Some Results on the Identity \( d(x) = \lambda x + \zeta(x) \)

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Abstract The main purpose of this paper is study and investigate some results concerning a derivation \( d \) on a 2-torsion free semiprime ring \( R \) with the center \( Z(R) \), when \( R \) admits \( d \) to satisfy some conditions, then there exist \( \lambda \in \mathbb{C} \) and an additive mapping \( \zeta: R \rightarrow \mathbb{C} \) such that \( d(x) = \lambda x + \zeta(x) \) for all \( x \in R \).

Keywords: semiprime rings, derivations, generalized derivation, biadditive mapping


1. Introduction

Some people ask why study derivation? at first we can say derivations on rings help us to understand rings better and also derivations on rings can tell us about the structure of the rings. For instance a ring is commutative if and only if the only inner derivation on the ring is zero. Also derivations can be helpful for relating a ring with the set of matrices with entries in the ring (see, [6]). Derivations play a significant role in determining whether a ring is commutative, see [1,3,4]. Derivations can also be useful in other fields. For example, derivations play a role in the calculation of the eigen values of matrices (see, [2]) which is important in mathematics and other sciences, business and engineering. Derivations also are used in quantum physics (see, [5]). Derivations can be added and subtracted and we still get a derivation, but when we compose a derivation with itself we do not necessarily get a derivation. In 1957 in [7] Posner proved that if \( d_1 \) and \( d_2 \) are two non-zero derivations on a prime ring whose characteristic is not 2, then \( d_1d_2 \) is not a derivation. Thus in a prime ring \( R \) whose characteristic is not 2 if \( d_2 \) is a derivation then \( d \) must be zero. In particular when \( d_2 \) is the zero derivation then \( d = 0 \). This means that the only nilpotent derivation with degree of nilpotency 2 on a prime ring whose characteristic is not 2 is the zero derivation. The usual derivative operator is additive and \((fg)' = f'g + fg'\). This definition has been generalized for every ring as follows. For a ring \( R \), an additive map \( d: R \rightarrow R \) is called a derivation on \( R \) if it satisfies the product rule \( d(ab) = d(a)b + ad(b) \). Many authors of the have studied centralizing derivations, endomorphisms, and some related additive mappings. Matej Bresar [9] proved, let \( R \) be a ring with center \( Z(R) \), mapping \( F \) of \( R \) into itself is called centralizing if \( F(x)x - xF(x) \in Z(R) \) for all \( x \in R \), then every additive centralizing mapping \( F \) of a von Neumann algebra \( R \) is of the form \( F(x) = cx + \zeta(x), x \in R \), where \( c \in Z(R) \) and \( \zeta \) is an additive mapping from \( R \) into \( Z(R) \), and also consider \( \alpha \)-derivations and some related mappings, which are centralizing on rings and Banach algebras. The history of commuting and centralizing mappings goes back to (1955) when Divinsky [10] proved that a simple Artinian ring is commutative if it has a commuting nontrivial automorphism. Two years later, Posner [7] has proved that the existence of a non-zero centralizing derivation on prime ring forces the ring to be commutative (Posner's second theorem). Muhammad A.C. and Mohammed S.S. [11] proved, let \( R \) be a semiprime ring and \( d: R \rightarrow R \) a mapping satisfy \( d(xy) = xyd(y) \) for all \( x,y \in R \). Then \( d \) is a centralizer. Muhammad A.C. and A. B.Thaheem [12] proved, let \( d \) and \( g \) be a pair of derivations of semiprime ring \( R \) satisfying \( d(y)x + xg(x) \in Z(R) \), then \( cd \) and \( cg \) are central for all \( c \in Z(R) \). B.Zalar [13] has proved, let \( R \) be a 2-torsion free semiprime ring and \( d: R \rightarrow R \) an additive mapping which satisfies \( d(x^2) = d(x)x \) for all \( x \in R \). Then \( d \) is a left centralizer. Recently, Mehsin Jabel [18,19,20] proved some results concerning generalized derivations on prime and semiprime rings.

