
Guoyu Zhang, Lijun Huang, Zudeng Yu, Yang Liu*, Hong Li*, Min Zhang

School of Mathematical Sciences, Inner Mongolia University, Hohhot, China

*Corresponding author: mathliuyang@aliyun.com; smslih@imu.edu.cn

Received January 26, 2014; Revised February 20, 2014; Accepted February 28, 2014

Abstract In this article, a fully discrete two-step $H^1$-Galerkin mixed method is presented for nonlinear pseudo-hyperbolic equation. The spatial direction and time direction are approximated by $H^1$-Galerkin mixed method and two-step difference method, respectively. Some a priori error results are analyzed for the scalar unknown function $u$ and its flux $q = a(x)\nabla u + a(x)\nabla u_t$. Moreover, a numerical test is made to verify our theoretical error analysis.

Keywords: two-step discrete method, $H^1$-Galerkin mixed method, nonlinear pseudo-hyperbolic equation, a priori error analysis, fully discrete scheme


1. Introduction

In this article, we consider the following nonlinear pseudo-hyperbolic equation [1-9] with initial and boundary conditions

\[
\begin{align*}
&u_t + u_t - \nabla \cdot (a(x)\nabla u + a(x)\nabla u_t) = f(u), (x,t) \in \Omega \times (0,T], \\
&u(x,t) = 0, (x,t) \in \partial \Omega \times [0,T], \\
&u(x,0) = u_0(x), u_t(x,0) = u_1(x),
\end{align*}
\]

where $\Omega$ is a bounded domain in $R^n (n=1,2,3)$, which boundary $\partial \Omega$ is smooth. The initial values $u_0(x)$ and $u_1(x)$ are known functions, the coefficient $a(x)$ is a bounded and smooth function. $f(u)$ is a nonlinear bounded function of $u$ with $f(0) = 0$ and a bounded derivative $f_u(u)$.

As we know, the pseudo-hyperbolic equations, which describe many physical phenomena, such as heat and mass transfer and nerve conduction, are a kind of important hyperbolic wave equations. In view of the importance of the pseudo-hyperbolic equations, a lot of numerical methods have been studied, such as splitting of positive definite mixed element method [1,2], least-squares mixed element method [3,4], $H^1$-Galerkin mixed element methods [5,6,7,8], mixed covolume method [9] and so forth.

Pani [10] presented and analyzed an $H^1$-Galerkin mixed method for solving the linear parabolic equation with initial and boundary condition and derived some semi-discrete and fully discrete a priori error results in detail. This method holds some good features: free of LBB condition, the freedom of approximation space’s selection and the better order of convergence for the flux in $L^2$-norm. Based on the above value points, the $H^1$-Galerkin mixed element method for many evolution equations (pseudo-hyperbolic equations [5,6], integro-differential equation [11], hyperbolic equation [12], Sobolev equation [13], RLW equation [14,15], Schrodinger equation [16], Burgers’ equation [20]) were formulated. At the same time, some good numerical methods (Nonconforming $H^1$-Galerkin mixed method [7,17], Splitting $H^1$-Galerkin mixed method [8], $H^1$-Galerkin expanded mixed methods [18,19]) based on the $H^1$-Galerkin mixed element method were developed by many authors. Liu and Li [5], Zhou [6] gave the theoretical analysis of a priori errors for $H^1$-Galerkin mixed method, respectively. Zhang et al. [7] studied a semi-discrete nonconforming $H^1$-Galerkin mixed scheme. Liu et al. [8] proposed a new splitting $H^1$-Galerkin mixed method, then analyzed a Crank-Nicolson fully discrete a priori error results. But in [5,6,7,8], the considered pseudo-hyperbolic equation is only a linear problem.

In this article, our aim is to consider a nonlinear pseudo-hyperbolic problem and to propose a fully discrete two-step difference method combining with an $H^1$-Galerkin mixed element method. Compared to backward Euler discrete method [5,6] based on the linear problem, our method get some optimal second-order results of convergence in time direction for nonlinear pseudo-hyperbolic equation (1). As far as we know, the study of the $H^1$-Galerkin mixed method based on two-step difference scheme is fairly limited in the literatures, so our
study is meaningful. We will give some a priori error analysis in detail and make some numerical simulation.

