

Massive Electrodynamic Gravity

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Abstract In this paper, an efficient combination of the diverse theoretical approaches, such like the Einstein gravitational waves, the Lifshitz cosmological perturbation theory, the Veltman perturbative quantum gravity, and the Maxwell electrodynamics, leads to an essentially new discussion of either massless and massive gravitons. The new force law of gravitation is established. The case of the Planckian particle is considered in the context of the Markov hypothesis.

Keywords: graviton mass, Maxwell electrodynamics, perturbative quantum gravity, gauge field theories, cosmological perturbation theory, gravitational waves, massless graviton, massive graviton, Planckian particle, Markov hypothesis, law of gravitation

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1. Introduction

Existence of gravitational radiation is one of the proposals due to General Relativity, for the first time considered through Albert Einstein about one hundred years ago [1,2]. Let us present the core basis of this idea, the detailed derivations can be found in handbooks, Cf. the Refs. [3,4,5].

Gravitational waves are defined through the weak field condition

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \\ h^{\mu\nu} &= \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}, \det h_{\mu\nu} \ll 1, \end{aligned} \quad (1)$$

where $\eta_{\mu\nu} = \text{diag}(1,1,1,1)$ is the Minkowski metric in the relativistic convention. Assuming the coordinate transformation $x'_\mu = \Lambda_\mu^\nu x_\nu$, where Λ_μ^ν are the Lorentz matrices, the Lorentz invariance

$$g_{\mu\nu} = \Lambda_\mu^\alpha \Lambda_\nu^\beta g_{\alpha\beta}, \eta_{\mu\nu} = \Lambda_\mu^\alpha \Lambda_\nu^\beta \eta_{\alpha\beta}, \quad (2)$$

leads to

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, h_{\alpha\beta} = \Lambda_\alpha^\mu \Lambda_\beta^\nu h_{\mu\nu} \quad (3)$$

The change of coordinates

$$x^\alpha = x^\alpha + \xi^\alpha(x^\beta), \quad (4)$$

generates the Lorentz transformation α

$$\Lambda_\beta^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} = \delta_\beta^\alpha + \partial_\beta \delta^\alpha, \quad (5)$$

which leads to the gauge transformation for Einstein's gravitational waves α

$$\dot{g}'_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - 2\partial_{(\beta} \xi_{\alpha)}, \xi_\beta = \eta_{\alpha\beta} \xi^\alpha. \quad (6)$$

In such an approximation

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} \eta^{\alpha\beta} (\partial_\mu h_{\nu\beta} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}), \quad (7)$$

$$R_{\mu\nu\alpha\beta} = 2\eta_{\mu\nu} \partial_{[\alpha} \Gamma_{\beta]}^\lambda = \partial_\alpha \partial_{[\nu} h_{\mu]\beta} + \partial_\beta \partial_{[\mu} h_{\nu]\alpha}, \quad (8)$$

$$R_{\mu\nu} = \eta^{\alpha\beta} R_{\mu\nu\alpha\beta} = \partial_\alpha \partial_{(\nu} h_{\mu)}^\alpha - \frac{1}{2} (\partial_\mu \partial_\nu h + \square h_{\mu\nu}), \quad (9)$$

$$R = \eta^{\mu\nu} R_{\mu\nu} = \partial_\alpha \partial^\mu h_{\mu}^\alpha - \square h, \quad (10)$$

$$\begin{aligned} G_{\mu\nu} &= \partial_\alpha \partial_{(\nu} h_{\mu)}^\alpha - \frac{1}{2} (\partial_\mu \partial_\nu h + \square h_{\mu\nu} \\ &+ \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \square h), \end{aligned} \quad (11)$$

where $h = \eta^{\mu\nu} h_{\mu\nu} = h_\mu^\mu$ and $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$, and the Einstein field equations for gravitational waves are

$$\begin{aligned} \partial_\mu \partial_\nu h + \square h_{\mu\nu} + \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} + \eta_{\mu\nu} \square h \\ - 2\partial_\alpha \partial_{(\nu} h_{\mu)}^\alpha = -2\kappa T_{\mu\nu}, \end{aligned} \quad (12)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of Matter fields. Making use of the tensor

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad (13)$$

and require the Lorenz-like gauge

$$\partial^\mu \bar{h}_{\mu\nu} = 0, \quad (14)$$

the field equations (12) become the wave equation

$$\square \bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}, \quad (15)$$

whose 00-component and the spherical symmetry

$$\bar{h}_{00} = \bar{h}_{00}(r), \quad (16)$$

for the energy-momentum tensor with

$$T_{00} = \rho c^2, \quad (17)$$

gives the Poisson equation

$$\Delta\phi = \frac{4\pi G}{c^2} \rho, \quad (18)$$

with the gravitational potential

$$\phi = -\frac{1}{4}\bar{h}_{00}, \quad (19)$$

which establishes the Newton law of universal gravitation.

