Anisotropic Bianchi Type-II Viscous Fluid Models with Time-Dependent Gravitational and Cosmological Constants

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Abstract

This paper deals with totally anisotropic Bianchi type-II cosmological models filled with a bulk viscous fluid in the presence of time-varying gravitational and cosmological constants. Exact solutions of the field equations are obtained by applying a special law of variation for Hubble’s parameter which yields a constant value of the deceleration parameter. Two different physically viable models of the universe are presented in two types of cosmologies, one with power-law expansion and other with exponential expansion. Cosmological model with power-law expansion has an initial big-bang type singularity at \( t = 0 \), whereas the model with exponential expansion has a singularity in infinite past. The physical and dynamical properties of the models are discussed.

Keywords: Bianchi II, bulk viscosity, gravitational constant, cosmological constant


1. Introduction

In recent years, experimental studies of CMBR and many other effects have stimulated much theoretical interest in anisotropic cosmological models. The interest has been due to the realization that the standard cosmological models, which are in very good agreement with the present-day universe, do not provide a clear description of the early phase of evolution of the universe, while physically realistic description of the early evolution of universe is best given by anisotropic models. The spatially homogeneous and anisotropic Bianchi models present a ‘middle way’ between FRW models and completely inhomogeneous and anisotropic models, and thus play an important role in modern cosmology. This is due to the fact that close to the big-bang singularity, neither the assumption of spherical symmetry nor of isotropy can be strictly valid. This stimulates researchers to obtain exact anisotropic solutions of Einstein’s field equations, which yield cosmologically acceptable physical model of the universe. There has been considerable interest in the study of spatially homogeneous and anisotropic cosmological models of Bianchi type I-IX [1].


In order to study the evolution of universe, many authors constructed cosmological models containing viscous fluids. The presence of viscosity in the fluid introduces many interesting features in the dynamics of homogeneous cosmological models. The roles played by the viscosity and the consequent dissipative mechanism in cosmology have been discussed by several authors. The heat represented by the large entropy per baryon in the microwave background provides a useful clue to the early universe, and a possible explanation for this huge entropy per baryon is that it was generated by physical dissipative processes acting at the beginning to the evolution of the universe. These dissipative processes may indeed be responsible for the smoothing out of initial anisotropies [12]. Misner [13] suggested that neutrino viscosity acting in the early era might have considerably reduced the present anisotropy of the blackbody radiation during the

The cosmological constant \( \Lambda \) has been one of the most mysterious and fascinating objects for both cosmologist and theoretical physicists since its introduction in cosmology by Albert Einstein. The explanation of its origin is considered one of the most fundamental issues for our comprehension of general relativity and quantum field theory. In recent years there has been a lot of interest in the study the role of the cosmological constant \( \Lambda \) at every early and later stages of the evolution of the universe. A wide range of observations suggest that the universe possesses a non-zero cosmological constant. The \( \Lambda \) term has been interpreted in terms of Higgs scalar field by Bergmann [16]. Drietlein [17] suggested that the mass of Higgs boson is connected with \( \Lambda \) being a function of temperature and is related to the process of broken symmetries, and therefore it could be a function of time in an expanding universe. In quantum field theory, the cosmological constant is considered as the vacuum energy density. The general speculation is that the universe might have been created from an excited vacuum fluctuation (absence of inflationary scenario) followed by super cooling and reheating subsequently due to vacuum energy.