2. Two-Step Mixed Scheme and A Priori Error Estimates

2.1. Mixed Weak Formulation

By a similar selection to [5,6], we consider an auxiliary variable \( q = a(x) \nabla u + a(x) \nabla u_t \), then (1) can be split into a coupled system by:

\[
\begin{align*}
(a(x)\nabla u + a(x)\nabla u_t) &= q, \\
u_{tt} + u_t - \nabla \cdot q &= f(u).
\end{align*}
\]

(2)

In (2), by making an inner product for \( \nabla v (\forall v \in H^1_0) \) and \( \nabla \cdot w \), we use Green’s formula to obtain the mixed weak formulation: find \([u, q] : [0, T] \to H^1_0 \times H(\text{div};\Omega)\) such that:

\[
\begin{align*}
(a(x)\nabla u_t, \nabla v) + (a(x)\nabla u, \nabla v) &= (q, \nabla v), \forall v \in H^1_0, \\
(a(q^t, w) + (\nabla \cdot q, \nabla \cdot w) &= -(f(u), \nabla \cdot w), \forall w \in H(\text{div};\Omega).
\end{align*}
\]

(3)

where \( \alpha(x) = \frac{1}{a(x)} \).

2.2. Fully Discrete Scheme and Some Important Lemmas

We give a partition \( 0 = t_0 < t_1 < \cdots < t_N = T \) with mesh length \( \Delta t = T / N \), \( N \) is a positive integer) of the time interval \([0, T]\). Furthermore, we define \( \phi^n = \phi(t_n) \) for a smooth function \( \phi \) on \([0, T]\). In the following, we will get two-step time discrete scheme based on the above expression.

At \( t = t_{n+1} \), we get an equivalent form of (3) as follows:

\[
\begin{align*}
(a(x)\nabla u_{t,n}^{n+1}, \nabla v) + (a(x)\nabla u_t^{n+1}, \nabla v) &= (q^{n+1}, \nabla v), \forall v \in H^1_0, \\
(a(q^t, w) + (\nabla \cdot q, \nabla \cdot w) &= -(f^{n+1}(u), \nabla \cdot w), \forall w \in H(\text{div};\Omega).
\end{align*}
\]

By two-step discrete method, (4) can be rewritten as:

\[
\begin{align*}
(a(x)\nabla u_{t,n}^{n+1}, \nabla v) &+ (a(x)\nabla u_t^{n+1}, \nabla v) \\
&= (q^{n+1}, \nabla v) - (R_{u}^{n+1}, \nabla v), \forall v \in H^1_0, \\
&= (a(x)\nabla u_t^{n+1} - 4a(x)\nabla u^n + a(x)\nabla u^{n-1}], \nabla v) \\
&+ (a\alpha q^{n+1} + a\alpha q^n, \nabla \cdot w) + (\nabla \cdot q^{n+1}, \nabla \cdot w) \\
&= -(f^{n+1}(u), \nabla \cdot w) - (f^n(u) - f^{n-1}(u), \nabla \cdot w), \forall w \in H(\text{div};\Omega).
\end{align*}
\]

(5)

where

\[
\begin{align*}
R_{u}^{n+1} &= a(x)\nabla u_{t,n}^{n+1} - a(x)\nabla u_t^{n+1} - 3a(x)\nabla u^n - 4a(x)\nabla u^{n-1}] \frac{2\Delta t}{2\Delta t}, \\
R_{q}^{n+1} &= a\alpha q^{n+1} - 4a\alpha q^n + a\alpha q^{n-1}] \frac{2\Delta t}{2\Delta t}, \\
R_{f}^{n+1} &= f^{n+1}(u) - (2f^n(u) - f^{n-1}(u)),
\end{align*}
\]

(6)

Based on the weak formulation (5), the fully discrete two-step mixed scheme is to find \([U^{n+1}, Q^{n+1}] \) in \( V_h \times W_h \) such that:

\[
\begin{align*}
(a(x)\nabla U^{n+1}, \nabla v_h) &+ (a(x)\nabla U^n + a(x)\nabla U^{n-1}], \nabla v_h) \\
&= (Q^{n+1}, \nabla v_h), \forall v_h \in V_h, \\
(a\alpha Q^{n+1} - 4a\alpha Q^n + a\alpha Q^{n-1}], w_h) &+ (\nabla \cdot Q^{n+1}, w_h) \\
&= -(2f^n(U) - f^{n-1}(U), \nabla \cdot w_h), \forall w_h \in W_h.
\end{align*}
\]

We now introduce two important lemmas with \( \eta = u - \tilde{u}^h \) and \( \rho = q - \tilde{q}^h \) to analyze some a priori error results.

