Prediction of Yielding Onset and Spread Pattern in Functionally Graded Thick-Walled Cylindrical Vessel Subjected to Thermo-Mechanical Loading

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Abstract This research is concerned with the prediction of yield behavior of functionally graded (FG) thick-walled cylindrical pressure vessel under the combination of internal pressure and temperature gradient. In order to obtain the yield onset location and plastic growth zone, we applied Von-Mises yield criteria besides Prantle-Reuss flow rule in plane strain condition. This approach results in defining new formulation in order to determine the elastic limit pressure. In such cases as internal pressure with/without thermal gradient, the start and the spread of plastic zone are predicted. Moreover, the distributions of stress components are obtained along with the radial direction in elastic and plastic regions. The analysis results show that both FG power index and temperature gradient influence the commencement of plastic deformation. Using the appropriate choice of the FG parameters and the specified thermal gradient, the plastic zone can commence simultaneously from inside, outside or intermediate radius.

Keywords: thermo-elastoplastic analysis, pressure cylindrical vessel, von-mises yield criterion, functionally graded material


1. Introduction

In modern industries, when the elements of a structure are exposed to high temperature gradient, employing the homogeneous materials in fabrication of the elements cannot meet the demands of the designer. At high temperatures, metals are weak against oxidation, corrosion, creep, etc. whilst using the pure ceramics as a good material coping well with the favorite thermal barrier properties may not satisfy such properties as toughness and high strength. In recent years, functionally graded materials have been broadly extended in many engineering fields such as aerospace structures, nuclear reactors, chemical plants, micro-scale and nano-scale devices, semiconductors, and biomedical industries. FGMs are a special kind of modern composites that are microscopically nonhomogeneous where continuous variation of the mechanical properties from one surface to the other are without any sudden changes. Studying the behavior of the thick-walled cylindrical vessel under thermal and mechanical loading is important in engineering applications. However, the mechanical and thermo-mechanical properties of the functionally graded materials are improved when compared with pure metal or pure ceramics because the stress concentration is reduced and the yield onset is influenced by thermal and mechanical loading. Several studies have been carried out on FG pressure vessel. For the sake of brevity we focus on those dealing with the elasto-plastic analysis of pressure vessels. Whalley [1] determined the distribution of stresses in a long elastic cylinder when subjected to pressure and thermal loading. In his study, the greatest shear stress under combined loading was calculated with respect to variation of thickness ratios and end conditions. Tresca's criterion was employed for yield onset with independency of elastic constants of temperature. An exact solution was proposed for thick-walled cylindrical vessel made of elastic linear-hardening material. The distribution of the hoop and equivalent stresses was optimized in the elasto-plastic analysis by considering the Bauschinger effect and the yield criterion of Tresca [2]. Darjani, Kargarnovin, and Naghdabadi [3] analyzed the steady thermal stresses in a functionally graded hollow cylinder in their paper. They converted the boundary value problem into a Fredholm integral equation. By solving the resulting equation numerically, they found that distribution of the thermal stresses may be smoother by using appropriate gradient for distribution of the FG material. Nayak [4] proposed a general analytical solution to obtain the thermo-mechanical stresses in a thick spherical vessel made of FGM. The governing equations were derived in terms of displacement components called Navier equations. Sadeghian and Ekhteraei-Toussi [5] solved the axisymmetric elastic-plastic problem of radially nonhomogeneous cylindrical shells under combination of thermal gradient and internal pressure based one Tresca.
plastic criteria and small deformation theory. Nayebi and Sadrabadai [6] reported the elasto-plastic analysis of a spherical shell of functional graded materials under thermo-mechanical loading. They applied the linear strain hardening behavior in plastic region. Carrea and Soave [7] explored the possibility of using a thin layer of FGM in the interface of a two-layered wall pressure cylindrical vessel in order to diminish the normal shear stresses across the wall thickness. Shariati, Sadeghi, Ghanmad, and Gharooni [8] evaluated the stress and the displacement in FG thick-walled cylindrical pressure vessel under internal pressure. They found out that the mechanical properties of pressure vessel changed in longitudinal direction. Furthermore, there are some researches which have been published in the domain of theoretical modeling and simulation of FGM structures that are subjected to mechanical and thermo-mechanical loadings [9,10,11,12,13].