In general relativity the Newtonian constant of gravitation \( G \) plays the role of coupling constant between geometry of the space-time and matter in Einstein’s field equations. Variation of Newton’s gravitational parameter \( G \) was originally suggested by Dirac on the basis of his large number hypothesis [18]. The implications of time varying \( G \) are important only at the early stages of evolution of the universe. There have been numerous modifications of general relativity in which \( G \) varies with time in order to achieve possible unification of gravitation and elementary particle physics. A number of authors such as Beesham [19], Berman [20], Kalilgas et al. [21], Arbab [22], Abdussattar and Vishwakarma [23] proposed the linking of variation of \( G \) and \( \Lambda \) within the framework of general relativity and studied several models with the Friedmann-Robertson-Walker (FRW) metric. This approach is appealing since it leaves the form of Einstein equations formally unchanged by allowing a variation of \( G \) to be accompanied by change in \( \Lambda \). Barrow and Parsons [24] presented a detailed analysis of FRW universes in a wide range of scalar-tensor theories of gravitation. Pradhan and Chakraborty [25] discussed LRS Bianchi type models with the varying gravitational and cosmological constants. Pradhan and Yadav [26] presented bulk viscous anisotropic cosmological models with time-varying \( G \) and \( \Lambda \). Singh and Kotambkar [27] discussed cosmological models with time-dependent \( G \) and \( \Lambda \) in higher dimensions. Pradhan et al. [28] obtained FRW universes with variable \( G \) and \( \Lambda \). Singh et al. [29] discussed Bianchi type-I cosmological models with variable \( G \) and \( \Lambda \) term in general relativity. Padmanabhan and Chitre [30] investigated the effect of bulk viscosity on the evolution of the universe at large. Beesham [31] presented a cosmological model with variable \( G \) and \( \Lambda \) in the presence of a bulk viscous fluid. Singh and Kale [32] discussed Bianchi type-I, Kantowski-Sachs and Bianchi type-III cosmological models filled with bulk viscous fluid together with variable \( G \) and \( \Lambda \). Singh et al. [33], Pradhan and Kumhar [34] studied anisotropic viscous fluid models with time varying \( \Lambda \) term. Singh et al. [35] obtained solution of Einstein’s Field equations with variable \( G \) and \( \Lambda \) in the presence of a perfect fluid for Bianchi type-III universe. Chakraborty and Roy [36] derived cosmological models with bulk viscosity with time varying \( G \) and \( \Lambda \). Bali and Tinker [37] presented Bianchi type-II bulk viscous fluid cosmological models with variable \( G \) and \( \Lambda \) by assuming the power law forms of the scale factors. Verma and Shri Ram [38,39,40] discussed bulk viscous fluid hyper-surface homogeneous, Bianchi type-III and Bianchi type-VI0 cosmological models with time-dependent \( G \) and \( \Lambda \). Amirhashchi [41] presented LRS Bianchi type-II stiff fluid cosmological models with decaying vacuum energy density \( \Lambda \) in general relativity. Singh and Sharma [42] investigated some Bianchi type-II cosmological models in the context of Brans-Dicke theory of gravitation in the absence and presence of a magnetic field. Recently, Jotania et al. [43] investigated magnetized string cosmology in anisotropic Bianchi-II space time with variable cosmological term. Baghel and Singh [44] obtained a spatially homogeneous Bianchi type-V cosmological model with bulk viscous fluid and time dependent \( G \) and \( \Lambda \). Recently Priyanka et al. [45] investigated a Bianchi type-III space time in the presence of a bulk viscous fluid within the frame work of Lyra’s geometry with time-dependent displacement vector.

In this paper we obtain Bianchi type-II bulk viscous fluid cosmological models with time-dependent gravitational and cosmological constants. The plan of the paper is as follows. We present the metric and field equations in Section 2. In Section 3, we obtain exact solutions of the field equations by applying the law of variation for Hubble’s parameter which yields a constant value of the deceleration parameter. These solutions correspond to singular and non-singular models in two types of cosmologies. The physical and kinematical behaviors of the cosmological models are discussed. Finally, conclusions are summarized in the Section 4. It is shown that the models are compatible with the recent observations in cosmology.