**Lemma 1** [10,21] Define a Ritz projection \( \tilde{u}^h \) in \( V_h \) of \( u \) by:

\[
(V(u - \tilde{u}^h), \nabla v_h) = 0, \forall v_h \in V_h.
\]

(7)

and the estimates hold

\[
\| \eta \|_{L^2} + \| \eta_t \|_{L^2} \leq C h^{k+1-j} (\| u \|_{L^2} + \| u_t \|_{L^2}), j = 0, 1
\]

(8)

**Lemma 2** [10] Let \( \tilde{q}^h \) be the standard finite element interpolant of \( q \), then holds

\[
\| \rho \|_{L^2} + \| \rho \|_{H(\text{div};\Omega)} \leq C h^{r+1} \| q \|_{r+1}.
\]

(9)

For the need for error analysis, we decompose

\[
\begin{align*}
u(t_n) - U^n &= (u(t_n) - \tilde{u}^h(t_n)) + (\tilde{u}^h(t_n) - U^n) = \eta^n + \zeta^n, \\
q(t_n) - Q^n &= (q(t_n) - \tilde{q}^h(t_n)) + (\tilde{q}^h(t_n) - Q^n) = \rho^n + \xi^n.
\end{align*}
\]

(10)

Combine (5) and (7) with (8) to get the error equations:

\[
\begin{align*}
(a(x)\nabla \zeta^{n+1}, \nabla v_h) &+ (a(x)\nabla \zeta^n - 4a(x)\nabla \zeta^{n-1}, \nabla v_h) \\
&= -(f^{n+1}(u) - (2f^n(u) - f^{n-1}(u)), \nabla \cdot w_h), \forall w_h \in V_h,
\end{align*}
\]

(11)
and bounded
hold
in (12) to get
in (20), sum
, the following equality holds
(14)

\[
\begin{align*}
&\text{Lemma 4 [15]} \quad \text{For } R_u^{n+1}, R_q^{n+1} \text{ and } R_f^{n+1}, \text{ the following errors hold:} \\
&\quad \| R_u^{n+1} \| \leq C \Delta t \| V_u \| _{L^2}^\infty (L^2), \\
&\quad \| R_q^{n+1} \| \leq C \Delta t \| V_q \| _{L^2}^\infty (L^2), \\
&\quad \| R_f^{n+1} \| \leq C \Delta t \| f_u (u) \| _{L^2}^\infty (L^2). 
\end{align*}
\]

\[
\text{Lemma 4 [15]} \quad \text{For a sequence } \{ \delta^n \}, \text{ the following equality holds}
\]
\[
\frac{\kappa}{2} \frac{3 \delta^{n+1} - 4 \delta^n + \delta^{n-1}}{2} = \frac{1}{\Delta t} \left[ \left( \kappa^2 \delta^{n+1} \right)^2 + \left( \kappa^2 (2 \delta^{n+1} - \delta^n) \right)^2 ight]
\]

\[
\begin{align*}
&\leq \frac{1}{\Delta t} \left( \left( \kappa^2 \delta^n \right)^2 + \left( \kappa^2 (2 \delta^n - \delta^{n-1}) \right)^2 \right) \\
&= \frac{1}{\Delta t} \left( \left( \kappa^2 \delta^n \right)^2 + \left( \kappa^2 (2 \delta^n - \delta^{n-1}) \right)^2 \right) \\
&\quad \text{by applying Cauchy-Schwarz inequality and Young inequality, we arrive at}
\end{align*}
\]

\[
\begin{align*}
&\frac{\alpha}{2} \| V_{\xi}^{n+1} \|^2 + \frac{1}{4 \Delta t} \left[ \left( \frac{1}{2} \right)^2 \right] \\
&\leq \frac{\alpha}{2} \| V_{\xi}^{n+1} \|^2 + \frac{1}{4 \Delta t} \left[ \left( \frac{1}{2} \right)^2 \right]
\end{align*}
\]

\[
\begin{align*}
&\text{The Error Estimates of Two-step Mixed Element} \\
&\text{Theorem 1} \quad \text{Assuming that } U^0, U^1 \in V_h \text{ and } Q^0, Q^1 \in W_h, \text{ then, for } 1 \leq J \leq M, j = 0,1 \text{ hold}
\end{align*}
\]

\[
\begin{align*}
&\| u^{J+1} - U^{J+1} \| \leq C \| h^{min(k+1-J, r)} + 2 \Delta t \|, j = 0,1 \\
&\| q^{J+1} - Q^{J+1} \| + \left( \sum_{n=0}^{J} \| q^{n+1} - Q^{n+1} \| \right)^2 + \frac{1}{\text{H(div, } \Omega)} \leq C \| h^{min(k+1-j, r)} + \Delta t^2 \|.
\end{align*}
\]

