + \mu f(a)
\]
\[
+ (1 - \lambda) f(b) - \frac{1}{\ln(b/a)} \int_a^b f(u) \frac{du}{u}
\]
where $a, b \in I$ with $a < b$ and $\lambda, \mu \in \mathbb{R}$.

In order to prove our main results we need the following identity.

**Lemma 1.** Let $f: I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^*$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. Then for all $\lambda, \mu \in \mathbb{R}$ we have:
\[
I_f(\lambda, \mu, a, b) = \ln(b/a) \left\{ \frac{1}{2} \int_0^{1/2} (t - \mu)a^{1-t}b^{t} f'(a^{1-t}b^{t}) dt + \int_{1/2}^1 (1 - \lambda)\frac{f'(a^{1-t}b^{t}) dt}{2} \right\}.
\]

**Proof.** By integration by parts and changing the variable, we can state
\[
\ln(b/a) \left\{ \frac{1}{2} \int_0^{1/2} (t - \mu)\frac{df}{a^{1-t}b^{t}} + \mu f(a) - \frac{1}{\ln(b/a)} \int_a^b f(u) \frac{du}{u} \right\}
\]
and similarly we get
\[
\ln(b/a) \left\{ \frac{1}{2} \int_0^{1/2} (t - \lambda)\frac{df}{a^{1-t}b^{t}} - \mu f(b) - \frac{1}{\ln(b/a)} \int_{ab}^b f(u) \frac{du}{u} \right\}
\]

Adding the resulting identities we obtain the desired result.

**Theorem 3** Let $f: I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^*$ such that $f' \in L[a, b]$, where $a, b \in I^*$ with $a < b$. If $|f'|^q$ is quasi-geometrically convex on $[a,b]$ for some fixed $q \geq 1$ and $0 \leq \mu \leq 1/2 \leq \lambda \leq 1$, then the following inequality holds
\[
I_f(\lambda, \mu, a, b) \leq \ln(b/a) \left\{ \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right\}^{1/q}
\]
\[
\left\{ C_1^{-1/q}(\mu) C_1^q(\mu, q, a, b) + C_2^{-1/q}(\lambda) C_4^q(\lambda, q, a, b) \right\}
\]
where
\[
C_1(\mu) = \mu^2 - \frac{\mu}{2} + \frac{1}{8},
\]
\[
C_2(\lambda) = \lambda^2 - \frac{3\lambda}{2} + \frac{5}{8},
\]
\[
C_3(\mu, q, a, b) = \frac{1}{2q \ln(b/a)} \left\{ (1 - 2\mu)(ab)^{q/2} + 4\mu a^{q/2}b^{q/2} \right\} L(a^{q/2}, b^{q/2}),
\]
\[
-\lambda^2 a^{q/2}b^{q/2} - 2\mu a^{q/2},
\]
\[
\frac{1}{2q \ln(b/a)} \left\{ 2(1 - \lambda) b^{q/2} - (2\lambda - 1)(ab)^{q/2} - 2L(a^{q/2}, b^{q/2}) \right\},
\]
and $L(a, b)$ is logarithmic mean defined by $L(a, b) = (b - a)/(\ln b - \ln a)$.

**Proof.** Since $|f'|^q$ is quasi-geometrically convex on $[a,b]$, for all $t \in [0,1]$
\[
|f'(a^{1-t}b^{t})|^q \leq \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\},
\]
Hence, using Lemma 1 and power mean inequality we get
\[
I_f(\lambda, \mu, a, b) \leq \ln(b/a) \left\{ \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right\}^{1/q}
\]
\[
\times \left\{ \frac{1}{2} \int_0^{1/2} \left| t - \mu \right| dt \right\}^{1-1/q} \left\{ \frac{1}{2} \int_0^{1/2} \left| t - \mu \right|^q dt \right\}^{1/q}
\]
\[
+ \left\{ \frac{1}{2} \int_{1/2}^1 \left| t - \mu \right| dt \right\}^{1-1/q} \left\{ \frac{1}{2} \int_{1/2}^1 \left| t - \mu \right|^q dt \right\}^{1/q}
\]
\[
\leq \ln(b/a) \left\{ \sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right\}^{1/q}
\]
\[
\times \left\{ \frac{1}{2} \int_0^{1/2} \left| t - \mu \right| dt \right\}^{1-1/q} \left\{ \frac{1}{2} \int_0^{1/2} \left| t - \mu \right|^q dt \right\}^{1/q}
\]
\[
+ \left\{ \frac{1}{2} \int_{1/2}^1 \left| t - \mu \right| dt \right\}^{1-1/q} \left\{ \frac{1}{2} \int_{1/2}^1 \left| t - \mu \right|^q dt \right\}^{1/q},
\]
where