2. The Metric and Field Equations

The totally anisotropic Bianchi type-II metric is given by

\[
ds^2 = -dt^2 + \Lambda^2 (dx^2 + dy^2 + dz^2) + \frac{B^2}{\Lambda^2} dy^2 + \frac{C^2}{\Lambda^2} dz^2
\]

where \( A(t), B(t) \) and \( C(t) \) are the cosmic scale factors. The energy-momentum tensor for bulk viscous fluid is given by

\[
T_{ij} = (\rho + \pi) + \eta \nu_{ij} + \pi g_{ij}
\]

where

\[
\pi = \rho - \xi \nu_{ij}.
\]
Here $\rho, p, \bar{p}$ and $\xi$ are respectively energy-density of matter, isotropic pressure, effective pressure, bulk viscosity coefficient and $v_1$, the four-velocity of the fluid satisfying

$$v_1v^j = -1. \quad (4)$$

A semicolon denotes covariant differentiation. The role played by viscosity and consequent dissipative mechanism in cosmology has been discussed extensively. The coefficient of bulk viscosity determines the magnitude of the viscous stress relative to expansion. In comoving coordinates we take $v^j = (0, 0, 0, 1)$.

The Einstein’s field equations with time-dependent $G$ and $\Lambda$, in suitable unit, read as

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T g_{ij} + \Lambda g_{ij} \quad (5)$$

where $R_{ij}$ is the Ricci tensor and $R$ is the scalar curvature.

For the line-element (1) with a bulk viscous fluid distribution, the field equations (5) in comoving coordinates give rise to the following equations:

$$8\pi G \rho + \Lambda = \frac{A}{A + B + C} \left( \frac{B}{C} + \frac{C}{B} \right) - \frac{3A^2}{4B^2C^2} \quad (6)$$

$$8\pi G p + \Lambda = \frac{A}{A + C} + \frac{AC}{AC} + \frac{A^2}{4B^2C^2} \quad (7)$$

$$8\pi G \bar{p} + \Lambda = \frac{A}{A + B} + \frac{AB}{AB} + \frac{A^2}{4B^2C^2} \quad (8)$$

$$8\pi G \xi + \Lambda = \frac{A}{A + C} + \frac{AC}{AC} - \frac{A^2}{4B^2C^2} \quad (9)$$

where a dot denotes differentiation with respect to $t$. An additional equation for time changes of $G$ and $\Lambda$ is obtained by the divergence of Einstein tensor i.e.

$$\frac{\dot{R}}{2} - \frac{1}{2} \frac{\dot{R}}{R} \frac{\dot{g}}{g} = 0 \quad (10)$$

which yields

$$8\pi G \rho + \Lambda + 8\pi G \left( \rho + \frac{\dot{\rho}}{\rho + p} \left( \frac{A}{A + B + C} + \frac{\dot{A}}{A} \right) \right) = 0. \quad (11)$$

The conservation of energy Equation (10), after using Equation (3), splits into two equations

$$\dot{\rho} + (\rho + p) \left( \frac{A}{A + B + C} + \frac{\dot{A}}{A} \right) = 0, \quad (12)$$

$$\dot{\Lambda} + 8\pi G \rho = 8\pi G \xi \left( \frac{A}{A + B + C} + \frac{\dot{A}}{A} \right)^2. \quad (13)$$

An average expansion scale factor can be defined by $R(t) = (ABC)^{1/3}$ implying that the Hubble parameter $H = \frac{\dot{R}}{R}$.

An important observational quantity is the deceleration parameter $q$ which is defined as

$$q = -\frac{\ddot{R}}{R^2}. \quad (15)$$

The sign of $q$ indicates whether the model inflates or not. The positive sign corresponds to standard decelerating model whereas the negative sign indicates acceleration.