\[
\begin{align*}
&\text{Proof: Set } \hat{v}^{n+1} = v^{n+1} \text{ in (11) and use lemma 4 to get}
\end{align*}
\]

\[
\begin{align*}
&\| a^2 V_{\xi}^{n+1} \|^2 + \phi (x) \left[ \frac{3 \delta^{n+1} - 4 \delta^n + \delta^{n-1}}{2 \Delta t}, V_{\xi}^{n+1} \right] \\
&= \| a^2 V_{\xi}^{n+1} \|^2 + \frac{1}{4 \Delta t} \left[ \left( \frac{1}{2} \right)^2 \right] \\
&\quad \text{by applying Cauchy-Schwarz inequality and Young inequality to get}
\end{align*}
\]

\[
\begin{align*}
&\frac{\alpha}{2} \| V_{\xi}^{n+1} \|^2 + \frac{1}{4 \Delta t} \left[ \left( \frac{1}{2} \right)^2 \right] \\
&\leq C \| \nabla \cdot \rho^{n+1} \| \|^2 + \frac{\alpha}{2 \Delta t} \left[ \left( \frac{1}{2} \right)^2 \right] \\
&\quad \text{by applying Cauchy-Schwarz inequality and Young inequality to get}
\end{align*}
\]

\[
\begin{align*}
\left( \frac{3 \delta^{n+1} - 4 \delta^n + \delta^{n-1}}{2 \Delta t}, V_{\xi}^{n+1} \right) + \| \nabla \cdot \xi^{n+1} \| \leq C \| \nabla \cdot \rho^{n+1} \| \|^2 + \frac{\alpha}{2 \Delta t} \left[ \left( \frac{1}{2} \right)^2 \right]
\end{align*}
\]
Use lemma 4 to get
\[
\frac{3}{2} \frac{3^{n+1} - 4^{n} + 2^{n-1}}{2^{n+1}}
\]
\[
= \frac{1}{4 \Delta} \left( \left\| \sqrt{2} \alpha \varepsilon_{n+1} \right\|^2 + 2 \left\| \sqrt{2} \alpha \varepsilon_{n+1} - \sqrt{2} \alpha \varepsilon_{n} \right\|^2 \right)^{2}
\]
\[
- \left( \left\| \sqrt{2} \alpha \varepsilon_{n+1} \right\|^2 + 2 \left\| \sqrt{2} \alpha \varepsilon_{n+1} - \sqrt{2} \alpha \varepsilon_{n-1} \right\|^2 \right)
\]
\[
+ \left\| \sqrt{2} \alpha \varepsilon_{n+1} - 2 \sqrt{2} \alpha \varepsilon_{n} + \sqrt{2} \alpha \varepsilon_{n-1} \right\|^2 + 2 \left\| \varepsilon_{n+1} \right\|^2
\]
\[
\leq C \left( \left\| \eta \right\|^2 + \left\| \eta^{n-1} \right\|^2 + \left\| \eta^{n-2} \right\|^2 \right)
\]
(25)

Using differential mean value theorem, we have for \( u_1 \) and \( u_2 \)
\[
\left( 2 \left( f^n(u) - f^n(U) - (f^n(u) - f^n(U), \nabla \cdot \varepsilon_{n+1} \right) \right)
\]
\[
= \left( 2 \left( f^n(u_1)(u^n - U^n) \right) \right)
\]
\[
- \left( 2 \left( f^n(u_2)(u^n - U^n), \nabla \cdot \varepsilon_{n+1} \right) \right)
\]
\[
\leq C \left( \left\| \eta \right\|^2 + \left\| \eta^{n-1} \right\|^2 \right) \left\| \varepsilon_{n+1} \right\|^2
\]
(26)

As the result in [15], we have
\[
\left\| \frac{3}{2} \frac{3^{n+1} - 4^{n} + 2^{n-1}}{2^{n+1}} \right\|^2 \leq C \frac{(\int_{t_0}^{t_{n+1}} \| \rho \|^2 \, ds)}{\Delta t}
\]
(27)