\[
\int_0^{1/2} \left| f (a) \right| dt = C_1 (\mu) = \mu^2 - \frac{\mu}{2} + \frac{1}{8}, \\
\int_0^{1/2} \left| f (a) \right| dt = C_2 (\lambda) = \lambda^2 - \frac{3\lambda}{2} + \frac{5}{8},
\]

\[
\int_0^{1/2} \left| f (a) \right| \left| f (b) \right| dt = C_3 (\mu, q, a, b), \\
\int_0^{1/2} \left| f (a) \right| \left| f (b) \right| dt = C_4 (\lambda, q, a, b),
\]

which completes the proof.

**Corollary 1** Under the assumptions of Theorem 3 with \( \lambda = \mu = 1/2 \), the inequality (2) reduced to the following inequality

\[
\left| f (a) + f (b) \right| - \left| \frac{f (u)}{u} \right| \left| f (a) \right| \left| f (b) \right| dt = \left( \frac{1}{8} \right) \ln (b / a) \left\{ \sup \left| f (a) \right| \right\} \left| f (b) \right| \left\{ \sup \left| f (b) \right| \right\} \left( 1/2, q, a, b \right) \\
\times \left\{ C_3^{1/q} (0, q, a, b) + C_4^{1/q} (1, q, a, b) \right\}.
\]

**Corollary 2** Under the assumptions of Theorem 3 with \( \mu = 0 \) and \( \lambda = 1 \), the inequality (2) reduced to the following inequality

\[
\left| f (\sqrt{ab}) \right| - \left| \frac{f (u)}{u} \right| \left| f (a) \right| \left| f (b) \right| dt = \left( \frac{1}{8} \right) \ln (b / a) \left\{ \sup \left| f (a) \right| \right\} \left| f (b) \right| \left\{ \sup \left| f (b) \right| \right\} \left( 1/2, q, a, b \right) \\
\times \left\{ C_3^{1/q} (0, q, a, b) + C_4^{1/q} (1, q, a, b) \right\}.
\]

**Corollary 3** Under the assumptions of Theorem 3 with \( \mu = 1/6 \) and \( \lambda = 5/6 \), the inequality (2) reduced to the following inequality

\[
\left| \frac{1}{3} f (a) + f (b) \right| + 2 f (\sqrt{ab}) \left| \frac{f (u)}{u} \right| \left| f (a) \right| \left| f (b) \right| dt = \left( \frac{5}{72} \right) \ln (b / a) \left\{ \sup \left| f (a) \right| \right\} \left| f (b) \right| \left\{ \sup \left| f (b) \right| \right\} \left( 1/2, q, a, b \right) \\
\times \left\{ C_3^{1/q} (1/6, q, a, b) + C_4^{1/q} (5/6, q, a, b) \right\}
\]

**Theorem 4** Let \( f : I \subset (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( f^\ast \) such that \( f^\ast \in L[a,b] \), where \( a,b \in I^\ast \) with \( a < b \). If \( \left| f^\ast \right|^q \) is quasi-geometrically convex on \([a,b]\) for some fixed \( q \geq 1 \) and \( 0 \leq \mu \leq 1/2 \leq \lambda \leq 1 \), then the following inequality holds.

