On Generalized Trigonometric Functions

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Abstract A new trigonometric functions called generalized trigonometric functions are perversely defined by a system of first order nonlinear ordinary differential equations with initial conditions. This system is related to the Hamilton system. In this paper, we define these functions using the equation \( |x|^m + |y|^m = 1 \), for \( m > 0 \). We study the graphs, the trigonometric identities and some of common properties of these functions. We find the first derivatives which have different forms when \( m \) is even and when \( m \) is odd.

Keywords: trigonometric functions, generalized trigonometric functions, trigonometric identities


1. Introduction

Ordinary trigonometry studies triangles in the Euclidean plane \( \mathbb{R}^2 \). There are some ways to defining the ordinary trigonometric functions on real numbers such as right-angled triangle definition, unit-circle definition, series definition, definitions via differential equations, and definition using functional equations. Trigonometric functions are one of the important group of the elementary functions. Using them, we can solve geometric problems, complex analytic problems and problems involving Fourier series. Also they are important because they are periodic. All the six trigonometric functions can defined through the sine and cosine functions.

In many papers, (see [1,2,3,4]), a new trigonometric functions are defined using a system of first order nonlinear ordinary differential equations with initial conditions. This system is related to the Hamilton system. The new functions are called generalized trigonometric functions and denoted by \( \sin_m, \cos_m, \tan_m, \cot_m, \sec_m, \csc_m \) for \( m > 0 \). It was proved that if \( x = \cos_m \theta \) and \( y = \sin_m \theta \), then \( |x|^m + |y|^m = 1 \). In this paper, we define these functions directly using the equation \( |x|^m + |y|^m = 1 \), for \( m > 0 \). We study the graphs and the trigonometric identities of these functions. Then we study the first derivative for special cases when \( m \) is natural number. Since trigonometric functions are used in Fourier series, Fourier transform, and signal processing, we look to improve the efficiency of signal processing and reduce the noise effects by using the generalized trigonometric functions. Moreover, the generalized trigonometric functions can be used to obtain analytic solutions to the equation of a nonlinear spring-max system.

Now, consider the equation \( |x|^m + |y|^m = 1 \). The graph of this equation in the Cartesian plan is symmetric about the axes. For a special case, if \( m = 1 \), the graph of \( |x| + |y| = 1 \) is a unit square centered at \((0,0)\) with vertices at \((0,0),(1,0),(0,1),(-1,0),(0,-1)\). For \( m = 2 \) the graph is the unit circle. For any \( m > 0 \), let \( S_m \) be the graph of the equation \( |x|^m + |y|^m = 1 \). We call \( S_m \) a unit semi-square. For an angle \( \theta \) placed in the center of \( S_m \) with initial ray on the positive \( x \)-axis and terminal ray intersects the graph of \( S_m \) at a point \( P(x,y) \), we say that \( \theta \) is placed in the standard position of \( S_m \). Suppose that the angle \( \theta = \frac{\pi}{4} \) is placed in the standard position of \( S_m \) and \( P_m(x,y) \) is the point of intersection of the terminal ray and \( S_m \). Then along the terminal ray, we have the following:
1. \( P_m \to (0,0) \) as \( m \to 0 \), and
2. \( P_m \to (1,1) \) as \( m \to \infty \).

In Figure 1, we graph the equation \( |x|^m + |y|^m = 1 \) for several values of \( m > 0 \) showing the point \( P_m \).

Figure 1. Graph of \( |x|^m + |y|^m = 1 \) for \( m = \frac{1}{3}, 1, 2, 3, 4, 100 \).
**Definition 1.1** In the Cartesian plane, if \( \theta \) is an angle placed in the center of the plane with initial ray at the positive \( x \)-axis, then we say that:

1. \( \theta \in Q_1 \) if the terminal ray lies between the positive \( x \)-axis and the positive \( y \)-axis. \( \theta \in Q_1 \) if \( \theta = 2n\pi, n \in \mathbb{Z} \).
2. \( \theta \in Q_2 \) if the terminal ray lies between the negative \( x \)-axis and the positive \( y \)-axis. \( \theta \in Q_2 \) if \( \theta = \pi + 2n\pi, n \in \mathbb{Z} \).
3. \( \theta \in Q_3 \) if the terminal ray lies between the negative \( x \)-axis and the negative \( y \)-axis. \( \theta \in Q_3 \) if \( \theta = \pi + 2n\pi, n \in \mathbb{Z} \).
4. \( \theta \in Q_4 \) if the terminal ray lies between the positive \( x \)-axis and the negative \( y \)-axis. \( \theta \in Q_4 \) if \( \theta = \frac{3\pi}{2} + 2n\pi, n \in \mathbb{Z} \).