Then, neglecting other components of $\bar{h}_{\mu\nu}$,

$$h = \bar{h}_{00} \quad (20)$$

one has

$$h^{00} = h^{11} = h^{22} = h^{33} = -2\phi, \quad (21)$$

and, consequently, the space-time is characterized by the interval

$$ds^2 = -(1+2\phi)dx^0dx_0 + (1-2\phi)dx^i dx_i. \quad (22)$$

Gravitational waves are one of the most intriguing questions of modern physics [6,7,8]. In 2011, V.F. Mukhanov presented the lecture on massive gravity [9] following from Einstein's gravitational waves, cosmological perturbation theory, and perturbative quantum gravity. This paper shows that the situation is really much more sophisticated than Mukhanov indicated.

2. Cosmological Perturbations

In the cosmological perturbation theory [10,11]

$$ds^2 = -(1+2\Phi)(dx^0)^2 + 2\partial_i B dx^i dx^0 + [(1-2\Psi)\delta_{ij} + 2s_{ij}] dx^i dx^j, \quad (23)$$

where the scalar and the tensor perturbations are

$$\Psi = -\frac{1}{6}h_k^k, \quad (24)$$

$$s_{ij} = \frac{1}{2}(h_{ij} + 2\Psi\delta_{ij}), \quad (25)$$

while Φ is the gravitational potential responsible for the appropriate law of gravitation, and $\partial_i B$ is the vector perturbation. Then, the weak field is

$$h_{00} = -2\Phi, h_{0i} = \partial_i B, h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}, \quad (26)$$

while the condition $\det h_{\mu\nu} \ll 1$ gives

$$\det h_{ij} \leq \frac{1+|\partial_i B|}{2|\Phi|}. \quad (27)$$

However, this state of affairs differs from General Relativity, because the cosmological perturbation theory corresponds with fixation of an inertial reference frame in the background Minkowski space-time and the scalar-vector-tensor decomposition of the perturbation metric components is made according to their transformation properties under spatial rotations, and is similar to decomposition of the electromagnetic field into electric and magnetic fields. The (0,2) spatial tensor can be further decomposed into a trace and a trace-free part, which corresponds with looking for an irreducible representations of the rotation group.

For the metric (23), one has

$$\Gamma_{00}^0 = \partial_0\Phi, \Gamma_{i0}^0 = \partial_i\Phi, \Gamma_{ij}^0 = -\partial_i\partial_j B + \frac{1}{2}\partial_0 h_{ij}, \quad (28)$$

$$\Gamma_{j0}^i = \frac{1}{2}\partial_0 h_{ij}, \Gamma_{00}^i = \partial^i(\Phi + \partial_0 B), \quad (29)$$

$$\Gamma_{jk}^i = \partial_{(i} h_{k)}^i - \partial^i h_{jk},$$

and

$$R_{0i0j} = \partial_i\partial_j\partial^i(\Phi + \partial_0 B) - \frac{1}{2}\partial_0^2 h_{ij}, R_{0ijk} = -\partial_0\partial_{[j} h_{k]i}, \quad (30)$$

$$R_{ijkl} = 2\partial_{[i}\partial_{j} h_{kl]},$$

and

$$R_{00} = \Delta(\Phi + \partial_0 B) + 3\partial_0^2\Psi, \quad (31)$$

$$R_{0i} = \partial_0\partial_k\left(3\delta_i^k\Psi + \frac{1}{2}h_i^k\right), \quad (32)$$

$$R_{ij} = -\partial_i\partial_j(\Phi - 3\Psi + \partial_0 B) - \frac{1}{2}\square h_{ij} + \partial_k\partial_{(i} h_{j)}^k. \quad (33)$$

