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Quartic B – Spline Method for Solving a Singular Singularly Perturbed Third-Order Boundary Value Problems

Hradyesh Kumar Mishra*, Sonali saini

Department of Mathematics Jaypee University of Engineering & Technology Guna-473226(M.P) India
*Corresponding author: hk.mishra@juet.ac.in

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Abstract In this paper, we study the numerical solution of singular singularly perturbed third-order boundary value problems (BVPs) by using Quartic B-spline method. An efficient algorithm is presented here to solve the approximate solution of the given problem. To understand our method, we introduce the Quartic B-spline basis function in the form of $B_i(x)$ at the different knots. After that we derive our method by using numerical difference formulas to construct the approximate values. Then we use the linear sequence of Quartic B-spline to get the numerical solution of the system of equations. These systems of equations are solved by using MATLAB. Three examples are illustrated to understand the present method.

Keywords: *Singular singularly perturbed, two-point problem, third-order BVPs, Basis function, Quartic B-spline*

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1. Introduction

In this paper, we consider the following class of singular singularly perturbed boundary value problems (BVPs):

$$\varepsilon y'''(x)+\frac{\xi}{x}y''(x)+\alpha(x)y'(x)+\beta(x)y(x)=f(x) \quad (1)$$

$$y(0)=p,\ y'(0)=q,\ y'(1)=r \quad (2)$$

where $\varepsilon > 0$ is a small parameter, $\alpha(x), \beta(x)$ and $f(x)$ are smooth functions and ξ, p, q, r are constant. The existence and uniqueness of such type problem is given by [23].

As we know the given problem is singular because of regular singularity at the coefficient of derivative terms. These types of problem are different from singular perturbation problem and a very little literature is available for these types of problems. Singular perturbation problem appear in many branches of applied mathematics and so many engineers and researcher works on them. These types of problems also commonly occur in the fluid mechanics, quantum mechanics, optimal control theory geophysics, reaction-diffusion equation etc.

In today's era of mathematics, there are so many special methods to find the accurate solution of the given problems in terms of ordinary and partial differential equations [8,14,19]. For detail one may refer to the book C.M. Bender and S.A. Orszag [2], J.J.H Miller, E. O' Riordan and G.I. Shiskin [13], R.E. O' Malley [17], H.G. Roos, M. Stynes and L. Tobiska [22] and the reference therein. There are so many authors who have given their contribution in the field of singular perturbation problem and with spline function [6,10,12,15,16,18]. Spline function is one of the most important methods to find the solution in computational mathematics. Using spline and its types; we can solve our problem in an efficient manner. To understand the spline function we can refer the book Carl De Boor [3] and P.M. Prenter [20] and the reference therein. M.K. Kadalbajoo and V.K. Aggarwal [9] used fitted mesh B spline method to solve a class of singular singularly perturbed Boundary value problem (BVPs). Yogesh Gupta, Pankaj Kumar Srivastava and Manoj Kumar [7] gave the application of B-Spline method for the numerical solution of a system of singularly perturbed boundary value problems. Gazala Akram [1] used Quartic spline methods to solve third-order SPBVPs. Jincai Chang, Qianli Yang and Long Zhao [4] explained the comparison of B-spline method and finite difference method to solve the BVPs of linear ODEs. M. Cui and F. Geng [5] used a computational method for solving third-order singularly perturbed boundary problem. M. Kumar and his coworkers [11] used cubic spline method to solve initial value technique for solving second order singular perturbation BVPs while J. Rashidnia, R. Mohammadi and M. Ghasemi [21] also used cubic spline method to solve their problems. Ikram A. Tirmizi, Fazal-i-Haq and Siraj-ul-islam [24] Used Quartic non-polynomial spline function to solve this type of problem.

There are a quite amount of work has been done for development of numerical methods for boundary value problems using spline and its types. The present paper describes the Quartic B-spline method to solve the singular singularly perturbed boundary value problem. Remaining part of this paper is organized as follows:

Section 2 describes the definition of basics of Quartic B-spline and value of its derivatives at nodal points. In Section 3, the Quartic B-spline method for third-order singular singularly perturbed boundary value method is described for solving equation (1) and (2). Section 4 of this paper consists of numerical solution of three problems. Finally paper is concluded in Section 5.

