An Inverse Coefficient Problem for a Parabolic Equation under Nonlocal Boundary and Integral Overdetermination Conditions

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Abstract This paper investigates the inverse problem of simultaneously determining the time-dependent thermal diffusivity and the temperature distribution in a parabolic equation in the case of nonlocal boundary conditions containing a real parameter and integral overdetermination conditions. Under some consistency conditions on the input data the existence, uniqueness and continuously dependence upon the data of the classical solution are shown by using the generalized Fourier method.

Keywords: heat equation, inverse problem, nonlocal boundary condition, integral overdetermination condition, Fourier method


1. Introduction

Suppose that one needs to determine simultaneously the temperature distribution \( u(x,t) \) as well as thermal diffusivity coefficient \( a(t) \) satisfying the heat equation

\[
\frac{\partial u}{\partial t} = a(t) \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 1, \quad 0 < t \leq T.
\]

with initial condition

\[
u(x,0) = \varphi(x), \quad 0 \leq x \leq 1,
\]

the boundary conditions

\[
u(0,t) = 0, \quad u_x(0,t) = u_x(1,t) + \alpha u(1,t), \quad 0 \leq t \leq T,
\]

and the energy condition

\[
\int_0^1 u(x,t)dx = E(t), \quad 0 \leq t \leq T.
\]

Where the parameter \( \alpha \) in an arbitrary real number and \( f(x,t), \varphi(x), E(t) \) are given functions. The nonlocal second boundary condition in (1.3) is the main specific feature of this problem; for \( \alpha = 0 \), it acquires the form

\[
\frac{\partial u}{\partial x}(0,t) = u_x(1,t), \quad u(0,t) = 0, \quad 0 \leq t \leq T,
\]

and was comprehensively studied in [1], are well-known as the Samarskii-Ionkin conditions, whilst (1.4) specifies an integral additional specification of the energy. The problem of finding a pair \((a(t), u(x,t))\) will be called an inverse problem.

Denote the domain \( Q_T \) by

\[
Q_T = \{(x,t) : \ 0 < x < 1, \ 0 < t \leq T\}.
\]

Definition 1. The pair \((a(t), u(x,t))\) from the class

\[
C[0,T] \times C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q_T})\]

for which conditions (1.1)-(1.4) are satisfied and \( a(t) \geq 0 \) on the interval \([0,T]\), is called the classical solution of the inverse problem (1.1)-(1.4):

Nonlocal boundary conditions like (1.4) arise from many important applications in heat transfer, thermoelasticity, control theory, life sciences and plays an important role in engineering and physics [8-17]. For example, for heat propagation in a thin rod in which the law of variation \( E(t) \) of the total quantity of heat in the rod is given in [1]. Various statements of inverse problems on determination of thermal coefficient in one-dimensional heat equation were studied in [15,16,17]. It is important to note that in the papers [15,16] the time-dependent thermal coefficient is determined from nonlocal overdetermination condition data. Besides, in [12,15] the coefficients of the heat equations are determined in the case of nonlocal boundary conditions.

In this paper the source control parameter \( a(t) \) needs to be determined by thermal energy \( E(t) \), and the existence and uniqueness of the classical solution of the problem (1.1) - (1.4) is reduced to fixed point principles by applying the Fourier method. The boundary conditions (1.3) admit the expansions by the system of eigenfunctions and associated functions corresponding to the spectral problem.
The paper is organized as follows. In Section 2, the eigenvalues and eigenfunctions of the auxiliary spectral problem and some of their properties are introduced by applying the Fourier method to the problem (1.1) - (1.4). In Section 3, the existence and uniqueness of the solution of the inverse problem (1.1) - (1.4) is proved. Finally, the continuous dependence upon the data of the solution of the inverse problem is shown in Section 4.

2. The Auxiliary Spectral Problem

The use of the Fourier method for solving problem (1.1)-(1.3) leads to the spectral problem for the operator given by the differential expression and boundary conditions

\[
\begin{cases}
LX(x) = X''(x) = -\lambda X(x), & 0 \leq x \leq 1 \\
X'(0) = X'(1) + \alpha X(1), & X(0) = 0.
\end{cases}
\] (2.1)

The boundary conditions in (2.1) are regular, but not strongly regular (2; pp: 66-67). The system of root functions of the operator is complete, but it does not form even a regular basis in \( L^2(0,1) \) [3]. However, as it was shown in [4], on the base of these eigenfunctions one can construct a basis allowing one to use the method of separation of variables for solving the initial-boundary value problem subject to the boundary condition (1.3).