For this reason

$$R = 6\square\Psi - 2\Delta(\Phi + \partial_0 B) + \partial_i\partial_j h^{ij}, \quad (34)$$

and, therefore,

$$G_{00} = 3\Delta\Psi + \frac{1}{2}\partial_k\partial_l h^{kl}, \quad (35)$$

$$G_{0i} = \partial_0\partial_k\left(3\delta_i^k\Psi + \frac{1}{2}h_i^k\right), \quad (36)$$

$$G_{ij} = (\delta_{ij}\Delta - \partial_i\partial_j)(\Phi + \partial_0 B) - 3(\delta_{ij}\square - \partial_i\partial_j)\Psi - \frac{1}{2}\square h_{ij} + \partial_k\partial_{(i} h_{j)}^k - \frac{1}{2}\delta_{ij}\partial_k\partial_l h^{kl}. \quad (37)$$

Consequently, the Einstein field equations are

$$3\Delta\Psi + \frac{1}{2}\partial_k\partial_l h^{kl} = \kappa T_{00}, \quad (38)$$

$$\partial_0\partial_k\left(3\delta_i^k\Psi + \frac{1}{2}h_i^k\right) = \kappa T_{0i}, \quad (39)$$

$$(\delta_{ij}\Delta - \partial_i\partial_j)(\Phi + \partial_0 B) - 3(\delta_{ij}\square - \partial_i\partial_j)\Psi - \frac{1}{2}\square h_{ij} + \partial_k\partial_{(i} h_{j)}^k - \frac{1}{2}\delta_{ij}\partial_k\partial_l h^{kl} = \kappa T_{ij}, \quad (40)$$

and, amazingly, can be solved. The 00 component and 0i component give

$$\partial_0 T_{00} = \partial^i T_{0i}, \quad (41)$$

while the 0j-component leads to

$$h_j^k = -6\delta_j^k \Psi + 2\kappa \int \int T_{0j} dx^0 dx^k, \quad (42)$$

Contraction of this solution with δ and application of (24) give

$$\Psi = \frac{\kappa}{6} \int \int T_{0k} dx^0 dx^k, \quad (43)$$

and, consequently, (42) becomes

$$h_j^i = -\kappa \delta_j^i \int \int T_{0k} dx^0 dx^k + 2\kappa \int \int T_{0j} dx^0 dx^i. \quad (44)$$

Noticing that

$$\partial_k \partial_l h^{kl} = \kappa T_{00}, \quad (45)$$

$$\partial_k \partial_{(i} h_{j)}^k = 2\kappa \int \partial_{(i} T_{0j)} dx^0 - \kappa \int \int \partial_i \partial_j T_{0k} dx^0 dx^k, \quad (46)$$

$$\begin{aligned} \square h_{ij} &= \kappa \left(\int \partial^0 T_{0k} dx^k - T_{00} \right) \delta_{ij} \\ &+ 2\kappa \left(\int \partial_i T_{0j} dx^0 - \int \partial^0 T_{0j} dx^i \right), \end{aligned} \quad (47)$$

$$\square \Psi = \frac{\kappa}{6} \left(T_{00} - \int \partial^0 T_{0k} dx^k \right), \quad (48)$$

$$\partial_i \partial_j \Psi = \frac{\kappa}{6} \int \int \partial_i \partial_j T_{0k} dx^0 dx^k, \quad (49)$$

one can see that the 00-component of the field equations becomes

$$\Delta \Psi = \frac{\kappa}{6} T_{00}. \quad (50)$$

Making use of the relation (41), one receives

$$\int \partial^0 T_{0j} dx_i = \int \int \partial_0^2 T_{00} dx_i dx_j, \quad (51)$$

$$\int \partial^0 T_{0k} dx^k = \int \int \partial_0^2 T_{00} dx^k dx_k, \quad (52)$$

$$\int \int \partial_i \partial_j T_{0k} dx^0 dx^k = \int \int \partial_i \partial_j T_{00} dx^k dx_k, \quad (53)$$

$$\int \partial_j T_{0i} dx^0 = \int \partial_j T_{00} dx_i, \quad (54)$$

and, consequently, the ij component of the field equations is

$$\begin{aligned} &(\delta_{ij} \Delta - \partial_i \partial_j)(\Phi + \partial_0 B) \\ &= \kappa T_{ij} + \frac{\kappa}{2} \left(T_{00} + \frac{2}{3} \int \int \partial_0^2 T_{00} dx^k dx_k \right) \delta_{ij} \\ &+ \frac{\kappa}{2} \int \int \partial_i \partial_j T_{00} dx^k dx_k - \kappa \int \int \partial_0^2 T_{00} dx_i dx_j - \kappa \int \partial_j T_{00} dx_i, \end{aligned} \quad (55)$$