2. Basics of Quartic B-spline Method

Consider equally spaced knots of a partition $\pi : a = x_0 < x_1 < x_2 < ... < x_N = b$ on $[a,b]$. Let $S_4[\pi]$ be the space of continuously-differentiable, piecewise fourth degree polynomial on π, that is $S_4[\pi]$ is the space of fourth-degree splines on π. Consider the Quartic B-spline basis in $S_4[\pi]$. The forth-degree B-splines are defined as:

$$B_{-4}(x) = \frac{1}{24h^4}\begin{cases} (x_0 - x)^4, & x \in [x_0, x_1] \\ 0, & otherwise \end{cases}$$

$$B_{-3}(x) = \frac{1}{24h^4}\begin{cases} h^4 + 4h^3(x_0 - x) + 6h^2(x_0 - x)^2 \\ +4h(x_0 - x)^3 - 4(x_0 - x)^4, & x \in [x_0, x_1] \\ (x_1 - x)^4, & x \in [x_1, x_2] \\ 0, & otherwise \end{cases}$$

$$B_{-2}(x) = \frac{1}{24h^4}\begin{cases} 11h^4 + 12h^3(x - x_0) - 6h^2(x - x_0)^2 \\ -12h(x - x_0)^3 + 6(x - x_0)^4, & x \in [x_0, x_1] \\ h^4 + 4h^3(x_1 - x) + 6h^2(x_1 - x)^2 \\ +4h(x_1 - x)^3 - 4(x_1 - x)^4, & x \in [x_1, x_2] \\ (x_2 - x)^4, & x \in [x_2, x_3] \\ 0, & otherwise \end{cases}$$

$$B_{-1}(x) = \frac{1}{24h^4}\begin{cases} h^4 + 4h^3(x - x_0) + 6h^2(x - x_0)^2 \\ +4h(x - x_0)^3 - 4(x - x_0)^4, & x \in [x_0, x_1] \\ 11h^4 + 12h^3(x - x_1) - 6h^2(x - x_1)^2 \\ -12h(x - x_1)^3 + 6(x - x_1)^4, & x \in [x_1, x_2] \\ h^4 + 4h^3(x_2 - x) + 6h^2(x_2 - x)^2 \\ +4h(x_2 - x)^3 - 4(x_2 - x)^4, & x \in [x_2, x_3] \\ (x_3 - x)^4, & x \in [x_3, x_4] \\ 0, & otherwise \end{cases}$$

$$B_0(x) = \frac{1}{24h^4}\begin{cases} (x - x_0)^4, & x \in [x_0, x_1] \\ h^4 + 4h^3(x - x_1) + 6h^2(x - x_1)^2 \\ +4h(x - x_1)^3 - 4(x - x_1)^4, & x \in [x_1, x_2] \\ 11h^4 + 12h^3(x - x_2) - 6h^2(x - x_2)^2 \\ -12h(x - x_2)^3 + 6(x - x_2)^4, & x \in [x_2, x_3] \\ h^4 + 4h^3(x_3 - x) + 6h^2(x_3 - x)^2 \\ +4h(x_3 - x)^3 - 4(x_3 - x)^4, & x \in [x_3, x_4] \\ (x_4 - x)^4, & x \in [x_4, x_5] \\ 0, & otherwise \end{cases}$$

$$B_1(x) = \frac{1}{24h^4}\begin{cases} (x - x_1)^4, & x \in [x_1, x_2] \\ h^4 + 4h^3(x - x_2) + 6h^2(x - x_2)^2 \\ +4h(x - x_2)^3 - 4(x - x_2)^4, & x \in [x_2, x_3] \\ 11h^4 + 12h^3(x - x_3) - 6h^2(x - x_3)^2 \\ -12h(x - x_3)^3 + 6(x - x_3)^4, & x \in [x_3, x_4] \\ h^4 + 4h^3(x_4 - x) + 6h^2(x_4 - x)^2 \\ +4h(x_4 - x)^3 - 4(x_4 - x)^4, & x \in [x_4, x_5] \\ (x_5 - x)^4, & x \in [x_5, x_6] \\ 0, & otherwise \end{cases}$$