In recent times a special attention has been paid to the study of direct and inverse problems for various classes of partial differential equations in particular cases of boundary conditions which are not strongly regular. Let us cite only those of them that are most close to the subject under consideration [5,6,7].

In this paper we propose to use the results described in [4] for solving the inverse problem (1.1) -(1.4). So, for construction of the basis of eigenfunctions of problem (2.1), we cite the necessary results from [4].

Consider the case in which \( \alpha \neq 0 \). We seek eigenvalues in the set of real numbers. Note that \( \lambda = 0 \) is not an eigenvalue, since problem (2.1) for this value of \( \lambda \) has only the trivial solution. Suppose that \( \lambda > 0 \). Then the eigenfunction should have the form \( X(x) = \sin \sqrt{\lambda} x \). By taking into account the nonlocal boundary condition, we obtain two equations

\[
\sin \sqrt{\lambda} = 0, \quad \cos \sqrt{\lambda} + \frac{\alpha}{\sqrt{\lambda}} \sin \sqrt{\lambda} x = 1.
\]

Solutions of the first equation form a series of eigenvalues and eigenfunctions of the operator (1.4) of the form

\[
\lambda^{(k)}_\alpha = (2k\pi)^2, \quad k = 1, 2, \ldots \quad (2.2)
\]

\[
X^{(k)}_\alpha(x) = \sin 2k\pi x, \quad k = 1, 2, \ldots
\]

The second equation can be represented as

\[
\tan \beta = \frac{\alpha}{2\beta}, \quad \beta = \frac{\sqrt{\lambda}}{2} > 0,
\]

with \( \alpha \neq 0 \), this equation has countably many solutions \( \beta_k \), satisfying the inequalities \( \pi k < \beta_k < \pi k + \pi/2 \), \( k = 0, 1, 2, \ldots \), for positive \( \alpha \):

\[
\frac{\alpha}{2k\pi} \left( 1 - \frac{1}{2k\pi} \right) + \frac{\alpha}{2k\pi} < \beta_k < \frac{\alpha}{2k\pi} \left( 1 + \frac{1}{2k\pi} \right)
\]

for sufficiently large \( k \). So, that there exists a second series of eigenvalues and eigenfunctions of the form

\[
\lambda^{(2)}_k = (2\beta_k)^2, \quad X^{(2)}_k(x) = \sin 2\beta_k x, \quad k = 0, 1, 2, \ldots \quad (2.3)
\]

This system is almost normed, but it does not form even a regular basis in \( L^2(0,1) \). The corresponding auxiliary system

\[
\begin{align*}
X_0(x) &= X^{(2)}_0(x)(2\beta)^{-1} \\
X_{2k}(x) &= X^{(1)}_k(x), \quad k = 1, 2, \\
X_{2k-1}(x) &= \left(X^{(2)}_k(x) - X^{(1)}_k(x)\right)(2(\beta_k - \pi k))^{-1}, \\
k &= 1, 2, \ldots
\end{align*}
\] (2.4)

Form a Riesz basis in \( L^2(0,1) \), and to find the biorthogonal system of \( \{X_k(k), k = 0, 1, 2, \ldots\} \), consider the adjoint differential operator \( L^* \) of \( L \) in the sense of the inner product of the space \( L^2(0,1) \):

\[
L^*Y(x) = Y'' = -\lambda Y(x), \quad 0 \leq x \leq 1.
\]

So, for two arbitrary twice continuously differentiable function \( X(x) \) and \( Y(x) \) on \([0,1]\) such that \( X'(0) = X'(1) + \alpha X(1), \quad X(0) = 0 \), we have

\[
(Lu, v) = \int_0^1 (-X''(x))Y(x)dx = \int_0^1 X(x)(-Y''(x))dx = (u, L^*v),
\]

this implies

\[
-\lambda X(1)Y(1) = -X'(1)Y(1) + X'(0)Y(0), \quad (2.5)
\]

we obtain \( [X'(1) - X(1)]Y(1) = X'(0)Y(0) \), by using \( X'(0) = X'(1) + \alpha X(1) \), we have \( [X'(1) - X(1)]Y(1) = [X'(1) + \alpha X(1)]Y(0) \), since \( \alpha \) is arbitrary, we get \( X(1) = Y(0) \). Now by using (2.5) and \( X'(0) = X'(1) + \alpha X(1) \), we obtain

