A Fixed Point Result of Expanding Mappings in Complete Cone Metric Spaces

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Abstract In this paper, we prove a fixed point theorem for expanding onto self-mappings in complete cone metric spaces. Our results improve and extend some comparable results in the literature.

Keywords: cone metric space, fixed point, expanding mapping


1. Introduction

In 2007, Huang and Zhang [5] introduced cone metric spaces replacing the real numbers by an ordered Banach space, and they have proved some fixed point theorems for self-mapping satisfying different types of contractive conditions in cone metric spaces. Later on, many authors have generalized and extended Huang and Zhang [5] fixed point theorems (see, e.g., [1,2,3,7,8]). In 1984, the concept of expanding mappings was introduced by Wang et. al. [9]. In 1992, Daffer and Kaneko [4] defined expanding mappings for pair of mappings in complete metric spaces and proved some fixed point theorems. In 2012, X. Huang, Ch. Zhu and Xi Wen [6] proved some fixed point theorems for expanding mappings cone metric spaces and they have also extended the results of Daffer and Kaneko [4]. The main aim of this paper is we proved a fixed point theorem for expanding mappings in cone metric spaces, our result extends and improves the results of [6].

The following definitions and properties are due to Huang and Zhang [5].

Definition 1.1. Let B be a real Banach space and 0 is the zero element of B, P a subset of B. The set P is called a cone if and only if:

(i) P is closed, non–empty and \( \{0\} \neq P \neq B \);

(ii) \( a, b \geq 0, x, y \in P \) implies \( ax + by \in P \);

(iii) \( P \cap (-P) = \{0\} \).

For a cone P in a Banach space B, define partial ordering \( \preceq \) with respect to P by \( x \preceq y \) if and only if \( y - x \in P \). We shall write \( x < y \) to indicate \( x \leq y \) but \( x \neq y \), while \( x \ll y \) will stand for \( y - x \in \text{Int} \, P \), where Int P denotes the interior of the set P. This cone P is called an order cone.

Let B be a Banach space and \( P \subseteq B \) be an order cone .The order cone P is called normal if there exists \( K > 0 \) such that for all \( x, y \in B \),

\[ 0 \leq x \leq y \implies |x| \leq K |y| \]

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.2. Let \( (X, d) \) be a cone metric space and \( T: X \rightarrow X \) is called an expanding mapping, if for every \( x, y \in X \) there exists a number \( k > 1 \) such that

\[ d(Tx, Ty) \geq k d(x, y) \]

Definition 1.3. Let \( (X, d) \) be a cone metric space .We say that \( \{x_n\} \) is

(i) a Cauchy sequence if for every \( c \in B \) with \( c > \theta \), there is \( N \) such that for all \( n, m \geq N \), \( d(x_n, x_m) \ll c \);

(ii) a convergent sequence if for any \( c > \theta \), there is an \( N \) such that for all \( n > N \), \( d(x_n, x) < c \), for some fixed \( x \in X \). We write \( x_n \rightarrow x \) (as \( n \rightarrow \infty \)).

The space \( (X, d) \) is called a complete cone metric space if every Cauchy sequence is convergent [5].

Definition 1.4. [5] Let \( (X, d) \) be a cone metric space and \( T: X \rightarrow X \), then T is called an expanding mapping, if for every \( x, y \in X \) there exists a number \( k > 1 \) such that

\[ d(Tx, Ty) \geq k d(x, y) \]

2. Main Result

In this section, we prove a fixed point theorem for expanding mappings in complete cone metric spaces.

We prove a Lemma which is useful in the main theorem.

Lemma 2.1. Let \( (X, d) \) be a cone metric space and \( \{x_n\} \) be a sequence in X. If there exists a number \( \lambda \in (0,1) \) such that

\[ d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}), \quad n=1,2,\ldots \quad (1) \]
then \( \{x_n\} \) is a Cauchy sequence in \( X \).