In this paper we study and investigate some results concerning derivation \( d \) on semiprime ring \( R \), we give some results about that.

2. Preliminaries

Throughout this paper will represent an associative ring with identity the center \( Z(R) \). We recall that \( R \) is semiprime if \( xRx = (o) \) implies \( x=0 \) and it is prime if \( xRy = (o) \) implies \( x=0 \) or \( y=0 \). A prime ring is semiprime but the converse is not true in general. A ring \( R \) is 2-torsion free in case \( 2x = o \) implies \( x = o \) for any \( x \in R \). An additive mapping \( d: R \rightarrow R \) is called a derivation if \( d(xy) = d(x)y + xd(y) \) holds for all \( x,y \in R \). A mapping \( d \) is called centralizing if \( [d(x),x] \in Z(R) \) for all \( x \in R \), in particular, if \( [d(x),x] = 0 \) for all \( x \in R \), then it is called commuting, and is called central if \( d(x) \in Z(R) \) for all \( x \in R \). Every central mapping is obviously commuting but not conversely in general. In [14] Q. Deng and H.E. Bell extended the notion of commutativity to one of \( n \)-commutativity, where \( n \) is an arbitrary positive integer, by...
defining a mapping \( d \) to be \( n \)-commuting on \( U \) if 
\[
[x^d, (dx)] = 0
\]
for all \( x \in U \), where \( U \) is a non empty subset of \( R \). A biadditive mapping \( B : R \times \longrightarrow R \) is called a biderivation if for every \( u \in R \) the mappings \( x \rightarrow B(x, u) \) and \( x \rightarrow B(u,x) \) are derivations of \( R \). For any semiprime ring \( R \) one can construct the ring of quotients \( Q \) of \( R \) \([15]\). As \( R \) can be embedded isomorphically in \( Q \), we consider \( R \) as a subring of \( Q \). If the element \( a \in Q \) commutes with every element in \( R \) then \( q \) belongs to \( C \), the center of \( Q \). \( C \) contains the centroid of \( R \) and it is called the extended centroid of \( R \). In general, \( C \) is a von Neumann regular ring, and \( C \) is a field if and only if \( R \) is prime \([15]\). For any semiprime ring \( R \) one can construct the ring of quotients \( Q \) of \( R \) \([15]\), the equation (1), reduces to 
\[
\left[ d^a (x) o(y), z \right] + 2 [x, r] \left( d^a (y) o(z) \right) = 0
\]
for all \( x, y, z \in R \). 

According to our main relation \( d(xo(y)+yo(z)) \in Z(R) \), replacing \( y \) by \( (yo(z)) \), we get 
\[
\left[ d^a (y) o(z), x \right] + 2 [y, r] \left( d^a (z) o(x) \right) = 0
\]
for all \( x, y, z \in R \).

Replacing \( x \) by \( (yo(z)) \) in (2), we get 
\[
\left[ d^a ((yo)(z)) o(yo), r \right] + 2 [yo(z), r] \left( d^a (y) o(z) \right) = 0
\]
for all \( y, z, r \in R \).

Form the main relation \( d^a (xo(y)+yo(z)) \in Z(R) \) for all \( x, y, z \in R \), we obtain 
\[
\left[ d^a (x) o(yo), r \right] + 2 [x, r] \left( d^a (y) o(z) \right) = 0
\]
for all \( x, y, z \in R \).

Then substituting \( a=0 \) in (3), we get 
\[
2 \left[ d^a (yo)(z), r \right] \left( d^a (y) o(z) \right) = 0
\]
for all \( z, y, r \in R \). 

Then 
\[
2 \left[ d^a (yo)(z) r - d^a (y) o(z) r + (yo) rd^a (z) \right] + [yo, r] \left( yo \right) = 0
\]
for all \( y, z, r \in R \). 