3. Solution of the Field Equations

The field equations (6) – (9) are a system of four highly non-linear equations in seven unknown parameters $A, B, C, \rho, p, \xi$ and $\Lambda$. For the complete determination of the model of the universe, three additional constraints relating these parameters are required. We first assume that the component $\sigma_i^1$ of the shear tensor $\sigma_i^j$ is proportional to the expansion scalar $\Theta$. This condition leads to

$$A = (BC)^m \quad (16)$$

where $m$ is a positive constant, which is a physically plausible condition. We also make the certain physically valid assumption of the Hubble parameter $H$ as

$$H = l (ABC)^{-n/3} \quad (17)$$

where $l > 0$ and $n \geq 0$ are constant. This type of relation, which gives a constant value of deceleration parameter, was initially considered by Berman [46], Berman and Gomide [47] for solving FRW cosmological models. The same concept of constant deceleration parameter was used, later on, by many workers (Shri Ram et al. [48], Saha and Rikhvitsky [49], Saha [50] and references therein). Shri Ram et al. [51,52] have further applied the same technique of the law of variation for Hubble parameter for solving the field equations in
Bianchi type-V in the presence of cosmic matter in the frame-work of Lyra’s geometry.

Considering \((ABC)^{1/3}\) as the average scale factor of anisotropic Bianchi-II space-time, the average Hubble’s parameter may be written as

\[
H = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right). \tag{18}
\]

From Equations. (17) and (18), and integrating, we obtain

\[
ABC = (ntl + c_1)^{3^n}, \quad (n \neq 0), \tag{19}
\]

\[
ABC = c_2^n \exp(3lt), \quad (n = 0). \tag{20}
\]

where \(c_1\) and \(c_2\) are integration constants. Without loss of any generality, we can take \(c_1 = 0\) and \(c_2 = 1\) so that

\[
ABC = (ntl)^{3^n}, \quad (n \neq 0), \tag{21}
\]

\[
ABC = \exp(3lt), \quad (n = 0). \tag{22}
\]

Thus, the law Equation. (17) provides power-law (21) and exponential-law (22) for expansion of the universe.

Now, subtracting Equation. (8) from Equation. (7), and taking integral of the resulting equation two times, we get

\[
\frac{C}{B} = \exp \left[ \int \frac{c_3}{ABC} \frac{dl}{n} \right] + c_4 \tag{23}
\]

where \(c_3\) and \(c_4\) are constants of integration.

We now obtain exact solutions of the field equations in two type of cosmologies with \(n \neq 0\) and \(n = 0\).

### 3.1. Solution with \(n \neq 0\)

We use the power-law solutions of the average-scale factor obtained in Equations. (16), (21) and (23), we obtain the explicit solutions of \(A, B\) and \(C\) can be written as

\[
A = (ntl)^{\frac{3m}{n(m+1)}}, \tag{24}
\]

\[
B = (ntl)^{\frac{3}{2n(m+1)}} \exp \left[ \frac{c_3(nlt)^{\frac{n-3}{n}}}{2(3-n)} - \frac{c_4}{2} \right], \tag{25}
\]

\[
C = (ntl)^{\frac{3}{2n(m+1)}} \exp \left[ \frac{c_3(nlt)^{\frac{n-3}{n}}}{2(1-n-3)} + \frac{c_4}{2} \right], \tag{26}
\]

where \(n \neq 3\).

In most investigations in cosmology, the bulk viscosity is assumed to be a simple power function of the energy density i.e.

\[
\xi = \xi_0 \rho^\beta \tag{27}
\]

where \(\xi_0\) and \(\beta\) are constant. Murphy \[15\] assumed \(\beta = 1\) in the case small density which corresponds to a radiative fluid. We also assume that the fluid obeys the barotropic equation of state

\[
p = \gamma \rho, \quad 0 \leq \gamma \leq 1. \tag{28}
\]

From Equations. (24) – (26), (11) and (28), we obtain

\[
\rho = k t^{\frac{-3(1+\gamma)}{n}}. \tag{29}
\]

Differentiating Equation. (29), we obtain

\[
\rho = \frac{-3k t^{(1+\gamma) - \frac{3(1+\gamma)}{n}}}{n}. \tag{30}
\]