\[
I_f (\lambda, \mu, a, b) \leq \frac{1}{q} \left( \left| f^\ast (a) \right|^q + \left| f^\ast (b) \right|^q \right) \left( 1/2, q, a, b \right) \\
\times \left\{ C_3^{1/q} (p, q, a, b) + C_6^{1/q} (p, \lambda, a, b) \right\}
\]

where

\[
C_3 (p, \mu) = \frac{1}{p+1} \left( \mu^{p+1} + \left( \frac{1}{2} - \mu \right)^{p+1} \right), \\
C_6 (p, \lambda) = \frac{1}{p+1} \left( \lambda^{p+1} + \left( 1 - \lambda \right)^{p+1} \right), \\
C_7 (q, a, b) = \frac{q-1}{2} \left( a^{q/2} + b^{q/2} \right), \\
C_8 (q, a, b) = L \left( a^{q/2}, b^{q/2} \right) - C_7 (q, a, b),
\]

and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** Since \( \left| f^\ast \right|^q \) is quasi-geometrically convex on \([a,b]\) and using Lemma 1 and Hölder inequality, we get

\[
I_f (\lambda, \mu, a, b) \leq \frac{1}{q} \left( \left| f^\ast (a) \right|^q + \left| f^\ast (b) \right|^q \right) \left( 1/2, q, a, b \right) \\
\times \left\{ \int_0^{1/2} \left| \frac{\ln (b/a)}{u} \right| \left| f (u) \right| \left( \sup \left| f (a) \right| \right) \left( 1/2, q, a, b \right) \left| f (b) \right| \left( \sup \left| f (b) \right| \right) \left( 1/2, q, a, b \right) \right. \\
\times \left. \left\{ C_3^{1/q} (0, q, a, b) + C_4^{1/q} (1, q, a, b) \right\} \right.
\]

\[
\times \left\{ \int_0^{1/2} \left| \frac{\ln (b/a)}{u} \right| \left| f (u) \right| \left( \sup \left| f (a) \right| \right) \left( 1/2, q, a, b \right) \left| f (b) \right| \left( \sup \left| f (b) \right| \right) \left( 1/2, q, a, b \right) \right. \\
\times \left. \left\{ C_3^{1/q} (0, q, a, b) + C_4^{1/q} (1, q, a, b) \right\} \right.
\]

here it is seen by simple computation that

\[
\int_0^{1/2} \left| f (u) \right|^p dt = \frac{1}{p+1} \left( \mu^{p+1} + \left( \frac{1}{2} - \mu \right)^{p+1} \right), \\
\int_0^{1/2} \left| f (u) \right|^p dt = \frac{1}{p+1} \left( \lambda^{p+1} + \left( 1 - \lambda \right)^{p+1} \right), \\
\int_0^{1/2} \left( a^{q/2} b^{q/2} \right) dt = \frac{a^{q/2} b^{q/2}}{2} L \left( a^{q/2}, b^{q/2} \right), \\
and \int_0^{1/2} \left( a^{q/2} b^{q/2} \right) dt = L \left( a^{q/2}, b^{q/2} \right) - \frac{a^{q/2} b^{q/2}}{2} L \left( a^{q/2}, b^{q/2} \right).
\]
Hence, the proof is completed.  

**Corollary 4** Under the assumptions of Theorem 4 with $\lambda = \mu = 1/2$, the inequality (4) reduced to the following inequality

$$
\frac{f(a)+f(b)}{2} = \frac{1}{\ln(b/a)} \int_a^b \frac{f(a)}{u} du \leq \ln(b/a)\left(\sup \left\{ f'(a)^q, f'(b)^q \right\} \right)^{\frac{1}{q}}
$$

$$
\times \left\{ \frac{1}{2^{p+1}(p+1)} \right\}^{\frac{1}{p}} \left\{ C_7^{1/q}(q,a,b) + C_8^{1/q}(q,a,b) \right\}.
$$

**Corollary 5** Under the assumptions of Theorem 4 with $\mu = 0$ and $\lambda = 1$, the inequality (4) reduced to the following inequality.

$$
\frac{\int f(ab) \cdot \frac{1}{\ln(b/a)} \int_a^b \frac{f(a)}{u} du}{(a,b)} \leq \ln(b/a)\left(\sup \left\{ f'(a)^q, f'(b)^q \right\} \right)^{\frac{1}{q}}
$$

$$
\times \left\{ \frac{1}{2^{p+1}(p+1)} \right\}^{\frac{1}{p}} \left\{ C_7^{1/q}(q,a,b) + C_8^{1/q}(q,a,b) \right\}.
$$