Replacing \( r \) by \( (yo) \) using (4) in above equation, we get 
\[
2 d^a (yo)(z) r - 2d^a (y) o(z) r + (yo) rd^a (z) + [yo, r] \left( yo \right) = 0
\]
for all \( z, y, r \in R \). 

Reidt-multiplying by \( [x, r] \), we obtain 
\[
2 d^a (yo)(z) r - 2 d^a (y) o(z) r + (yo) rd^a (z) + [yo, r] \left( yo \right) = 0
\]
for all \( y, z, r \in R \).

According to Lemma 2, there exists an ideal \( U \) of \( R \) such that 
\[
d^a (yo)(z) r - 2 d^a (y) o(z) r + (yo) rd^a (z) + [yo, r] \left( yo \right) = 0
\]
for all \( y, z, r \in R \). 

Apply d to both sided, we obtain 
\[
\left[ d^a (x), y \right] + \left[ d^a (y), x \right] = 0
\]
for all \( x, y \in U \), \( y e R \). Since \( U \) is a central ideal, we get 
\[
\left[ d^a (x), y \right] = 0
\]
for all \( x, y \in U \), \( y e R \). Thus, we get that d is commuting of \( R \), which implies 
\[
\left[ z, d(z) \right] = 0
\]
for all \( z e R \). 

Hence, we see that the mapping \( (z, y) \rightarrow \left[ d(z), y \right] \) is a biderivation. By Lemma 4, there exist an idempotent \( e \in C \) and an element \( \mu \in C \) such that the algebra \( (1 - e)R \) is commutative and 
\[
e B(x, y) = \mu [x, y] \quad \text{for all } x, y \in R.
\]

Lemma 5 \([21]\), Main Theorem

Let \( R \) be a semiprime ring, \( d \) a non-zero derivation of \( R \), and \( U \) a non-zero left ideal of \( R \). If for some positive integers \( t_0, t_1, \ldots, t_n \) and all \( x \in U \), the identity 
\[
[\ldots[[d(x_0)^{t_0}d(x_{t_0})d(x_{t_0})\ldots]]d(x_n)^{t_n} = 0
\]
holds, then either \( d(U) = 0 \) or \( U \) and \( d(R)U \) are contained in non-zero central rings \( R \). In particular when \( R \) is a prime ring, \( R \) is commutative.

3. The main results

Theorem 3.1.

Let \( R \) be a 2-torsion free semiprime ring and \( d: R \rightarrow R \) be a derivation on \( R \) such that \( d(xo(y)+yo(z)) \in Z(R) \) for all \( x, y, z \in R \). Then \( d^a (y) o(z) = 0 \) for all \( x, y, z \in R \).

Proof: At first we suppose that \( d \) in non-zero derivation in our main relation \( d(xo(y)+yo(z)) \in Z(R) \) for all \( x, y, z \in R \), then we have 
\[
d^{(x)} d(xo(y)) + (xy) \in Z(R) \]
for all \( x, y, z \in R \). Replacing \( y \) by \( (yo(z)) \), we get 
\[
d^a (d(xo(yo(z)))+xo(yo(z))) \in Z(R) \]
for all \( x, y, z \in R \). Then 
\[
d^a (xoyo(z))d(x)+2(xd^a(yo(z))yo(z)) = 0
\]
in \( Z(R) \) for all \( x, y, z \in R \).

According to our main relation \( d(xo(y)+yo(z)) \in Z(R) \), the equation (1), reduces to 
\[
\left[ d^a (x) o(yo), z \right] + 2 [x, r] \left( d^a (y) o(z) \right) = 0
\]
for all \( x, y, z \in R \).
When \(d = 0\), we have \([x,y] \in Z(R)\) for all \(x,y \in R\), then by same technique in the first part of proof, we completes the proof.