Substituting Equations. (24) – (26) in Equation. (9), we have

\[
\frac{36m+9}{4n^2(m+1)^2t^2} \frac{c_3}{4(nlt)^{\frac{6}{n} - \frac{3(1-m)}{n}}} - \frac{8\pi G \rho + 8\pi G \rho + \Lambda.}{8\pi G \rho + 8\pi G \rho + \Lambda.} \tag{31}
\]

Differentiation of Equation. (31) gives

\[
\frac{3c_3}{2} \frac{(nlt)^{-\frac{6}{n}}}{n} \left[ \frac{3(1-m)^l}{2(1+m)} \left( \frac{nlt}{m+1} \right)^n \right] - \frac{36m+9}{2n^2(m+1)^2t^3} = \frac{8\pi G \rho + 8\pi G \rho + \Lambda.}{8\pi G \rho + 8\pi G \rho + \Lambda.} \tag{32}
\]

Substituting Equation. (12) in (32) yields

\[
8\pi G \rho + 8\pi G \rho \frac{9\xi}{(nt)^2} = \frac{3c_3}{2} \frac{(nlt)^{-\frac{6}{n}}}{n} \left[ \frac{3(1-m)^l}{2(1+m)} \left( \frac{nlt}{m+1} \right)^n \right] - \frac{36m+9}{2n^2(m+1)^2t^3}. \tag{33}
\]

Again, substituting Equations. (27) and (30) into (33), we obtain

\[
G \left[ \frac{3c_3}{2} \frac{(nlt)^{-\frac{6}{n}}}{n} \left[ \frac{3(1-m)^l}{2(1+m)} \left( \frac{nlt}{m+1} \right)^n \right] - \frac{36m+9}{2n^2(m+1)^2t^3} \right] \tag{34}
\]

From Equations. (29), (31) and (34), we find that

\[
\Lambda = \frac{3c_3}{2} \frac{(nlt)^{-\frac{6}{n}}}{n} \left[ \frac{3(1-m)^l}{2(1+m)} \left( \frac{nlt}{m+1} \right)^n \right] - \frac{36m+9}{2n^2(m+1)^2t^3} \tag{35}
\]

From Equation. (29), it is observed that the energy density is decreasing function of time \(t\) and it approaches a small positive value at present epoch. From Equation. (34), we observe that the gravitational constant \(G\) is an increasing function of time and it approaches large positive value at late time. From Equation. (35), it can be
seen that the cosmological term $\Lambda$ is a decreasing function of time.

The expressions for the kinematical parameters, the generalized Hubble’s parameter, the expansion scalar, proper volume and shear scalar are obtained as

$$V = (nt)^n, \quad (36)$$

$$H = \frac{1}{nt}, \quad (37)$$

$$\theta = \frac{3}{nt}, \quad (38)$$

$$\sigma^2 = \frac{9l^2m^2 - 18l^2m}{2(m+1)^2t^2} + \frac{c_4^2}{2}\exp(-6lt). \quad (39)$$

From the above set of solutions, it is clear that in the expanding model, the spatial volume is zero at $t = 0$. This shows that the universe starts evolving with zero volume at $t = 0$ and expands with cosmic time $t$. At this epoch the physical and kinematical quantities $\rho, \vec{\xi}, \Lambda, \theta, H$ and $\sigma$ all tend to infinity. This means that the model of universe has a big-bang singularity at $t = 0$. The physical parameters are decreasing function of time and ultimately tend to zero for large time. The scalar expansion and shear scalar are extremely large at the origin of the universe ($t = 0$) and are decreasing monotonically with the passage of time and will take zero value for late future.

Since $\frac{\sigma}{\theta} = \text{constant as } t \to \infty$, the anisotropy in the universe is maintained throughout.

The deceleration parameter $q$ is given by

$$q = n - 1. \quad (40)$$

For $n > 1$, $q > 0$, therefore this model decelerates in the standard way. It deserves mention that the decelerating models are also consistent with the recent CMB observations made by WMAP as well as with the high redshift supernovae Ia data including SN 1997 ff at $Z = 1.755$ [53]. For $0 < n < 1$, $q < 0$ which corresponds to an accelerating model of the universe.