and after contraction with δ gives

$$\Delta(\Phi + \partial_0 B) = \frac{\kappa}{2} (T + T_{00}), \quad (56)$$

where $T = \delta^{ij} T_{ij}$. Finally, one obtains

$$h_{ij} = 2\kappa \int \int T_{00} dx_i dx_j - \kappa \delta_{ij} \int \int T_{00} dx_k dx^k, \quad (57)$$

$$\Psi = \frac{\kappa}{6} \int \int T_{00} dx_k dx^k, \quad (58)$$

and the condition (27) for non-zero T_{00} can be rewritten in the following form

$$\frac{\kappa^3 \left(\int \int T_{00} dx_k dx^k \right)^3}{\det \left(\delta_{ij} - 2 \frac{\int \int T_{00} dx_i dx_j}{\int \int T_{00} dx_k dx^k} \right)} \ll \frac{1 + |\partial_i B|^2}{2|\Phi|}. \quad (59)$$

In the equation (56), the evolutions of B and Φ can not be separated, but, consistency with the Newton gravitation suggests the Poisson equation

$$\Delta \Phi = \frac{4\pi G}{c^2} \rho, \quad (60)$$

and then

$$\Delta B = \frac{\kappa}{2} \int T dx^0, \quad (61)$$

that is the problem is a choice of suitable $T_{\mu\nu}$.

3. Perturbative Graviton

Gauge field theories offer the perturbative point of view, wherein graviton is a spin-2 particle [12]. Then, $T_{\mu\nu}$ of graviton is derived from a Lagrangian

$$L_M = L_M(\eta_{\mu\nu}) + \frac{1}{2} t^{\mu\nu} (h_{\mu\nu} - \eta_{\mu\nu}) + \dots, \quad (62)$$

or, explicitly

$$T^{\mu\nu} = 2 \frac{\delta L_M}{\delta h_{\mu\nu}} \Big|_{h^{\mu\nu} = \eta^{\mu\nu}} \quad (63)$$

The Lagrangian of Matter fields has the form

$$L_M = \frac{c^6}{h^2 G} \frac{m^2}{4} (h_\mu^\mu h_\nu^\nu - h_{\mu\nu} h^{\mu\nu}), h_\mu^\mu = h^{\mu\nu} h_{\mu\nu}, \quad (64)$$

where m is a graviton mass. Straightforward calculations gives

$$T^{\mu\nu} = \frac{m^2 c^6}{h^2 G} \eta^{\mu\nu}, \quad (65)$$

and, consequently,

$$T = \delta^{ij} T_{ij} = 3 \frac{m^2 c^6}{h^2 G}, T_{00} = -\frac{m^2 c^6}{h^2 G}. \quad (66)$$

Implementing the massless limit, one has

$$\Psi = 0, h_{ij} = 0, \Delta(\Phi + \partial_0 B) = 0, \quad (67)$$

and, consequently, the metric (23) becomes

$$ds^2 = -(1 + 2\Phi)(dx^0)^2 + 2\partial_i B dx^i dx^0 + dx^i dx^j, \quad (68)$$

while the condition (27) is equivalent to

$$\frac{1+|\partial_i B|^2}{2|\Phi|} \gg 0. \quad (69)$$

4. Massless Limit

The Laplace equation (67) can be rewritten in the form

$$\Delta \Phi = -\Delta \partial_0 B, \quad (70)$$

and can be solved by the most general separation of variables $\Phi = C$, $\partial_0 B = C$,

$$\Delta \Phi = C, \Delta \partial_0 B = -C, \quad (71)$$

where C is a separation constant. In such a situation,

$$\begin{aligned} \Phi(r) &= C_0 - \frac{C_1}{r} + \frac{C}{6} r^2, \\ B(r, t) &= \left(C_0' - \frac{C_1'}{r} - \frac{C}{6} r^2 \right) (t - t_0). \end{aligned} \quad (72)$$

Taking into account the boundary conditions

$$\Phi(r_0) = \Phi_0, \partial_i \Phi(r_0) = \partial_i \Phi_0, \quad (73)$$

one receives

$$\Phi(r) = \Phi_0 + \left(1 - \frac{r_0}{r} \right) r_0^i \partial_i \Phi_0 + \frac{C}{6} \left(r^2 - 3r_0^2 + \frac{2r_0^3}{r} \right) \quad (74)$$