$$B_{n-5}(x) = \frac{1}{24h^4}\begin{cases} (x - x_{n-5})^4, & x \in [x_{n-5}, x_{n-4}] \\ h^4 + 4h^3(x - x_{n-4}) + 6h^2(x - x_{n-4})^2 \\ +4h(x - x_{n-4})^3 - 4(x - x_{n-4})^4, & x \in [x_{n-4}, x_{n-3}] \\ 11h^4 + 12h^3(x - x_{n-3}) - 6h^2(x - x_{n-3})^2 \\ -12h(x - x_{n-3})^3 + 6(x - x_{n-3})^4, & x \in [x_{n-3}, x_{n-2}] \\ h^4 + 4h^3(x_{n-2} - x) + 6h^2(x_{n-2} - x)^2 \\ +4h(x_{n-2} - x)^3 - 4(x_{n-2} - x)^4, & x \in [x_{n-2}, x_{n-1}] \\ (x_{n-1} - x)^4, & x \in [x_{n-1}, x_n] \\ 0, & otherwise \end{cases}$$

$$B_{n-4}(x) = \frac{1}{24h^4}\begin{cases} (x - x_{n-4})^4, & x \in [x_{n-4}, x_{n-3}] \\ h^4 + 4h^3(x - x_{n-3}) + 6h^2(x - x_{n-3})^2 \\ +4h(x - x_{n-3})^3 - 4(x - x_{n-3})^4, & x \in [x_{n-3}, x_{n-2}] \\ 11h^4 + 12h^3(x - x_{n-2}) - 6h^2(x - x_{n-2})^2 \\ -12h(x - x_{n-2})^3 + 6(x - x_{n-2})^4, & x \in [x_{n-2}, x_{n-1}] \\ h^4 + 4h^3(x_{n-1} - x) + 6h^2(x_{n-1} - x)^2 \\ +4h(x_{n-1} - x)^3 - 4(x_{n-1} - x)^4, & x \in [x_{n-1}, x_n] \\ 0, & otherwise \end{cases}$$

$$B_{n-3}(x)=\frac{1}{24h^4}\begin{cases}(x-x_{n-3})^4, & x\in[x_{n-3},x_{n-2}]\\ h^4+4h^3(x-x_{n-2})+6h^2(x-x_{n-2})^2\\ +4h(x-x_{n-2})^3-4(x-x_{n-2})^4, & x\in[x_{n-2},x_{n-1}]\\ 11h^4+12h^3(x-x_{n-1})-6h^2(x-x_{n-1})^2\\ -12h(x-x_{n-1})^3+6(x-x_{n-1})^4, & x\in[x_{n-1},x_n]\\ 0, & otherwise\end{cases}$$

$$B_{n-2}(x)=\frac{1}{24h^4}\begin{cases}(x-x_{n-2})^4, & x\in[x_{n-2},x_{n-1}]\\ h^4+4h^3(x-x_{n-1})+6h^2(x-x_{n-1})^2\\ +4h(x-x_{n-1})^3-4(x-x_{n-1})^4, & x\in[x_{n-1},x_n]\\ 0, & otherwise\end{cases}$$

$$B_{n-1}(x)=\frac{1}{24h^4}\begin{cases}(x-x_{n-1})^4, & x\in[x_{n-1},x_n]\\ 0, & otherwise\end{cases}$$

and for $i=2,3,\dots,n-6$.

$$B_i(x)=\frac{1}{24h^4}\begin{cases}(x-x_i)^4, & x\in[x_i,x_{i+1}]\\ h^4+4h^3(x-x_{i+1})+6h^2(x-x_{i+1})^2\\ +4h(x-x_{i+1})^3-4(x-x_{i+1})^4, & x\in[x_{i+1},x_{i+2}]\\ 11h^4+12h^3(x-x_{i+2})-6h^2(x-x_{i+2})^2\\ -12h(x-x_{i+2})^3+6(x-x_{i+2})^4, & x\in[x_{i+2},x_{i+3}]\\ h^4+4h^3(x_{i+3}-x)+6h^2(x_{i+3}-x)^2\\ +4h(x_{i+3}-x)^3-4(x_{i+3}-x)^4, & x\in[x_{i+3},x_{i+4}]\\ (x_{i+4}-x)^4, & x\in[x_{i+4},x_{i+5}]\\ 0, & otherwise\end{cases} \tag{3}$$

To solve the singular singularly perturbed third-order boundary value problems $B_i(x), B_i'(x), B_i''(x), B_i'''(x)$, are evaluated at the different knots which are summarized in Table 1.