**2. Generalized Trigonometric Functions; Definitions and Graphs**

**Definition 2.1** For a given \( \theta \) and for a unit semi-square \( S_m \), let \( \theta \) be an angle placed in the standard position. Suppose that the terminal ray intersects \( S_m \) in a point \( (x, y) \) (as seen in Figure 2). We define the six generalized trigonometric functions of \( \theta \) as follows:

- **1. g-sine of \( \theta \):**
  \[
  \sin_m \theta = \frac{y}{\sqrt{x^m + y^m}}
  \]
  provided \( x \neq 0 \).

- **2. g-cosine of \( \theta \):**
  \[
  \cos_m \theta = \frac{x}{\sqrt{x^m + y^m}}
  \]
  provided \( y \neq 0 \).

- **3. g-tangent of \( \theta \):**
  \[
  \tan_m \theta = \frac{x}{y}
  \]
  provided \( x \neq 0 \).

- **4. g-cosecant of \( \theta \):**
  \[
  \csc_m \theta = \frac{1}{y}
  \]
  provided \( y \neq 0 \).

- **5. g-secant of \( \theta \):**
  \[
  \sec_m \theta = \frac{1}{x}
  \]
  provided \( x \neq 0 \).

- **6. g-cotangent of \( \theta \):**
  \[
  \cot_m \theta = \frac{x}{y}
  \]
  provided \( y \neq 0 \).

The following table gives some values of g-trigonometric functions for some special angles:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( 0 )</th>
<th>( \frac{\pi}{2} )</th>
<th>( \pi )</th>
<th>( \frac{3\pi}{2} )</th>
<th>( 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin_m \theta )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \cos_m \theta )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \tan_m \theta )</td>
<td>Undefined</td>
<td>0</td>
<td>Undefined</td>
<td>0</td>
<td>Undefined</td>
</tr>
</tbody>
</table>

**Theorem 2.2** For all \( \theta \in \mathbb{R} \), we have the following:

1. \( |\cos_m \theta|^m + |\sin_m \theta|^m = 1 \).
2. \( 1 + |\tan_m \theta|^m = |\sec_m \theta|^m \).
3. \( 1 + |\cot_m \theta|^m = |\csc_m \theta|^m \).

**Theorem 2.3** For any \( \theta \in \mathbb{R} \), we have the following:

1. \( \sin_m \theta = \frac{\sin \theta}{\sqrt{x^m + y^m}} \).
2. \( \cos_m \theta = \frac{\cos \theta}{\sqrt{x^m + y^m}} \).
3. \( \tan_m \theta = \tan \theta \), provided \( \cos \theta \neq 0 \).
4. \( \csc_m \theta = \frac{1}{\sqrt{x^m + y^m}} \), provided \( \sin \theta \neq 0 \).
5. \( \sec_m \theta = \frac{1}{\sqrt{x^m + y^m}} \), provided \( \cos \theta \neq 0 \).
6. \( \cot_m \theta = \cot \theta \), provided \( \sin \theta \neq 0 \).

**Corollary 2.4** Let \( y = f(\theta) \) be a g-trigonometric function. For any \( k \in \mathbb{Z} \) and for any \( \theta \in \mathbb{R} \), we have that \( f(\theta + 2k\pi) = f(\theta) \). That is, g-trigonometric functions are periodic functions.

**Theorem 2.5** Let \( \theta \in \mathbb{R} \). Then:
1. \( \sin_m(\theta + \frac{\pi}{2}) = \cos_m \theta \).

2. \( \cos_m(\theta + \frac{\pi}{2}) = -\sin_m \theta \).

3. \( \sin_m(\theta \mp \pi) = -\sin_m \theta \).

4. \( \cos_m(\theta \mp \pi) = \pm \cos_m \theta \).