Because the constant in the gravitational potential

$$\Phi_0 - r_0^i \partial_i \Phi_0 - \frac{C}{2} r_0^2, \quad (75)$$

vanishes when differentiation is performed, it does not play a role for the force law of gravitation $F_i = -m \partial_i \Phi$, and, consequently, it can be identically vanishing. This condition gives

$$C = \frac{2}{r_0^2} \left(\Phi_0 - r_0^i \partial_i \Phi_0 \right), \quad (76)$$

and, for this reason, one obtains finally

$$\Phi(r) = \frac{2}{3} \left(\Phi_0 - \frac{5}{2} r_0^i \partial_i \Phi_0 \right) \frac{r_0}{r} + \frac{\Phi_0 - r_0^i \partial_i \Phi_0}{3} \frac{r^2}{r_0^2}. \quad (77)$$

Considering

$$\frac{2}{3} \left(\Phi_0 - \frac{5}{2} r_0^i \partial_i \Phi_0 \right) = -\frac{\Omega_0^2}{2} r_0^2, \frac{\Phi_0 - r_0^i \partial_i \Phi_0}{3} = \frac{\omega_0^2}{2} r_0^2. \quad (78)$$

where Ω_0 and ω_0 are two constant angular frequency parameters, for which

$$\Phi_0 = \frac{1}{2} (5\omega_0^2 + \Omega_0^2) r_0^2, r_0^i \partial_i \Phi_0 = \frac{1}{2} (2\omega_0^2 + \Omega_0^2) r_0^2, \quad (79)$$

and the force law of gravitation is

$$F_i = \lim_{m \rightarrow \infty} \left(-\frac{m \Omega_0^2}{2} \frac{r_0^3}{r^3} r_i - m \omega_0^2 r_i \right), \quad (80)$$

one sees that the force law is non-trivial if and only if

$$\Omega_0 = \sqrt{\frac{K_0}{m}}, \omega_0 = \sqrt{\frac{k_0}{m}}, \quad (81)$$

where K_0 and k_0 are positive constants.

5. Massive Gravitons

Let m be a positive graviton mass parameter. Then, one has

1. The Newton law of universal gravitation, if

$$\omega_0 = 0, \Omega_0 = \frac{GM}{r_0^3}, \quad (82)$$

where $M > 0$ is a mass of a spherically-symmetric body of radius r_0

2. The oscillatory law of gravitation, if

$$\omega_0 \neq 0, \Omega_0 = 0. \quad (83)$$

3. The force law of gravitation $\sim \frac{1}{r^2}$, if

$$r_0^i \partial_i \Phi_0 = \Phi_0, \frac{2}{3} \left(\Phi_0 - \frac{5}{2} r_0^i \partial_i \Phi_0 \right) = \frac{M \Phi_1}{m}, \quad (84)$$

where M is as previously, and Φ_1 is a certain constant reference value of the potential Φ . Then,

$$\Phi_0 = \frac{M}{m} \Phi_1, r_0^i \partial_i \Phi_0 = -\frac{M}{m} \Phi_1, \quad (85)$$

and, for consistency, one has

$$M = \beta m = \text{const}, \quad (86)$$

where β is a dimensionless constant, which means that the mass of a body generating gravitation is multiplication of the graviton mass.

Therefore,

$$\Phi(r) = \beta \Phi_1 \frac{r_0}{r}. \quad (87)$$

where the constant $\beta \Phi_1 r_0$ can be established from experimental data.

4. The most general force law of gravitation, which is non-trivial in the massless limit,

$$\Phi(r) = \frac{M_1 \Phi_1 r_0}{m} \frac{1}{r} + \frac{M_2 \Phi_2}{m r_0^2} r^2, \quad (88)$$

where $M_1 \Phi_1$ and $M_2 \Phi_2$ are certain reference constants, is given by

$$\Phi_0 = \frac{5M_2 \Phi_2 - M_1 \Phi_1}{m}, r_0^i \partial_i \Phi_0 = \frac{2M_2 \Phi_2 - M_1 \Phi_1}{m}. \quad (89)$$

gives

$$M_2 \Phi_2 = \frac{1}{2} M_1 \Phi_1, \quad (90)$$

and, consequently,

$$\Phi_0 = \frac{3}{2} \frac{M_1 \Phi_1}{m}, r_0^i \partial_i \Phi_0 = 0. \quad (91)$$

Therefore,

$$\Phi(r) = \frac{2\Phi_0}{3} \frac{r_0}{r} + \frac{\Phi_0}{3} \frac{r^2}{r_0^2}, \quad (92)$$