Table 1. Values of $B_i(x), B_i'(x)$, $B_i''(x)$ and $B_i'''(x)$ at knots

$B(x)$	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i+5}
$B_i(x)$	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
$B_i'(x)$	0	$\frac{4}{24h}$	$\frac{12}{24h}$	$\frac{-12}{24h}$	$\frac{-4}{24h}$	0
$B_i''(x)$	0	$\frac{12}{24h^2}$	$\frac{-12}{24h^2}$	$\frac{-12}{24h^2}$	$\frac{12}{24h^2}$	0
$B_i'''(x)$	0	$\frac{24}{24h^3}$	$\frac{-72}{24h^3}$	$\frac{72}{24h^3}$	$\frac{-24}{24h^3}$	0

3. Description of the Method

Consider the singular singularly perturbed third-order boundary value problem of the form:

$$\varepsilon y'''(x)+\frac{\xi}{x}y''(x)+\alpha(x)y'(x) +\beta(x)y(x)=f(x) \tag{4}$$

$$y(0)=p,\ y'(0)=q,\ y'(1)=r \tag{5}$$

where ε is a small positive parameter $(0<\varepsilon\le 1)$ and ξ, p, q, r are constants. $\alpha(x), \beta(x)$ and $f(x)$ are sufficiently smooth functions.

When we remove the singularity of given equation (4), then the modified form of equation is given as

$$(\varepsilon+\xi)y'''(x)+\alpha(x)y'(x) +\beta(x)y(x)=f(x) \tag{6}$$

Let $S(x)$ be the Quartic B-spline function at the nodal points. Then $S(x)$ can be written as

$$y(x)=S(x)=\sum_{i=-4}^{n-1}c_iB_i(x) \tag{7}$$

be the approximate solution of boundary value problem (4), where c_i's are unknown coefficient and $B_i(x)$'s are fourth degree B-spline function. Then let $x_0, x_1 \dots x_n$ be $n+1$ grid points in the interval [0, 1]. So that we have, $x_i=x_0+ih$, $x_0=0$, $x_n=1$ $i=1,2\dots n$, $h=1/n$ at the knots. Using the table of B-spline function, we get approximate value of $y(x_i), y'(x_i), y''(x_i)$ and $y'''(x_i)$ as

$$S(x_i)=y(x_i)=\frac{c_{i-2}+11c_{i-1}+11c_i+c_{i+1}}{24} \tag{8}$$

$$S'(x_i)=y'(x_i)=\frac{c_{i-2}+3c_{i-1}-3c_i-c_{i+1}}{6h} \tag{9}$$

$$S''(x_i)=y''(x_i)=\frac{c_{i-2}-c_{i-1}-c_i+c_{i+1}}{2h^2} \tag{10}$$

$$S'''(x_i)=y'''(x_i)=\frac{c_{i-2}-3c_{i-1}+3c_i-c_{i+1}}{h^3} \tag{11}$$

Putting above values from equations (8)-(11) in equation (4), we get

$$\begin{aligned}&\varepsilon\left\{\frac{24c_{i-2}-72c_{i-1}+72c_i-24c_{i+1}}{24h^3}\right\}\\ &+\frac{\xi}{x_i}\left\{\frac{12c_{i-2}-12c_{i-1}-12c_i+12c_{i+1}}{24h^2}\right\}\\ &+\alpha(x_i)\left\{\frac{4c_{i-2}+12c_{i-1}-12c_i-4c_{i+1}}{24h}\right\}\\ &+\beta(x_i)\left\{\frac{c_{i-2}+11c_{i-1}+11c_i+c_{i+1}}{24}\right\}\\ &=f(x_i)\\ &i=1,2,\dots,n\end{aligned} \tag{12}$$

which is arranged in the following manner:

$$
\begin{aligned}
&\left\{24\varepsilon+\frac{12\xi h}{x_i}+4h^2\alpha(x_i)+\beta(x_i)h^3\right\}c_{i-2}\\
&+\left\{-72\varepsilon-\frac{12\xi h}{x_i}+12\alpha(x_i)h^2+11\beta(x_i)h^3\right\}c_{i-1}\\
&+\left\{72\varepsilon-\frac{12\xi h}{x_i}-12\alpha(x_i)h^2+11\beta(x_i)h^3\right\}c_{i}\\
&+\left\{-24\varepsilon+\frac{12\xi h}{x_i}-4h^2\alpha(x_i)+\beta(x_i)h^3\right\}c_{i+1}\\
&=24h^3 f(x_i),\\
&i=1,2,\ldots,n
\end{aligned}
\tag{13}
$$