Using the relations between g-trigonometric functions and usual trigonometric functions and using the Graph-4.4.2 grapher program, we give graphs of g-trigonometric functions for \( m = \frac{1}{3}, 1 \) and 4. We neglect the graphs of \( \tan_m \theta \) and \( \cot_m \theta \) since they are exactly the graphs of \( \tan \theta \) and \( \cot \theta \) respectively.

### 3. Identities and Some Common Properties

Evidently, and for \( m > 0 \), the g-trigonometric functions have the following direct common properties:

1. \( |\sin_m x| \leq 1 \) and \( |\cos_m x| \leq 1 \).

2. All g-trigonometric functions are periodic. Moreover, the functions \( \sin_m x, \cos_m x, \sec_m x \) and \( \csc_m x \) have period \( 2\pi \), while the functions \( \tan_m x \) and \( \cot_m x \) have period \( \pi \).

3. The g-trigonometric functions \( y = \cos_m x \) and \( y = \sec_m x \) are even functions, while the other g-trigonometric functions are odd functions.

**Theorem 3.1** For any \( \theta_1, \theta_2 \in \mathbb{R}, \) and \( m > 0 \),

\[
\cos_m(\theta_1 - \theta_2) = \frac{\cos(\theta_1) \cos(\theta_2) + \sin(\theta_1) \sin(\theta_2)}{\sqrt{\cos^m(\theta_1) + \sin^m(\theta_1)}}
\]

**Proof.** Consider \( S_m \), the graph of \( |x|^m + |y|^m = 1 \) in the first quadrant, as shown Figure 7. Draw the two vectors \( \overrightarrow{oa} \) and \( \overrightarrow{oc} \) as terminal rays of the two angles \( \theta_1 \) and \( \theta_2 \) respectively. So, we have the following:

\[
\cos_m(\theta_1 - \theta_2) = \frac{\cos(\theta_1) \cos(\theta_2) + \sin(\theta_1) \sin(\theta_2)}{\sqrt{\cos^m(\theta_1) + \sin^m(\theta_1)}}
\]

\[
\cos_m(\theta_1) \cos_m(\theta_2) + \sin_m(\theta_1) \sin_m(\theta_2)
\]

\[
\sqrt{\cos(\theta_1) \cos(\theta_2) + \sin(\theta_1) \sin(\theta_2)}
\]

\[
\cos_m(\theta_1) \cos_m(\theta_2) + \sin_m(\theta_1) \sin_m(\theta_2)
\]

\[
\sqrt{\cos^m(\theta_1) + \sin^m(\theta_1)}
\]

\[
\overrightarrow{oa} \cdot \overrightarrow{oc}
\]

\[
\cos_m(\theta_1 - \theta_2) = \frac{\cos_m(\theta_1) \cos_m(\theta_2) + \sin_m(\theta_1) \sin_m(\theta_2)}{\sqrt{\cos^m(\theta_1) + \sin^m(\theta_1)}}
\]

\[
\sqrt{\cos(\theta_1) \cos(\theta_2) + \sin(\theta_1) \sin(\theta_2)}
\]

\[
\cos_m(\theta_1) \cos_m(\theta_2) + \sin_m(\theta_1) \sin_m(\theta_2)
\]

\[
\sqrt{\cos^m(\theta_1) + \sin^m(\theta_1)}
\]

\[
\overrightarrow{oa} \cdot \overrightarrow{oc}
\]
Corollary 3.2 For any $\theta_1, \theta_2 \in \mathbb{R}$,
\[
\cos_m(\theta_1 + \theta_2) = \cos_m(\theta_1) \cos_m(\theta_2) - \sin_m(\theta_1) \sin_m(\theta_2).
\]
1. \[
\cos_m(2\theta) = \cos_m^2(\theta) - \sin_m^2(\theta).
\]
2. \[
\sin_m(2\theta) = 2\sin_m(\theta) \cos_m(\theta) - \sin_m^2(\theta).
\]
3. \[
\tan_m(\theta_1 - \theta_2) = \frac{\tan_m(\theta_1) - \tan_m(\theta_2)}{1 + \tan_m(\theta_1) \tan_m(\theta_2)}.
\]
4. \[
\tan_m(\theta_1 + \theta_2) = \frac{\tan_m(\theta_1) + \tan_m(\theta_2)}{1 - \tan_m(\theta_1) \tan_m(\theta_2)}.
\]
5. \[
\tan_m(2\theta) = \frac{2\tan_m(\theta)}{1 - \tan_m^2(\theta)}.
\]
6. \[
\cos_m^2(\theta) + \sin_m^2(\theta) = 1.
\]