If $M_1 = M$ and $\Phi_1 = -\frac{GM}{r_0}$, then

$$\Phi_0 = -\frac{3}{2} \frac{GM^2}{mr_0}, \quad (93)$$

and, consequently,

$$\Phi(r) = \frac{\omega_0^2 r_0^3}{r} - \frac{\omega_0^2}{2} r^2, \omega_0 = \sqrt{\frac{GM^2}{mr_0^3}}. \quad (94)$$

For $M = \beta m$ and $r_0 = R$, where R is a radius of the spherical body,

$$\omega_0 = \sqrt{\beta} \Omega_0, \Omega_0 = \frac{GM}{R^3}. \quad (95)$$

Then, a graviton mass is

$$m = \left(\frac{\Omega_0}{\omega_0} \right)^2 M, \quad (96)$$

and suggests existence of the various gravitons emitted by a mass.

6. Electromagnetic Gravity

The resulting force law of gravitation

$$F_i = -\frac{m\Omega_0^2}{2} \frac{r_0^3}{r^3} r_i - m\omega_0^2 r_i. \quad (97)$$

can be considered as the analogue of the Lorentz force

$$F_i = mg_i + m\epsilon_i^{jk} v_j d_k, \quad (98)$$

where the field d_k plays a role analogous to magnetic field, gravitational acceleration g_i plays a role of electric field, and mass m of a moving particle plays a role of electric charge.

Introducing the analogue of the Faraday tensor of electromagnetic field

$$\mathcal{G}_{\mu\nu} = \frac{1}{\sqrt{4\pi\epsilon_0 G}} \begin{bmatrix} 0 & \frac{g_i}{c} \\ -\frac{g_i}{c} & \epsilon_{ij}^k d_k \end{bmatrix} \quad (99)$$

for which the dual tensor is

$$\tilde{\mathcal{G}}_{\mu\nu} = \epsilon_{\mu\nu k\lambda} \mathcal{G}^{k\lambda} = \frac{1}{\sqrt{4\pi\epsilon_0 G}} \begin{bmatrix} 0 & d_i \\ -d_j & -\epsilon_{ij}^k \frac{g_k}{c} \end{bmatrix} \quad (100)$$

and taking the four-current

$$J_\nu = [c\rho, J_i], \quad (101)$$

one obtains the analogue of the Maxwell equations

$$\partial^\mu \mathcal{G}_{\mu\nu} = \mu_0 J_\nu, \partial^\mu \tilde{\mathcal{G}}_{\mu\nu} = 0, \quad (102)$$

or, explicitly, the analogue of the Gauss law for electricity

$$\partial^i g_i = -\sqrt{\frac{4\pi G}{\epsilon_0}} \rho, \quad (103)$$

the analogue of the Ampere law

$$\epsilon_{ij}^k \partial^i d_k = \sqrt{\frac{4\pi G}{\epsilon_0}} \frac{1}{c^2} J_i - \frac{1}{c^2} \dot{g}_i, \quad (104)$$

the analogue of the Gauss law for magnetism

$$\partial^k d_k = 0, \quad (105)$$

and the analogue of the Faraday law

$$\dot{d}_j = \epsilon_{ij}^k \partial^i g_k. \quad (106)$$

Introducing the analogue of the electromagnetic four-potential A_μ

$$A_\mu = \frac{1}{\sqrt{4\pi\epsilon_0 G}} v_\mu, \quad (107)$$

where $v_\mu = [c, v_i]$ is the velocity four-vector, one has

$$\mathcal{G}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (108)$$

and, making use of (108), one receives

$$G_{0i} = -\frac{1}{\sqrt{4\pi\epsilon_0 G}} \frac{g_i}{c} = \partial_0 A_i - \partial_i A_0 = \frac{1}{\sqrt{4\pi\epsilon_0 G}} \frac{1}{c} \dot{v}_i, \quad (109)$$

$$\begin{aligned} G_{ij} &= -\frac{1}{\sqrt{4\pi\epsilon_0 G}} \epsilon_{ij}^k d_k = \partial_i A_j - \partial_j A_i \\ &= \frac{1}{\sqrt{4\pi\epsilon_0 G}} (\partial_i v_j - \partial_j v_i), \end{aligned} \quad (110)$$