As we know from equation (6),

$$
\begin{aligned}
&(\varepsilon+\xi)\,y'''+\alpha(x)y'(x)\\
&+\beta(x)y(x)=f(x)
\end{aligned}
$$

Equation (13) can be arranged in the form of above equation (6), we get

$$
\begin{aligned}
&\left\{24(\varepsilon+\xi)+4h^2\alpha(x_0)+h^3\beta(x_0)\right\}c_{-2}\\
&+\left\{-72(\varepsilon+\xi)+12h^2\alpha(x_0)+11h^3\beta(x_0)\right\}c_{-1}\\
&+\left\{72(\varepsilon+\xi)-12h^2\alpha(x_0)+11h^3\beta(x_0)\right\}c_{0}\\
&+\left\{-24(\varepsilon+\xi)-4h^2\alpha(x_0)+h^3\beta(x_0)\right\}c_{1}\\
&=24h^3 f(x_0),\\
&i=1,2,\ldots,n
\end{aligned}
\tag{14}
$$

The given boundary condition becomes

$$
\begin{aligned}
&c_{-2}+11c_{-1}\\
&+11c_0+c_1=24\alpha_0
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
&4c_{N-2}+12c_{N-1}-12c_N\\
&-4c_{N+1}=24h\beta_0
\end{aligned}
\tag{16}
$$

and

$$
\begin{aligned}
&c_{N-2}+11c_{N-1}\\
&+11c_N+c_{N+1}=24\beta_1
\end{aligned}
\tag{17}
$$

Equation (14), (15), (16) and (17) lead to a $(N+4)\times(N+4)$ system with $(N+4)$ unknowns.

Now, we write the above system of equations in the following form: $[S\,x_N=I_N]$, where $x_N=(c_{-4},c_{-3}c_{-2},c_{-1},c_0,c_1,c_2\ldots c_{n-2},c_{n-1})^T$ are unknown,

$$
I_N=\begin{pmatrix}24\alpha_0,24h^3f(x_0),\ldots..,\\ 24h^3f(x_i),24h\beta_0,24\beta_1\end{pmatrix}
$$

and the coefficient matrix S is given by:

$$
\left[\begin{array}{l}
1\quad 11\quad 11\quad 1\\
24(\varepsilon+\xi)+4h^2\alpha(x_0)+h^3\beta(x_0)\\
-72(\varepsilon+\xi)+12h^2\alpha(x_0)+11h^3\beta(x_0)\\
72(\varepsilon+\xi)-12h^2\alpha(x_0)+11h^3\beta(x_0)\\
-24(\varepsilon+\xi)-4h^2\alpha(x_0)+h^3\beta(x_0)\\
24\varepsilon+\frac{12\xi h}{x_1}+4h^2\alpha(x_1)+\beta(x_1)h^3\\
-72\varepsilon-\frac{12\xi h}{x_1}+12\alpha(x_1)h^2+11\beta(x_1)h\\
72\varepsilon-\frac{12\xi h}{x_1}-12\alpha(x_1)h^2+11\beta(x_1)h^3\\
-24\varepsilon+\frac{12\xi h}{x_1}-4h^2\alpha(x_1)+\beta(x_1)h^3\\
0\quad 24\varepsilon+\frac{12\xi h}{x_2}+4h^2\alpha(x_2)+\beta(x_2)h^3\\
-72\varepsilon-\frac{12\xi h}{x_2}+12\alpha(x_2)h^2+11\beta(x_2)h\\
72\varepsilon-\frac{12\xi h}{x_2}-12\alpha(x_2)h^2+11\beta(x_2)h^3\\
0\quad \vdots\quad \vdots\quad \vdots\\
\vdots\quad \cdots\quad \cdots\quad \cdots\\
0\quad \cdots\quad \cdots\quad \cdots\\
0\quad \cdots\quad \cdots\quad \cdots
\end{array}\right.
$$

$$
\left.\begin{array}{cccccc}
0 & 0 & \cdots & \cdots & \cdots & 0\\
0 & 0 & \vdots & \vdots & \vdots & \cdots\\
0 & \vdots & \vdots & \vdots & \vdots & \cdots\\
\multicolumn{6}{l}{-24\varepsilon+\frac{12\xi h}{x_2}-4h^2\alpha(x_2)+\beta(x_2)h^3\ \cdots\vdots\ \vdots\vdots\cdots}\\
\cdots & \cdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & \cdots & 4 & 12 & -12 & -4\\
\cdots & 0 & 1 & 11 & 11 & 1
\end{array}\right]
$$