**Theorem 3.5.**

Let \(R\) be a 2-torsion free semiprime ring and \(d: R \to R\) be a derivation on \(R\) such that \(d^n(xoy) + [x,y] \in Z(R)\) for all \(x,y \in R\), then there exist \(C\) and an additive mapping \(\zeta: R \to C\) such that \(d(x) = \lambda x + \zeta(x)\) for all \(x \in R\), where \(n\) is a fixed positive integer.

**Proof:** We have the relation \(d^n(xoy) + [x,y] \in Z(R)\) for all \(x,y \in R\). Then applying Lemma 3, we obtain \([d^n(xoy)], [x,y] \in Z(R)\) for all \(x,y \in R\). Then according to (7) and (6), we arrive at

\[
\left[d^n (x,y), r\right] = 0 \quad \text{for all } x,y,r \in R. \tag{9}
\]

Again right-multiplying of relation (7), (by \(x\)) gives \(d^n(xoy) = 0\) for all \(x,y \in R\). Then according to (9) and (6), we arrive at \(d^n(xoy) = 0\) for all \(x,y \in R\).

So, if we continue by same technique, arrive to the relation \(d^n(xoy) = 0\) for all \(x,y \in R\), which completes the proof of the corollary.

**Corollary 3.3.**

Let \(R\) be a 2-torsion free semiprime ring and \(d: R \to R\) be a non-zero derivation on \(R\) such that \(d^n(xoy) = 0\) for all \(x,y \in R\), then \(R\) contains a fixed positive integer.

**Theorem 3.4.**

Let \(R\) be a 2-torsion free semiprime ring and \(d: R \to R\) be a derivation on \(R\) such that \(d^n(xoy) = 0\) for all \(x,y \in R\), then there exist \(C\) and an additive mapping \(\zeta: R \to C\) such that \(d(x) = \lambda x + \zeta(x)\) holds for all \(x \in R\), where \(n\) is a fixed positive integer.

**Proof:** We have the relation \(d^n(xoy) = 0\) for all \(x,y \in R\). Now we suppose that \(d\) is a non-zero derivation on \(R\), then \(d^n(xoy) = 0\) for all \(x,y \in R\), then \([d^n(x,y)], [x,y] = 0 \quad \text{for all } x,y \in R. \tag{10}

Replacing \(r\) by \([x,y]\), we obtain \([d^n(x,y)], [x,y] = 0 \quad \text{for all } x,y \in R. \tag{11}\)

Applying Lemma 3, we obtain \(d^n([x,y]) = 0 \quad \text{for all } x,y \in R. \tag{12}\)

Replacing \(r\) by \([x,y]\) in (11) with apply Lemma 1, we have \([x,y] = 0 \quad \text{for all } x,y \in R. \tag{13}\)

Applying Lemma 3, we obtain \([d^n([x,y])], [x,y] = 0 \quad \text{for all } x,y \in R. \tag{14}\)

The substitution this fact in the relation (10), gives

\[
[d^n(x,y)], [x,y] = 0 \quad \text{for all } x,y \in R. \tag{15}\]

Replacing \(r\) by \(d(z)\) in (11) with apply Lemma 1, we arrive to \(d(z) = 0 \quad \text{for all } z \in R. \tag{16}\)

Applying \(d\) to both sides, we obtain \([d^n(x,y)], [x,d(y)] = 0 \quad \text{for all } x,y \in U, y \in R. \tag{17}\)

Hence, we see that the mapping \((x,y) \mapsto [x,d(y)]\) is a biderivation. By Lemma 4, there exist an idempotent \(e \in C\) and an element \(\mu \in C\) such that the algebra \((1 - e)R\) is commutative (hence, \((1 - e)R \subseteq C\), and \(e[d(z)], [x,y] = e\mu[x,y]\) holds for all \(x \in R\), by this we complete our proof.
holds for all $z,y \in R$. Thus, $\varepsilon d(z) - \mu z$ commutes with every element in $R$, so that $\varepsilon d(z) - \mu z \in C$. Now, let $\varepsilon d(z) = \mu z$ and define a mapping $\zeta$ by $\zeta(z) = (\varepsilon d(z) - \mu z) + (1 - \varepsilon) d(z)$. Note that $\zeta$ maps in $C$ and that $d(z) = \lambda z + \zeta(z)$ holds for every $z \in R$, by this we complete our proof. When $d=0$, we have $(xoy)\in Z(R)$ for all $x,y \in R$, the we completes the proof of the theorem by same method in part second of Theorem 3.1.