### 3.2. Solution with $n = 0$

In this case, we obtain an exponentially expanding nonsingular cosmological model. Solving Equations. (16), (22), and (23), we obtain the metric functions as

$$A = \exp\left(\frac{3lm}{m+1}t\right), \quad (41)$$

$$B = \exp\left[\frac{3lt}{2} + \frac{c_4}{6l}\exp(-3lt) - \frac{c_4}{2}\right], \quad (42)$$

$$C = \exp\left[\frac{3lt}{2} - \frac{c_4}{6l}\exp(-3lt) + \frac{c_4}{2}\right]. \quad (43)$$

The expressions for the kinematical parameters i.e. the spatial volume $V$, the generalized Hubble’s parameter, the expansion scalar, shear scalar and deceleration parameter are obtained as

$$V = \exp(3lt), \quad (44)$$

$$H = \frac{(2m+1)l}{(m+1)}, \quad (45)$$

$$\theta = \frac{3(2m+1)}{m+1}, \quad (46)$$

$$\sigma^2 = \frac{18l^2m^2 + 9l^2(m+1)^2}{4(m+1)^2} - \frac{c_4^2}{2}\exp(6lt). \quad (47)$$

$$q = -1. \quad (48)$$

Following the procedure as in section (3.1), we obtain the expression for energy density, gravitational constant and cosmological constant as under:

$$\rho = k_2 \exp\left[-\frac{3(1+\gamma)(2m+1)t}{m+1}\right], \quad (49)$$

$$G = \left[\frac{72\pi G^2 k^4 (2m+1)^2}{(m+1)^2} \exp\left[-\frac{3(1+\gamma)(2m+1)}{m+1}\right]\right]^{-1} \times \exp\left[\frac{3(1+\gamma)(2m+1)}{m+1}\right], \quad (50)$$

$$\Lambda = \frac{45l^2m + 9l^2}{4(m+1)} + \frac{c_4^2}{4}\exp(-6lt) - \frac{1}{4}\exp\left(-\frac{6lt}{m+1}\right), \quad (51)$$

We observe that this model has no finite time singularity. The spatial volume increases with time and tends to infinity for large $t$. The matter energy density $\rho$ is a defined function of time which ultimately tends to zero as. The expansion scalar $\theta$ is constant throughout the passage of time. The shear scalar $\sigma$ also tends to zero as $t \to \infty$. This shows that the universe starts expanding exponentially from infinite past with constant expansion rate. As $t \to \infty$, all the spatial scale factors diverge whereas the generalized Hubble’s parameter is constant throughout the evolution of the universe. The gravitational constant is constant initially and gradually increases and tends to infinity for large $t$. The cosmological constant $\Lambda$ is infinite at the beginning of the model and decreases as time increases and ultimately become zero at late times.
Also $\frac{\sigma^2}{\theta} = 0$ as $t \to \infty$, the model approaches isotropy for large time. For $q = -1$, we have $\frac{dH}{dt} = 0$, which implies the greatest value of Hubble parameter and fastest rate of expansion of the universe. It is evident that negative $q$ would accelerate and increase the age of the universe.

4. Conclusion

We have obtained exact solutions for a viscous fluid with time-dependent gravitational and cosmological constants in totally anisotropic Bianchi-type II space-time. Models of the universe in two types of cosmologies are obtained by applying the law of variation of the Hubble parameter. The nature of singularity in the models has discussed. The model with power-law expansion has a big-bang singularity at $t = 0$, whereas the model with exponential expansion has a singularity in the infinite past. The role of viscosity in the evolution of the universe is discussed. We observed that in both types of cosmological model the gravitational constant $G$ is an increasing function of time and approaches large positive value at late time, whereas the cosmological term is decreasing function of time and ultimately becomes zero at late time. The physical and kinematical properties of the cosmological models have discussed. It has been observed that the solutions are compatible with recent observations in cosmology.

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References