$$
\begin{bmatrix}c_{-4}\\ c_{-3}\\ c_{-2}\\ .\\ .\\ .\\ c_{n-2}\\ c_{n-1}\end{bmatrix}=\begin{bmatrix}24\alpha_0\\ 24h^3f(x_0)\\ .\\ .\\ .\\ 24h^3f(x_n)\\ 24h\beta_0\\ 24\beta_1\end{bmatrix}
\tag{18}
$$

To find the approximate solution of equation (4), we arrange the system of equations in above matrix form and solve them by using MATLAB.

4. Numerical Illustration

In this section, we have solved three examples by using Quartic B-spline Method to show the accuracy of present method. Results of these examples are calculated by Using MATLAB. A good compatibility is found between the exact and approximate solution.

Example 4.1: Consider the following singular singularly perturbed boundary value problem:

$$\varepsilon y'''(x)+\frac{1}{x}y''(x)+y(x)=f(x)$$

having boundary conditions

$$y(0)=0;\ y(1)=3\varepsilon\sin 3;\ y'(0)=9\varepsilon$$

where

$$f(x)=3\varepsilon\left[\sin 3x-27\cos 3x-\frac{9}{x}\sin 3x\right].$$

The exact solution of the given problem is

$$y=3\varepsilon\sin 3x.$$

The error analysis for the given problem 4.1 is written into the table (Table 4.1) for different values of ε and N.

Table 4.1. Maximum absolute error

$\varepsilon=2^{-k}$	N = 16	32	64	128
K=4	1.408641942707289 E-005	3.421696622846193 E-006	8.471215559024969 E-007	2.112224965766796 E-007
K=6	1.268640622345552 E-006	2.837670777938728 E-007	6.893078990238832 E-008	1.710704040230882 E-008
K=8	1.187132145225034 E-007	2.086508208050397 E-008	4.706519502253870 E-009	1.145222777720889 E-009
K=10	1.776966710971897 E-008	1.881190494178175 E-009	3.305320266620837 E-010	7.462028911442431 E-011
K=12	3.860779092148929 E-009	2.779118613840600 E-010	2.941789075793649 E-011	5.180643132367269 E-012
K=14	9.370918470450601 E-010	6.033533216228890 E-011	4.344903505992120 E-012	4.596763779080720 E-013
K=16	2.325168717436685 E-010	1.458473243570605 E-011	9.417747479220293 E-013	6.784774565088414 E-014
K=18	5.801947310313466 E-011	3.618841742586486 E-012	2.277608897788058 E-013	1.470452584565254 E-014
K=20	1.449800941759943 E-011	9.030016483194895 E-013	5.648037672796656 E-014	3.556167879159404 E-015

$\varepsilon=2^{-k}$	N = 256	512	1024
K=4	5.276607850657200 E-008	1.319012452039026 E-008	3.300913903325764 E-009
K=6	4.269443867888079 E-009	1.066775466573855 E-009	2.674169843897012 E-010
K=8	2.843509438327230 E-010	7.096327865607766 E-011	1.775297096828776 E-011
K=10	1.816591495285014 E-011	4.511353096647852 E-012	1.127476584317222 E-012
K=12	1.170129748304105 E-012	2.848949375022780 E-013	7.085749088170390 E-014
K=14	8.101405288805930 E-014	1.830233555856486 E-014	4.454959621688626 E-015
K=16	7.183083338205276 E-015	1.267561088643537 E-015	2.841355080905605 E-016
K=18	1.059633134817446 E-015	1.126706285778725 E-016	2.179246366695864 E-017
K=20	2.296937359552113 E-016	1.672551257648341 E-017	2.455124997616590 E-018

Example 4.2: Consider the following singular singularly perturbed boundary value problem:

$$\varepsilon y'''+\frac{5}{x}y''+8y'+3y=f(x)$$

having boundary conditions

$$y(0)=0, y'(0)=0, y(1)=0$$

where

$$f(x)=-3x^3-21x^2+16x-6\varepsilon+\frac{10}{x}-30.$$

The exact solution of the given problem is x^2-x^3.