**Corollary 3.7.**

Let $R$ be a 2-torsion free semi prime ring and $d: R \to R$ be a derivation on $R$. If $R$ admits $d$ to satisfy $d^n[(x,y)] + d^q[(x,y)] + (xoy) \in Z(R)$ for all $x,y \in R$, then $d(x) = \lambda x + \zeta(x)$ holds for every $x \in R$, where $\lambda$ and $\zeta$ are fixed positive integers.

**Proof:**

Let $R$ be a 2-torsion free semi prime ring and $d: R \to R$ be a derivation on $R$. If $R$ admits $d$ to satisfy $d^n[(x,y)] + d^q[(x,y)] + (xoy) \in Z(R)$ for all $x,y \in R$, then $d(x) = \lambda x + \zeta(x)$ holds for every $x \in R$, where $\lambda$ and $\zeta$ are fixed positive integers.

**Theorem 3.8.**

Let $R$ be a 2-torsion free semi prime ring and $d: R \to R$ be a derivation on $R$. If $R$ admits $d$ to satisfy $d^n[(x,y)] + d^q[(x,y)] + (xoy) \in Z(R)$ for all $x,y \in R$, then $d(x) = \lambda x + \zeta(x)$ holds for every $x \in R$, where $\lambda$ and $\zeta$ are fixed positive integers.

**Proof:**

Let $R$ be a 2-torsion free semi prime ring and $d: R \to R$ be a derivation on $R$. If $R$ admits $d$ to satisfy $d^n[(x,y)] + d^q[(x,y)] + (xoy) \in Z(R)$ for all $x,y \in R$, then $d(x) = \lambda x + \zeta(x)$ holds for every $x \in R$, where $\lambda$ and $\zeta$ are fixed positive integers.

**Theorem 3.9.**

Let $R$ be a 2-torsion free semi prime ring and $d: R \to R$ be a derivation on $R$. If $R$ admits $d$ to satisfy $\varepsilon d[(x,y)] + \mu d[(x,y)] + (xoy) \in Z(R)$ for all $x,y \in R$, then there exist $C$ and an additive mapping $\zeta: R \to C$ such that $d(x) = \lambda x + \zeta(x)$ holds for every $x \in R$, where $\lambda$ and $\zeta$ are fixed positive integers.

**Proof:**

Let $R$ be a 2-torsion free semi prime ring and $d: R \to R$ be a derivation on $R$. If $R$ admits $d$ to satisfy $\varepsilon d[(x,y)] + \mu d[(x,y)] + (xoy) \in Z(R)$ for all $x,y \in R$, then there exist $C$ and an additive mapping $\zeta: R \to C$ such that $d(x) = \lambda x + \zeta(x)$ holds for every $x \in R$, where $\lambda$ and $\zeta$ are fixed positive integers.
\[
R = \begin{pmatrix}
Z & Z \\
2Z & 2Z \\
Z & Z \\
2Z & 2Z
\end{pmatrix}, \quad a = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]

and \(d\) is the inner derivation induced by \(a\) (every derivation is inner derivation), that is, \(d(x) = [a, x]\), for all \(x \in R\). Then \(R\) is a non-commutative prime ring with \(\text{char } R = 2\), and \(d([x, y]) = [x, y] \in Z(R)\), for all \(x, y \in R\). Then by similar approach we can show otherwise.

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**References**