**Corollary 6** Under the assumptions of Theorem 4 with $\mu = 1/6$ and $\lambda = 5/6$, the inequality (4) reduced to the following inequality

$$
\frac{1}{3} \left\{ \frac{f(a)+f(b)}{2} \right\} + \frac{1}{\ln(b/a)} \int_a^b \frac{f(a)}{u} du \leq \ln(b/a)\left(\sup \left\{ f'(a)^q, f'(b)^q \right\} \right)^{\frac{1}{q}}
$$

$$
\times \left\{ \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right\}^{\frac{1}{p}} \left\{ C_7^{1/q}(q,a,b) + C_8^{1/q}(q,a,b) \right\}.
$$

**Theorem 5** Let $f: I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^*$ such that $f' \in L[a,b]$, where $a,b \in I^*$ with $a < b$. If $|f'| = q$ is quasi-geometrically convex on $[a,b]$ for some fixed $q > 1$ and $0 \leq \mu \leq 1/2 \leq \lambda \leq 1$, then the following inequality holds

$$
I_f(\lambda,\mu,a,b) = \ln(b/a)\left(\sup \left\{ f'(a)^q, f'(b)^q \right\} \right)^{\frac{1}{q}}
$$

$$
\times \left\{ C_7^{1/p}(p,a,b) C_8^{1/q}(q,\mu) + C_8^{1/p}(p,a,b) C_6^{1/q}(q,\lambda) \right\}
$$

where $C_5, C_6, C_7, C_8$ are defined as in Theorem 4 and $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** Since $|f'| = q$ is quasi-geometrically convex on $[a,b]$ and using Lemma 1 and Hölder inequality, we get

$$
I_f(\lambda,\mu,a,b) \leq \ln(b/a)\left(\sup \left\{ f'(a)^q, f'(b)^q \right\} \right)^{\frac{1}{q}}
$$

$$
\times \left\{ \frac{1}{2^{q+1}(q+1)} \right\}^{\frac{1}{q}} \left\{ C_7^{1/p}(p,a,b) + C_8^{1/p}(p,a,b) \right\}.
$$

Hence, the proof is completed.

**Corollary 7** Under the assumptions of Theorem 5 with $\lambda = \mu = 1/2$, the inequality (5) reduced to the following inequality

$$
\frac{f(a)+f(b)}{2} = \frac{1}{\ln(b/a)} \int_a^b \frac{f(a)}{u} du \leq \ln(b/a)\left(\sup \left\{ f'(a)^q, f'(b)^q \right\} \right)^{\frac{1}{q}}
$$

$$
\times \left\{ \frac{1}{2^{q+1}(q+1)} \right\}^{\frac{1}{q}} \left\{ C_7^{1/p}(p,a,b) + C_8^{1/p}(p,a,b) \right\}.
$$

**Corollary 8** Under the assumptions of Theorem 5 with $\mu = 0$ and $\lambda = 1$, the inequality (5) reduced to the following inequality

$$
\frac{f(a)+f(b)}{2} = \frac{1}{\ln(b/a)} \int_a^b \frac{f(a)}{u} du \leq \ln(b/a)\left(\sup \left\{ f'(a)^q, f'(b)^q \right\} \right)^{\frac{1}{q}}
$$

$$
\times \left\{ \frac{1}{2^{q+1}(q+1)} \right\}^{\frac{1}{q}} \left\{ C_7^{1/p}(p,a,b) + C_8^{1/p}(p,a,b) \right\}.
$$
Corollary 9 Under the assumptions of Theorem 5 with \( \mu = 1/6 \) and \( \lambda = 5/6 \), the inequality (5) reduced to the following inequality

\[
\left[ \frac{1}{3} \left( \frac{f(a)}{2} + f\left( \frac{a+b}{2} \right) \right) + 2f\left( \sqrt{ab} \right) \right] - \frac{b}{\ln(b/a)} \int_a^b \frac{f(u)}{u} du \\
\leq \frac{\ln(b/a)}{2} \left( \sup \left\{ \left| f'(a) \right|^p, \left| f'(b) \right|^q \right\} \right)^{1/p} \\
x \left( 1 + \frac{2^{q+1}}{6^{q+1}(q+1)} \right)^{1/q} \left\{ C_{7}^{1/p}(p,a,b) + C_{8}^{1/p}(p,a,b) \right\}.
\]

References