or, equivalently,

$$g_i = -\dot{v}_i, d_i = \epsilon_i^{kl} \partial_k v_l. \quad (111)$$

Also, the stress-energy tensor is the analogue of the electromagnetic one

$$T_{\mu\nu} = \frac{1}{\mu_0} \left(\eta^{\alpha\beta} \mathcal{G}_{\mu\alpha} \mathcal{G}_{\nu\beta} - \frac{1}{4} \eta^{\mu\nu} \mathcal{G}_{\alpha\beta} \mathcal{G}^{\alpha\beta} \right) = \begin{bmatrix} \epsilon & \frac{S_i}{c} \\ \frac{S_j}{c} & -\sigma_{ij} \end{bmatrix} \quad (112)$$

where ϵ is the field energy density

$$\epsilon = \frac{g^2 + c^2 d^2}{8\pi G}, \quad (113)$$

S_i is the analogue of the Poynting vector

$$S_i = \frac{c}{4\pi G} \epsilon_i^{kl} g_k d_l \quad (114)$$

and σ_{ij} is the analogue of the Maxwell stress tensor

$$\sigma_{ij} = \frac{1}{4\pi G} \left(g_i g_j + c^2 d_i d_j - \frac{g^2 + c^2 d^2}{2} \delta_{ij} \right). \quad (115)$$

Making use of the definitions (111) within the equation (104), one receives

$$\square v_j - \partial_j (\partial^i v_i) = \sqrt{\frac{4\pi G}{\epsilon_0}} \frac{1}{c^2} J_j, \quad (116)$$

and, similarly, the equation (103) leads to

$$\partial^i \dot{v}_i = \sqrt{\frac{4\pi G}{\epsilon_0}} \rho. \quad (117)$$

Consequently,

$$J_\mu = c^2 \sqrt{\frac{\epsilon_0}{4\pi G}} \left[\frac{1}{c} \partial^i \dot{v}_i, \square v_j - \partial_j (\partial^i v_i) \right]. \quad (118)$$

Particularly, for a graviton mass $m=0$, one has

$$J_\mu = -\frac{1}{\mu_0} \left(\frac{mc}{\hbar} \right)^2 A_\mu, \quad (119)$$

or, taking into account (107),

$$J_\mu = c^2 \sqrt{\frac{\epsilon_0}{4\pi G}} \left[-c \left(\frac{mc}{\hbar} \right)^2, -\left(\frac{mc}{\hbar} \right)^2 v_j \right]. \quad (120)$$

Comparing the relations (118) and (120), one obtains

$$\partial^i \dot{v}_i = -\left(\frac{mc^2}{\hbar} \right)^2, \square v_j - \partial_j (\partial^i v_i) = -\left(\frac{mc}{\hbar} \right)^2 v_j, \quad (121)$$

and, consequently, one receives the analogue of the Proca equation

$$\left(\square + \left(\frac{mc}{\hbar} \right)^2 \right) v_j = 0, m = \frac{\hbar}{c^2} \sqrt{\partial^i g_i}, \quad (122)$$

describing the massive graviton.

The analogy with the Maxwell electrodynamics gives the Lagrangian

$$L = -\frac{1}{4\mu_0} \mathbb{G}_{\mu\nu} \mathbb{G}^{\mu\nu} - A_\mu J^\mu. \quad (123)$$

Also, on the quantum level, one obtains the analogue of the quantum electrodynamics and the Feynman diagrams are fully analogous. Making use of (119), the Lagrangian (123) becomes

$$\begin{aligned} L &= -\frac{c^2}{4\pi G} \left[\frac{1}{2} (\partial_\mu v_\nu \partial^\mu v^\nu - \partial_\mu v_\nu \partial^\nu v^\mu) + \left(\frac{mc}{\hbar} \right)^2 v_\mu v^\mu \right] \\ &= -\frac{c^2}{4\pi G} \left[\frac{1}{2} (\partial_\mu v_i \partial^\mu v^i - \partial_i v_j \partial^j v^i) + \left(\frac{mc}{\hbar} \right)^2 v_i v^i \right] \end{aligned} \quad (124)$$

where we have made use of the four-velocity definition $v_\mu = [c, v_i]$, and the constant Lagrangian has been omitted. Noticing that the continuity equation

$$\partial^\mu J_\mu = 0, \quad (125)$$

should be satisfied, the equation (119) gives the Lorentz gauge

$$\partial^\mu A_\mu = 0, \quad (126)$$

which gives the divergence-free velocity field

$$\partial^\mu v_\mu = \partial^i v_i = 0, \quad (127)$$

which implies that $\partial^i \dot{v}_i = 0$ and, consequently, the graviton mass identically vanishes $m = 0$. However, because this is an additional relation which can be broken and which is not needed for consistency of the theory, one must not take this condition into account. When the continuity equation is non-valid, then a graviton mass is existent and fully argued.