The error analysis for the given problem 4.2 is written into the table (Table 4.2) for different values of ε and N.

Table 4.2. Maximum absolute error

$\varepsilon=2^{-k}$	N = 16	32	64	128
K=4	1.732924878958975	1.693749666730350	1.790414997256026	1.838508099755681
K=6	0.114209215003219	2.066664084000224	1.658980651586824	1.684084568221806
K=8	0.032094218187027	0.056285763371302	3.118526121505009	1.659234811467953
K=10	0.023625025801773	0.016486575595986	0.028086080952593	55.023331345600191
K=12	0.021889502398009	0.012184583046432	0.008355735754786	0.014045585292197
K=14	0.021476179304491	0.011299247171147	0.006183934090207	0.004206057066766
K=16	0.021374087675802	0.011088243480153	0.005736145810692	0.003114498163195
K=18	0.021348641521563	0.011036116438517	0.005629385393796	0.002889242254827
K=20	0.021342284769303	0.011023123339673	0.005603008757000	0.002835528028812

$\varepsilon=2^{-k}$	N = 256	512	1024
K=4	1.869596840022862	1.889526161855390	1.899489278084534
K=6	1.727618030463723	1.755650783719723	1.771522765260029
K=8	1.667286172510062	1.677525515905763	1.698830746469149
K=10	1.663330964088996	1.667347034698080	1.669352395470570
K=12	1.747038229913353	1.666359908228204	1.668365381623804
K=14	0.007025488510836	0.572801173097219	1.668118773338619
K=16	0.002110125118765	0.003513671604772	0.244580347201779
K=18	0.001562876315359	0.001056842874341	0.001757099074114
K=20	0.001449898253786	7.828444272189221E-004	5.288672874607766E-004

Example 4.3: Consider the following singular singularly perturbed boundary value problem:

$$\varepsilon y''' + \frac{2}{x} y'' + y' + y = f(x)$$

having boundary conditions

$$y(0) = 0,\ y(1) = 1, y'(1) = \frac{1}{\sqrt{\varepsilon}} \frac{\cos\left(\frac{1}{\sqrt{\varepsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\varepsilon}}\right)}$$

where

$$f(x) = \frac{\sin\left(\frac{x}{\sqrt{\varepsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\varepsilon}}\right)} \left[1 - \frac{2}{\varepsilon x}\right].$$

The exact solution of the given problem is

$$y(x) = \frac{\sin\left(\frac{x}{\sqrt{\varepsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\varepsilon}}\right)}.$$

The error analysis for the given problem 4.3 is written into the table (Table 4.3) for different values of ε and N.

Table 4.3.1. Maximum absolute error at $\varepsilon = 10^{-2}$

N	Exact Solution	Approximate Solution	Maximum absolute error
16	-1.075504720733982	-1.096536791543242	0.021032070809260
32	-0.565122397691108	-0.558537365118708	0.006585032572400
64	-0.830430037454106	-0.828926217849321	0.001503819604785
128	-0.699911090527186	-0.699533938252734	3.771522744527855 E-004
256	-0.356738183515779	-0.356642677153630	9.550636214861186 E-005
512	-0.250529290696717	-0.250505140897063	2.414979965409403 E-005
1024	-0.125557897412189	-0.125551821729412	6.075682777190172 E-006

Table 4.3.2. Maximum absolute error at $\varepsilon = 10^{-3}$

N	Exact Solution	Approximate Solution	Maximum absolute error
16	- 4.773189588966905	-5.199370483450619	0.426180894483714
32	- 4.742319633326208	- 4.780429449098111	0.038109815771903
64	2.309129262760391	2.265764705444567	0.043364557315824
128	1.190718013228124	1.197898616065857	0.007180602837733
256	0.599930294390805	0.601059593803903	0.001129299413099
512	2.569318099645588	2.569559755922741	2.416562771530018 E-004
1024	2.175669287205748	2.175727643086610	5.835588086178234 E-005

5. Conclusion

In this paper, Quartic B-spline Method for solving third-order singular singularly perturbed boundary value problem is proposed. There are three examples which are solved by using present method. The results obtained by this method are shown in Table 4.1, 4.2, 4.3.1 and 4.3.2 respectively. It is clear that from the numerical examples of the present method gives better agreement with the exact and approximate values. Hence Quartic B-spline method is very efficient and its implementation is also very easy and accurate to evaluate according to the given problem and boundary conditions.

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