Theorem 3.3 For any $\theta_1, \theta_2 \in \mathbb{R}$,
\[
\sin_m(\theta_1 - \theta_2) = \sin_m(\theta_1) \cos_m(\theta_2) - \cos_m(\theta_1) \sin_m(\theta_2).
\]
1. \[
\frac{d}{d\theta}(\cos_m(\theta)) = -\sin_m^{-1}(\cos_m(\theta) + \cos_m(\theta)).
\]
2. \[
\frac{d}{d\theta}(\sin_m(\theta)) = \cos_m^{-1}(\sin_m(\theta) + \cos_m(\theta)).
\]
3. \[
\frac{d}{d\theta}(\tan_m(\theta)) = \sec^2(\theta).
\]
4. \[
\frac{d}{d\theta}(\sec_m(\theta)) = \sin_m^{-1}(\tan^2(\theta) + 1).
\]
5. \[
\frac{d}{d\theta}(\csc_m(\theta)) = -\cos_m^{-1}(\cot_m(\theta) + 1).
\]
6. \[
\frac{d}{d\theta}(\cot_m(\theta)) = -\sec^2(\theta).
\]

Mathematically, if $m$ is even, then $|x|^m = x^m$ for all $x \in \mathbb{R}$. But if $m$ is odd, the value of $|x|^m$ depends on the sign of $x$. So, there is a quite difference between the forms of derivatives of the trigonometric functions when $m$ is even and when $m$ is odd. Moreover, we will see that in the case when $m$ is odd, the derivatives of the trigonometric functions have different forms when $m = 1$ and when $m > 1$. If $m$ is odd, the polar equation of $S_m$ has the following piecewise definition function with four cases depending on $\theta$:

In order to simplify the derivatives of the generalized trigonometric functions when $m$ is odd, define the following four functions:

So, we have that for $n \in \mathbb{Z}$, if $\theta \neq (2n+1)\frac{\pi}{2}$, then
\[
\delta(\theta) = \left\lfloor \frac{\sin(\theta)}{\cos(\theta)} \right\rfloor, \quad \text{and if } \theta \neq n\pi, \quad \gamma(\theta) = \frac{\cos(\theta)}{\sin(\theta)}.
\]

Theorem 4.2 Let $m$ be an odd natural number, and let
\[
r = \frac{1}{\sqrt{\cos^m(\theta) + \sin^m(\theta)}}
\]
be the polar equation of $S_m$. Then for $\theta \neq n\frac{\pi}{2}, n \in \mathbb{Z}$,
\[
\frac{dr}{d\theta} = \left(\gamma(\theta) \sin_m(\theta) \cos_m^{-1}(\theta) - \delta(\theta) \sin_m^{-1}(\theta) \cos_m(\theta)\right)
\]
Proof. Derive directly, we get that
\[
\frac{dr}{d\theta} = m \frac{\cos^m \theta + \sin^m \theta}{m!} \left( \cos \theta \sin \theta \right)^{m-1} \left( \cos \theta \sin \theta \right)\sin \theta \cos \theta
\]
As \( m \) is odd, \( m - 1 \) is even and we have that
\[
\frac{dr}{d\theta} = \frac{1}{m-1} \frac{\cos \theta \sin \theta}{\cos \theta - \sin \theta}
\]

Theorem 4.3 If \( m = 1 \), then \( \frac{dr}{d\theta} \) does not exist for all \( \theta = n\pi \), \( n \in \mathbb{Z} \).

Proof. For \( m = 1 \), the right hand and the left hand derivatives of \( r \) with respect to \( \theta \) at \( \theta = n\pi \) are
\[
\left( \frac{dr}{d\theta} \right)^+ = -\left( \frac{dr}{d\theta} \right)^- = 1.
\]
Hence, \( \frac{dr}{d\theta} \) does not exist for all \( \theta = n\pi \), \( n \in \mathbb{Z} \).