Substitute (25)-(27) into (24) and note that lemma 3 to get
\[
\frac{1}{4 \Delta} \left( \left\| \sqrt{2} \alpha \varepsilon_{n+1} \right\|^2 + 2 \left\| \sqrt{2} \alpha \varepsilon_{n+1} - \sqrt{2} \alpha \varepsilon_{n} \right\|^2 \right)^{2}
\]
\[
- \left( \left\| \sqrt{2} \alpha \varepsilon_{n+1} \right\|^2 + 2 \left\| \sqrt{2} \alpha \varepsilon_{n+1} - \sqrt{2} \alpha \varepsilon_{n-1} \right\|^2 \right)
\]
\[
+ \left\| \sqrt{2} \alpha \varepsilon_{n+1} - 2 \sqrt{2} \alpha \varepsilon_{n} + \sqrt{2} \alpha \varepsilon_{n-1} \right\|^2 + 2 \left\| \varepsilon_{n+1} \right\|^2
\]
\[
\leq C \left( \left\| \eta \right\|^2 + \left\| \eta^{n-1} \right\|^2 + \left\| \eta^{n-2} \right\|^2 \right)
\]
(28)

By a combination of (9) and (10) with Gronwall lemma, we have
\[
\left( \left\| \sqrt{2} \alpha \varepsilon_{n+1} \right\|^2 + 2 \left\| \sqrt{2} \alpha \varepsilon_{n+1} - \sqrt{2} \alpha \varepsilon_{n} \right\|^2 \right)^{2}
\]
\[
+ \left( \left\| \sqrt{2} \alpha \varepsilon_{n+1} - 2 \sqrt{2} \alpha \varepsilon_{n} + \sqrt{2} \alpha \varepsilon_{n-1} \right\|^2 + 2 \left\| \varepsilon_{n+1} \right\|^2
\]
\[
\leq C \left( h^2 \min(k+1, r) + \Delta t^4 \right)
\]
(30)

Substitute (30) into (21) and (22) to get
\[
\left\| \varepsilon_{n+1} \right\|^2 + 2 \Delta t \frac{J}{n=1} \left\| \varepsilon_{n+1} \right\|^2 + \left( \left\| \varepsilon_{n+1} \right\|^2 \right)^{2}
\]
\[
+ \left( \left\| \varepsilon_{n+1} - \varepsilon_{n} \right\|^2 \right)^{2}
\]
\[
\leq C \left( h^2 \min(k+1, r) + \Delta t^4 \right)
\]
(31)

By a combination (28), (31), (9) and (10) with triangle inequality, we get the conclusion of theorem 1.

3. Numerical Test

In this section, for verifying the theoretical analysis, we take 1-D nonlinear pseudo-hyperbolic equation with space-time domain \([0,1] \times [0,1] \]

Table 1. Optimal errors in \( L^2 \) norm

<table>
<thead>
<tr>
<th>( h = 2 \Delta t )</th>
<th>( \left| u_h - u \right|_{L^2} )</th>
<th>Order</th>
<th>( \left| q_h - q \right|_{L^2} )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>1.0396E-02</td>
<td>2.2453E-02</td>
<td>1.9998</td>
<td>5.5425E-04</td>
</tr>
<tr>
<td>1/20</td>
<td>2.6083E-03</td>
<td>1.9970</td>
<td>1.3753E-04</td>
<td>2.0108</td>
</tr>
<tr>
<td>1/40</td>
<td>6.5341E-04</td>
<td>1.19970</td>
<td>6.5425E-04</td>
<td>2.0059</td>
</tr>
<tr>
<td>1/80</td>
<td>1.6353E-04</td>
<td>2.0030</td>
<td>2.0083</td>
<td></td>
</tr>
<tr>
<td>1/160</td>
<td>4.0907E-05</td>
<td>2.0030</td>
<td>2.0083</td>
<td></td>
</tr>
</tbody>
</table>
4. Some Conclusions and Remarks

So far, the $H^1$-Galerkin mixed element method [10] has been considered to seek for the numerical solutions by more and more people. In the literatures on the $H^1$-Galerkin mixed method, the backward Euler method and Crank-Nicolson scheme are widely presented. However, the two-step difference method based on the $H^1$-Galerkin mixed method is sparingly studied and analyzed. In this article, we consider a two-step difference method in time direction combining $H^1$-Galerkin mixed method and obtain some optimal time second-order rates of convergence.

From the results of numerical calculations, we can clearly find that fully discrete two-step $H^1$-Galerkin mixed method is efficient. In the near future, the fully discrete two-step difference method combined with $H^1$-Galerkin mixed method for nonlinear pseudo-hyperbolic integro-differential equations is analyzed.

Acknowledgement

In this paper, the research was sponsored by the National Natural Science Fund (Project No. 11301258; 11361035), Natural Science Fund of Inner Mongolia (Project No. NJZZ12011; NJZY13199) and The National Undergraduate Innovative Training Project (201310126040).

References