\[
-\lambda X(1)Y(1) = -X'(1)Y(1) + [X'(1) + \alpha X(1)]Y(0) \Rightarrow -\lambda X(1)[Y(1) + \alpha Y(0)] = -X(1) [Y(1) - Y(0)],
\]

since \( Y(1) = Y(0) \), we get \( -\lambda X(1)[Y(1) + \alpha Y(0)] = 0 \), this implies \( Y(1) + \alpha Y(0) = 0 \). Then, the adjoint problem of (2.1) has the form

\[
\begin{align*}
L^*Y(x) &= Y''(x) - \lambda Y(x), \quad 0 \leq x \leq 1. \\
Y(1) + \alpha Y(0) &= 0, \quad Y(1) = Y(0).
\end{align*}
\] (2.6)

It is easy to verify that the eigenvalues of this problem are same as for the problem (2.1). The system of
Thus, we have

\[
y_k^{(2)}(x) = C_k^{(2)} \cos(\beta_k (1 - 2x)), \quad k = 0, 1, 2, \ldots;
\]

\[
y_k^{(1)}(x) = C_k^{(1)} \cos \left(\beta_k (1 - 2x) + \text{arctan} \left(\frac{\alpha}{2\pi k}\right)\right), \quad k = 0, 1, 2, \ldots
\]

where

\[
c_k^{(1)} = -2 \left(\sin \left(\text{arctan}\left(\frac{\alpha}{2\pi k}\right)\right)\right)^{-1}, \quad k = 0, 1, 2, \ldots
\]

\[
c_k^{(2)} = 2 \left(\sin \beta_k \left(1 + \frac{\sin(2\beta_k)}{2\beta_k}\right)\right)^{-1}, \quad k = 0, 1, 2, \ldots
\]

For the function system (2.4), there exists a biorthogonal normalized system, given by

\[
Y_0(x) = Y_0^{(2)}(x)(2\beta_0)
\]

\[
Y_{2k}(x) = Y_{2k}^{(2)}(x) + Y_{2k}^{(1)}(x), \quad k = 1, 2, \ldots \tag{2.7}
\]

\[
Y_{2k-1}(x) = (2(\beta_k - \pi k))Y_{2k}^{(2)}(x), \quad k = 1, 2, \ldots
\]

It is shown in [4] that the systems (2.4) and (2.7) form biorthogonal systems on the interval \([0, 1]\), i.e.

\[
\left(X_i, Y_j\right) = \int_0^1 X_i(x)Y_j(x)dx = \delta_{ij} = \begin{cases}0, & i \neq j \\ 1, & i = j \end{cases}
\]

3. Existence and Uniqueness of the Solution of the Inverse Problem

We have the following assumptions on the data of the problem (1.1) - (1.4):

\[
(A_1)_{h}: \varphi(x) \in C^2[0, 1];
\]

\[
(A_2): \begin{cases}
(A_2)_1: \varphi_0(0) = \varphi_1(1) + \alpha \varphi(1), \varphi(0) = 0; \\
(A_2)_2: \varphi_0 \geq 0, \varphi_{2k-1} \geq 0, \quad k = 1, 2, \ldots,
\end{cases}
\]

\[
(A_3): \begin{cases}
(A_3)_1: E(t) \in C^1[0, T]; \\
(A_3)_2: f(x, t) = f_s(x, t) + \alpha \varphi f_s(x, t), \\
f(0, t) = 0, 0 \leq t \leq T;
\end{cases}
\]

where \(\varphi_k = \int_0^1 \varphi(x)Y_k(x)dx, \quad F_k(t) = \int_0^t f(x, t)Y_k(x)dx, \quad k = 1, 2, \ldots\)

The main result is presented as follows.

**Theorem 1.** Let the assumptions (A1)-(A3) be satisfied. Then the inverse problem (1)-(4) has a unique classical solution.