**Proof.** By the induction and the condition (1), we have
\[
d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \leq \lambda^2 d(x_{n-1}, x_{n-2}) \leq \ldots \leq \lambda^n d(x_1, x_0).
\]
For \( n > m \)
\[
d(x_n, x_m) \leq d(x_n, x_{n-1}) \leq d(x_{n-1}, x_{n-2}) \leq \ldots \leq d(x_{m+1}, x_m) \leq (\lambda^{n-m} + \lambda^{n-m-1} + \ldots + \lambda^0) d(x_1, x_0).
\]
Thus, \( d(x_n, x_m) \leq \lambda^m / (1 + \lambda) d(x_1, x_0) \).

Let \( \theta < c \) be given. Choose \( r > 0 \) such that \( c + N_r(\theta) \subseteq P \), where \( N_r(\theta) = \{ x \in E : ||x|| < r \} \). Also choose a natural number \( N_1 \) such that \( \lambda^m / (1 + \lambda) d(x_1, x_0) \in N_1(\theta) \), for all \( m \geq N_1 \). Thus
\[
d(x_n, x_m) \leq \lambda^m / (1 + \lambda) d(x_1, x_0) < c, \quad \text{for all } m \geq N_1.
\]
Hence, \( \{x_n\} \) is a Cauchy sequence in \( X \).

The following theorem improved and extended the Theorem 2.1. of [6].

**Theorem 2.2.** Let \( (X, d) \) be a complete cone metric space and \( T : X \to X \) be a surjection. Suppose that there exists \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0 \) with \( \alpha_1 + \alpha_2 + \alpha_3 + 2 \alpha_4 > 1 \) such that
\[
d(Tx, Ty) \geq \alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(y, Tx)
\]
for all \( x, y \in X \), \( x \neq y \). Then \( T \) has a fixed point in \( X \).

**Proof.** By our assumption, it is clear that \( T \) is injective. Let \( F \) be the inverse mapping of \( T \).

Let \( x_0 \in X \), then \( x_1 = F(x_0), x_2 = F(x_1) = F^2(x_0), \ldots, x_{n+1} = F(x_n) = F^{n+1}(x_0), \ldots \)

We assume that \( x_{n+1} \neq x_n \) for all \( n = 1, 2, 3 \) otherwise \( x_{n+1} \neq x_n \), for some \( n \), \( x_0 \) is a fixed point of \( T \).

From the condition (2) it follows that
\[
d(x_{n-1}, x_n) = d(T^{-1} x_{n-1}, TT^{-1} x_n) \geq \alpha_1 d(T^{-1} x_{n-1}, T^{-1} x_n) + \alpha_2 d(T^{-1} x_{n-1}, TT^{-1} x_n) + \alpha_3 d(T^{-1} x_{n-1}, TT^{-1} x_n).
\]

By the Lemma 2.1, we get that \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( (X, d) \) is complete, the sequence \( \{x_n\} \) converges to a point \( z \in X \). Let \( z = Tp, p \in X \), we have
\[
d(x, z) = d(Tx, Tz) \geq \alpha_1 d(x, z) + \alpha_2 d(x, Ty) + \alpha_3 d(y, Ty) + \alpha_4 d(y, Tx).
\]

Letting \( n \to \infty \), we get that
\[
\theta \geq \alpha_1 d(z, p) + \alpha_2 d(z, Tz) + \alpha_3 d(p, z) + \alpha_4 d(p, Tz).
\]

That is, \( \alpha_1 + \alpha_2 + \alpha_3 + 2 \alpha_4 \geq 1 \).

Therefore, \( d(p, z) = \theta \). That is, \( z = p \).

Therefore, \( p = z = Tp \).

Therefore, \( z \) is a fixed point of \( T \).

**Remark 2.3.** If we choose \( \alpha_4 = 0 \) in Theorem 2.1, then we get that Theorem 2.1. of [6].

**Remark 2.4.** If we choose \( \alpha_1 = k \) and \( \alpha_2 = \alpha_3 = \alpha_4 = 0 \) in Theorem 2.1, then we get that Corollary 2.1. of [6].

**References**