Similarly, \( \left( \frac{dr}{d\theta} \right)^+ \neq \left( \frac{dr}{d\theta} \right)^- \) for all \( \theta = (2n + 1)\pi / 2 \).

Theorem 4.4 If \( m \) is odd, and \( m \neq 1 \), then \( \frac{dr}{d\theta} = 0 \) for all \( \theta = n\pi \), \( n \in \mathbb{Z} \).

Proof. If \( m > 1 \), then \( \forall \theta = n\pi \), \( n \in \mathbb{Z} \), both \( \cos^{m-1} \theta \sin \theta = 0 \) and \( \sin^{m-1} \theta \cos \theta = 0 \). This implies that
\[
\left( \frac{dr}{d\theta} \right)^+ = \left( \frac{dr}{d\theta} \right)^- = 0.
\]

Remark 4.5 For all \( m > 0 \) and for \( \theta \neq n\pi / 2 \), \( n \in \mathbb{Z} \), we have \( \delta(\theta) = \delta^*(\theta) \) and \( \gamma(\theta) = \gamma^*(\theta) \), so we get that
\[
\frac{dr}{d\theta} = r \left[ \gamma^*(\theta) \cos^{m-1} \theta \sin \theta - \delta^*(\theta) \sin^{m-1} \theta \cos \theta \right].
\]
Moreover, if \( m \neq 1 \) and at \( \theta = n\pi \), \( n \in \mathbb{Z} \), we have that,
\[
r = 1, \quad \delta^*(\theta) \text{ and } \gamma^*(\theta) \text{ have values } 1 \text{ or } -1, \text{ and }
\cos^{m-1} \theta \sin \theta = 0, \sin^{m-1} \theta \cos \theta = 0. \text{ Hence}
\[
\frac{dr}{d\theta} = r \left[ \gamma^*(\theta) \cos^{m-1} \theta \sin \theta - \delta^*(\theta) \sin^{m-1} \theta \cos \theta \right] = \delta^*(\theta) \cdot \frac{\sin^{m-1} \theta \cos \theta}{\cos \theta - \sin \theta}.
\]
Using this remark, we have the following summary theorem.

Theorem 4.6 Let \( m \) be an odd natural number, and let
\[
r = \left( \frac{\cos \theta \sin \theta}{\cos \theta - \sin \theta} \right)^m
\]
be the polar equation of the unit semi-square \( S_m \). Then
a. For \( m = 1 \), and for \( \theta \neq n\pi / 2 \), \( n \in \mathbb{Z} \),
\[
\frac{dr}{d\theta} = r(\gamma^*(\theta) \sin \theta - \delta^*(\theta) \cos \theta).
\]
b. For \( m \neq 1 \), and for all \( \theta \),
\[
\frac{dr}{d\theta} = r \left[ \gamma^*(\theta) \cos^{m-1} \theta \sin \theta - \delta^*(\theta) \sin^{m-1} \theta \cos \theta \right].
\]

Theorem 4.7
a. For \( m = 1 \), and for \( \theta \neq n\pi / 2 \), \( n \in \mathbb{Z} \),
1. \( \frac{d}{d\theta} (\cos_1 \theta) = -\delta(\theta)(\cos_1^2 \theta + \sin_1^2 \theta) \).
2. \( \frac{d}{d\theta} (\sin_1 \theta) = \gamma(\theta)(\cos_1^2 \theta + \sin_1^2 \theta) \).
3. \( \frac{d}{d\theta} (\sec_1 \theta) = \delta(\theta)(1 + \tan_1^2 \theta) \).
4. \( \frac{d}{d\theta} (\csc_1 \theta) = -\gamma(\theta)(1 + \cot_1^2 \theta) \).

b. For \( m \neq 1 \), and for all \( \theta \),
1. \( \frac{d}{d\theta} (\cos_m \theta) = -\sin^{m-2} \theta | \sin_m \theta | (\sin_m^2 \theta + \cos_m^2 \theta) \).
2. \( \frac{d}{d\theta} (\sin_m \theta) = \cos^{m-2} \theta | \cos_m \theta | (\cos_m^2 \theta + \sin_m^2 \theta) \).
3. \( \frac{d}{d\theta} (\sec_m \theta) = \sin^{m-2} \theta | \sin_m \theta | (1 + \tan_m^2 \theta) \).
4. \( \frac{d}{d\theta} (\csc_m \theta) = -\cos^{m-2} \theta | \cos_m \theta | (1 + \cot_m^2 \theta) \).
Proof. a) (1) For $m = 1$, nd for $\theta \neq n\frac{\pi}{2}, n \in \mathbb{Z}$, we have