**Proof.** By applying the standard procedure of the Fourier method, Any solution of equation (1.1) can represented as:

\[
u(x, t) = \sum_{k=0}^{\infty} u_k(t)X_k(x), \tag{3.1}
\]

So, by replacing \(u(x, t)\) in equation (1.1) by the representation (3.1), we get

\[
\sum_{k=0}^{\infty} u_k(t)X_k(x) = a(t)\sum_{k=0}^{\infty} u_k(t)X_k(x) + f(x, t), \quad \tag{3.2}
\]

multiplying the equation (3.2) by \(Y_k(x)\), and integrating over \((0, 1)\), we get the following system of equations:

\[
u_0(t) + \lambda_0^{(2)} a(t)u_0(t) = F_0(t),
\]

\[
u_{2k}(t) + \lambda_k^{(2)} a(t)u_{2k}(t) = F_{2k}(t), \quad k = 1, 2, \ldots
\]

\[
u_{2k-1}(t) + \lambda_k^{(2)} a(t)u_{2k-1}(t) = F_{2k-1}(t), \quad k = 1, 2, \ldots
\]

Substituting the solution of this system of equations and initial condition (1.2) in (3.1), we obtain the solution of the problem (1.1) - (1.3) in the following form:

\[
u(x, t) = \left[\begin{array}{c}
\varphi_0 e^{-\lambda_0^{(2)} \int_0^t a(s)ds} \\
\int_0^t F_0(\tau) e^{-\lambda_0^{(2)} \int_0^\tau a(s) ds} d\tau
\end{array}\right] X_0(x)
\]

\[
+ \sum_{k=1}^{\infty} \left[\begin{array}{c}
\varphi_{2k} e^{-\lambda_k^{(1)} \int_0^t a(s) ds} \\
\int_0^t F_{2k}(\tau) e^{-\lambda_k^{(1)} \int_0^\tau a(s) ds} d\tau
\end{array}\right] X_{2k}(x)
\]

\[
+ \sum_{k=1}^{\infty} \left[\begin{array}{c}
\varphi_{2k-1} e^{-\lambda_k^{(2)} \int_0^t a(s) ds} \\
\int_0^t F_{2k-1}(\tau) e^{-\lambda_k^{(2)} \int_0^\tau a(s) ds} d\tau
\end{array}\right] X_{2k-1}(x)
\]

Under the conditions (A1) and (A2), the series (3.3) and its x-partial derivative are uniformly convergent in \(\overline{Q_T}\) since their majorizing sums are absolutely convergent. Therefore, their sums \(u(x, t)\) and \(u_x(x, t)\) are continuous in \(\overline{Q_T}\). In addition, the t-partial derivative and the xx-second-order partial derivative series are uniformly convergent in \(\overline{Q_T}\). Thus, we have \(u(x, t) \in C^2(\overline{Q_T}) \cap C^1(\overline{Q_T})\). In addition, \(u_t(x, t)\) is continuous in \(\overline{Q_T}\). Differentiating (1.4) under the assumption (A2), we obtain

\[
\int_0^t u_t(x, t)dx = E'(t), \quad \tag{3.4}
\]

\[
0 \leq t \leq T,
\]

using (3.3) and (3.4), yield

\[
a(t) = P[a(t)], \quad \tag{3.5}
\]

where
From the last inequalities, we have \[
\int \lambda^2 \phi_k e^{-\lambda^2 \int_0^t u(s)ds} \leq \lambda \max_{t \in [0,T]} |a(t) - b(t)|.
\]
Let us denote
\[
C^+[0,T] = \{a(t) \in C[0,T] : a(t) \geq 0\}.
\]
It is easy to verify that under conditions \((A_1)_3, (A_2)_3\) and \((A_3)_1\),
\[
P : C^+[0,T] \to C^+[0,T].
\]
Let us show that \(P\) is a contraction mapping in \(C^+[0,T]\). Then, we have for \(a(t), b(t) \in C^+[0,T]\),
\[
P[a(t)] - P[b(t)] \leq K[a(t)]N[b(t)] - N[a(t)]
\]
and
\[
\|N[a(t)] - N[b(t)]\| \leq \xi \max_{t \in [0,T]} |a(t) - b(t)|,
\]
where
\[
K[a(t)] = \left(\frac{1 - \cos(2\beta_0)}{4\beta_0^2}\right)F_0(t) e^{-\lambda^2 \int_0^t u(s)ds}
\]
and
\[
N[a(t)] = \lambda \max_{t \in [0,T]} |a(t) - b(t)|.\]