The equations (97) and (98) can be used to find the explicit form of the fields g_i and d_i . There are two identifications

$$g_i = -\frac{\Omega_0^2}{2} \frac{r_0^3}{r^3} r_i, \epsilon_i^{jk} v_j d_k = -\omega_0^2 r_i, \quad (128)$$

$$g_i = -\omega_0^2 r_i, \epsilon_i^{jk} v_j d_k = -\frac{\Omega_0^2}{2} \frac{r_0^3}{r^3} r_i \quad (129)$$

For the first one

$$\partial^i g_i = 0, \quad (130)$$

and, consequently, $m = 0$, and the Lorentz gauge is consistent. For the second one,

$$\partial^i g_i = -3\omega_0^2, m = \sqrt{3} \frac{\hbar}{c^2} \sqrt{-\omega_0^2}, \quad (131)$$

and for $\omega_0^2 > 0$ a graviton is a tachyon, whereas for $\omega_0^2 = -|\omega_0^2| < 0$, that is an inverted harmonic oscillator, one has

$$m = \sqrt{3} \frac{\hbar |\omega_0|}{c^2}, \quad (132)$$

what compared with (96), gives

$$M\Omega_0^2 = -\sqrt{3} \frac{\hbar |\omega_0|^3}{c^2}. \quad (133)$$

Therefore, for non-tachyonic mass M , one should have $\Omega_0^2 = -\frac{GM}{R^3} = -|\Omega_0|^2$, that is a repulsive gravitation, and

$$|\omega_0| = \frac{1}{R} \left(\frac{GM^2 c^2}{\sqrt{3} \hbar} \right)^{1/3}. \quad (134)$$

Interestingly, for a sphere of radius equal to the Planck length $R = \ell_P$ and mass equal to the Planck mass $M = M_P$, one obtains

$$|\omega_0| = \frac{\omega_P}{3^{1/6}} \approx 1.544 \cdot 10^{19} \text{ YHz}, \quad (135)$$

$$|\Omega_0| = \omega_P \approx 1.855 \cdot 10^{19} \text{ YHz}, \quad (136)$$

$$m = 3^{1/3} M_P \approx 1.7610 \cdot 10^4 \frac{\text{YeV}}{c^2}, \quad (137)$$

where ω_P is the Planck frequency, while the force law of gravitation becomes

$$F_i = M_P \omega_P^2 \left(1 + \frac{3^{1/3}}{2} \frac{\ell_P^3}{r^3} \right) r_i \quad (138)$$

For emission of a graviton through the Planckian particle, $M = MP$, a hypothetical particle existing in the Planck scale energetic regime, a graviton mass exceeds MP , what in the light of the Markov hypothesis [13,14], that is $m \leq MP$, means that this graviton is undetectable.

Moreover, one can see that (111) gives

$$\epsilon_i^{jk} v_j d_k = v^l \partial_i v_l - v_l \partial^l v_i, \quad (139)$$

and when the velocity field v_i is known, for example as the solution to the equation (122), both (128) and (129) are the equations for r_i

$$r_i = R^2 \left(\frac{GM^2 c^2}{\sqrt{3} \hbar} \right)^{-2/3} \left(v^j \partial_i v_j - v_j \partial^j v_i \right), \quad (140)$$

$$\frac{r_i}{r^3} = \frac{2}{GM} \left(v^l \partial_i v_l - v_l \partial^l v_i \right). \quad (141)$$

7. Summary

Joining the various approaches to gravity - the Einstein gravitational waves following from General Relativity whose applicability is unclear, the Lifshitz cosmological perturbation theory which is currently applicable for astrophysical scales, the Veltman perturbative quantum gravity based on the gauge field theories which usually are applicable for high energy physics, and the Maxwell electrodynamics which is applicable at any scales - we have received an essential discussion on either massless and massive gravitons. The analysis of the Planckian particle in the context of the Markov hypothesis has been shown that a massive graviton emitted throughout such a particle is undetectable. The new force law of gravitation has been established. This theory, however, needs a

straightforward confrontation with a suitable experimental data. Its should be emphasized that the discussion of this model in more advanced detail was presented in the author's monograph [15].

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