$$\frac{d}{d\theta} (\cos_1 \theta) = \frac{d}{d\theta} (r \cos \theta) = -r \sin \theta + \cos \theta \frac{dr}{d\theta}$$

$$= -r \sin \theta + r \cos \theta (\gamma(\theta) \sin_1 \theta - \delta(\theta) \cos_1 \theta)$$

$$= -\sin_1 \theta + r \cos \theta |\sin_1 \theta| - \delta(\theta) \cos_1^2 \theta$$

$$= -\sin_1 \theta + r \sin_\theta - \delta(\theta) \cos_1^2 \theta$$

$$= -r \sin_1 \theta |\sin_1 \theta| - \delta(\theta) \cos_1^2 \theta$$

$$= -\sin_1 \theta (1 - |\cos_1 \theta|) - \delta(\theta) \cos_1^2 \theta$$

$$= -\sin_1 \theta \sin_\theta - \delta(\theta) \cos_1^2 \theta$$

So,

$$\frac{d}{d\theta} (\cos_1 \theta) = \frac{d}{d\theta} (r \cos \theta) = -r \sin \theta + \cos \theta \frac{dr}{d\theta}$$

In both cases, we have that

$$\frac{d}{d\theta} (\cos_1 \theta) = \frac{d}{d\theta} (r \cos \theta) = -r \sin \theta + \cos \theta \frac{dr}{d\theta}$$

b) (1) For $m \neq 1$, we have two cases:

Case 1: $\theta \neq n\frac{\pi}{2}, n \in \mathbb{Z}$.

In this case, $\delta^*(\theta) = |\sin_\theta|$ and $\gamma^*(\theta) = |\cos_\theta|$.

So,

$$\frac{d}{d\theta} (\cos_1 \theta) = \frac{d}{d\theta} (r \cos \theta) = -r \sin \theta + \cos \theta \frac{dr}{d\theta}$$

$$= -\sin_1 \theta + \cos \theta (r \gamma^*(\theta) \cos_1^{m-1} \theta \sin_1 \theta - \delta^*(\theta) \sin_1^{m-1} \theta \cos_1 \theta)$$

Since $m$ is odd, $m - 1$ is even, so

$$\frac{d}{d\theta} (\cos_1 \theta) = \frac{d}{d\theta} (r \cos \theta) = -r \sin \theta + \cos \theta \frac{dr}{d\theta}$$

Case 2: $\theta = n\frac{\pi}{2}, n \in \mathbb{Z}$. In this case, $\frac{dr}{d\theta} = 0$. So,

$$\frac{d}{d\theta} (\cos_1 \theta) = \frac{d}{d\theta} (r \cos \theta) = -r \sin \theta + \cos \theta \frac{dr}{d\theta} = -\sin_1 \theta \sin_\theta + \sin_1 \theta |\cos_1 \theta|$$

Now if $\theta = n\pi$, then $\frac{d}{d\theta} (\cos_1 \theta) = 0$. In this case,

$$(-\sin_1^{m-2} \theta |\sin_1 \theta| (\sin_1^2 \theta + \cos_1^2 \theta)) = 0.$$

And if $\theta = (2n+1)\frac{\pi}{2}, \frac{d}{d\theta} (\cos_1 \theta) = 1$. In this case,

$$(-\sin_1^{m-2} \theta |\sin_1 \theta| (\sin_1^2 \theta + \cos_1^2 \theta)) = 1.$$

In both cases, we have that

$$\frac{d}{d\theta} (\cos_1 \theta) = \sin_1^{m-2} \theta |\sin_1 \theta| (\sin_1^2 \theta + \cos_1^2 \theta).$$

Remark 4.8 Since $\tan_1 \theta = \tan \theta$, and $\cot_1 \theta = \cot \theta$, we have that

$$\frac{d}{d\theta} (\tan_1 \theta) = \sec^2 \theta$$

and

$$\frac{d}{d\theta} (\cot_1 \theta) = -\csc^2 \theta.$$