Since \(a(t) \geq 0, b(t) \geq 0\), the estimates
\[
\left| e^{-\lambda^2 \int_0^t u(s)ds} - e^{-\lambda^2 \int_0^t b(s)ds} \right| \leq \xi \max_{t \in [0,T]} |a(t) - b(t)|;
\]
are true by using the mean value theorem, where \(\xi = \lambda \max_{t \in [0,T]} |a(t)|\). From the last inequalities, we obtain
\[
\|P[a(t)] - P[b(t)]\| \leq \xi \max_{t \in [0,T]} |a(t) - b(t)|,
\]
and
\[
\|N[a(t)] - N[b(t)]\| \leq \xi \max_{t \in [0,T]} |a(t) - b(t)|.
\]
In the case \( \xi \left( \frac{c_3c_2 + c_4c_4}{c_2^2} \right) < 1 \). Equation (3.10) has a unique solution \( a(t) \in C^+ [0, T] \), by the Banach fixed point theorem.

Now, let us show that the solution \( (a, u) \), obtained for (1)-(4), is unique. Suppose that \((b, v)\) is also a solution pair of (1:1)-(1:4). Then from the representation (3.3) of the solution, we have:

\[
\begin{align*}
\int_{0}^{t} F_{0}(t) e^{-\int_{0}^{s} J_{0}^{2}(t_{s})^{2} ds} ds = X_{0}(x) \\
+ \int_{0}^{t} F_{2k}(t) e^{-\int_{0}^{s} J_{0}^{2}(t_{s})^{2} ds} ds = X_{2k}(x) \\
+ \sum_{k=1}^{\infty} \int_{0}^{t} F_{2k-1}(t) e^{-\int_{0}^{s} J_{0}^{2}(t_{s})^{2} ds} ds = X_{2k-1}(x).
\end{align*}
\]

From the equation (3.5), and (3.10), we obtain

\[
\max_{t \in [0, T]} \left| a(t) - b(t) \right| \leq \xi \left( \frac{c_3c_2 + c_4c_4}{c_2^2} \right) \max_{t \in [0, T]} \left| a(t) - b(t) \right|,
\]

since \( \xi \left( \frac{c_3c_2 + c_4c_4}{c_2^2} \right) < 1 \), implies that \( a = b \). By substituting \( a = b \) into (3.11), we get \( u = v \).

Theorem 1 has been proved.

4. Continuous Dependence of \((a,u)\) upon the Data

**Theorem 2.** Under assumption (A1) - (A3), the solution \((a,u)\) depends continuously upon the data.

**Proof.** Let \( \Phi(\varphi, F,E) \) and \( \Phi(\bar{\varphi}, \bar{F}, \bar{E}) \) be two sets of the data, which satisfy the assumptions (A1) - (A3). Then there exist positive constants \( M_{i}, i = 1, 2, 3 \) such that.

\[
\| \Phi \|_{L^{2}[0,1]} \leq M_1, \quad \| F \|_{L^{2}[0,1]} \leq M_2, \quad \| E \|_{C^{1}[0,T]} \leq M_3, (4.1)
\]

\[
\left\| \int_{0}^{\min_{t \in [0,T]} \| a(t) - b(t) \| \leq \xi \left( \frac{c_3c_2 + c_4c_4}{c_2^2} \right) \max_{t \in [0, T]} \left| a(t) - b(t) \right|,
\]

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\[
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\]

\[
\left\| \int_{0}^{\min_{t \in [0,T]} \| a(t) - b(t) \| \leq \xi \left( \frac{c_3c_2 + c_4c_4}{c_2^2} \right) \max_{t \in [0, T]} \left| a(t) - b(t) \right|,
\]

since \( \xi \left( \frac{c_3c_2 + c_4c_4}{c_2^2} \right) < 1 \), implies that \( a = b \). By substituting \( a = b \) into (3.11), we get \( u = v \).

Theorem 1 has been proved.
The inverse problem regarding the simultaneously identification of the time-dependent thermal diffusivity and the temperature distribution in one-dimensional heat equation with nonlocal boundary condition and integral overdetermination conditions has been considered. Where the nonlocal boundary conditions containing a real parameter. For the zero value of the parameter, this conditions is well known as the Samarskii-Ionkin conditions and has been comprehensively studiedand. This inverse problem has been investigated from both theoretical. In this study the conditions for the existence, uniqueness and continuous dependence upon the data of the problem have been established.

References