In the present research, for mathematical modeling of yielding onset and growth, elastic strain is taken into account as the calculation of total strain. To achieve this objective, Prandtl-Reuss flow rule is employed for the first time. The behavior of thermal elasto-plastic FG and thick-walled cylindrical vessel is studied based on Prandtl-Reuss flow rule. Using Von-Mises yield criteria, the behavior of thermal elastic-plastic FG thick-walled cylindrical vessel is described as following form [17]:

\[
\begin{align*}
E(r,T) &= E(b) \frac{r}{b} m_1 = E_0(T) r m_1 \\
\alpha(r,T) &= \alpha(b) \frac{r}{b} m_2 = \alpha_0(T) r m_2 \\
k(r) &= k(b) \frac{r}{b} m_3 = k_0 r m_3 \\
Y(r) &= Y(b) \frac{r}{b} m_4 = Y_0 r m_4
\end{align*}
\]

Where \(E_0, \alpha_0, k_0, Y_0\) are reference datum of Young modulus and thermal expansion coefficients, respectively. \(r\) and \(b\) denote the radial direction and outer radius, respectively. \(E_0, \alpha_0\) are depended to thermal gradient, according to Touloukian’s relation, as follows [14]:

\[
q_j = q_0 \left(q_{-1} T^{-1} + 1 + q_1 T + q_2 T^2 + q_3 T^3\right)
\]

In the above relation, \(T\) is the temperature gradient, \(q_0, q_1, q_2, q_3\) are constant parameters.

2.1. Governing Equations

Due to the symmetry of the shell geometry and loading, differential equation is governed in cylindrical coordinates, as follows:

\[
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0
\]

The constitutive equations of linear elastic materials in cylindrical coordinates are given as follows [16]:

\[
\begin{align*}
\sigma_r &= (\lambda(r) + 2\mu(r))\varepsilon_r + \lambda(r)\varepsilon_\theta - (3\lambda(r) + 2\mu(r))\alpha(r)T(r) \\
\sigma_\theta &= (\lambda(r) + 2\mu(r))\varepsilon_r + \lambda(r)\varepsilon_\theta - (3\lambda(r) + 2\mu(r))\alpha(r)T(r)
\end{align*}
\]

In above equations \(\sigma_r, \sigma_\theta\) and \(\varepsilon_r, \varepsilon_\theta\) denote the radial, circumferential stress and strain components, respectively. Also, \(\lambda, \mu\) denote the radial direction. For symmetric problems in cylindrical coordinates, the strain-displacement relations are reduced as:

\[
\varepsilon_r(r) = \frac{du(r)}{dr} = u', \varepsilon_\theta(r) = \frac{u(r)}{r} = \frac{u}{r}
\]

Based on (4), the equation (3) is rewritten as follows:

\[
\begin{align*}
r^2 u' + Aru' + Bu &= ((1 + n)\alpha_0 r m_2 / (1 - n)) \left[(m_1 + m_2)rT(r) + r^2 T'(r)\right]
\end{align*}
\]

The marks \((')\) and \((')\) denote the first and second derivatives with respect to radial coordinate. In above equation: \(A=1+m_1, B=\frac{(1+1)m_1-1}{1-v}\)

2.1.1. Thermal Analysis

Uniform one-dimensional heat conduction equation in FG cylindrical vessel is described as following form [17]:

\[
T'' + \left(k'(r) + \frac{1}{k(r)}\right) T' = 0
\]

Thermal boundary conditions (Dirichlet) imposed on the cylindrical vessel is presented as [17]:

\[
\begin{align*}
k_a \frac{dT}{dr} + h_a T &= q_a \quad \text{at} \quad r = a \\
k_b \frac{dT}{dr} + h_b T &= q_b \quad \text{at} \quad r = b
\end{align*}
\]

where \(k_a, k_b, h_a, h_b, q_a, q_b\) denote the thermal conductivity and convection coefficients in inner and outer radii, respectively.

By imposing relation (1c) into thermal differential equations (7), the general solution is obtained as:

\[
T(r) = D_1 r^{-m_3} + D_2
\]
\[ D_1 \text{ and } D_2 \text{ are constants which are depended to thermal boundary conditions, mechanical and physical properties of FG vessel (See appendix A).} \]

Based on (9) and substituting into (6), equation (6) is derived in terms of displacement as the following form:

\[ r^2u^* + Aru' + Bu = C_2r^{m_1+1} + C_1r^{m_2-m_3+1} \quad (10) \]

where \( C_1 = \frac{(1 + \nu) \alpha_0 (m_1 + m_2 - m_3)}{(1 - \nu)} \),

\[ C_2 = \frac{(1 + \nu) \alpha_0 (m_1 + m_2)}{(1 - \nu)} \].

Equation (10) is a non-homogenous Euler differential equation that is solved in terms of radial displacements, as follows [18]:

\[ u(r) = L_r r^{m_1} + L_2 r^{m_2} + L_3 r^{m_2-m_3+1} + L_4 r^{m_2+1} \quad (11) \]

In above equation, \( n_{1,2} = \frac{1 - A + \sqrt{A}}{2} \). By substituting of (11) into (5) and inserting the resulting relation into (4a) and (4b), the stress components can be obtained as follows:

\[ \sigma_r(r) = \frac{E_0 r^{m_1}}{(1 + \nu)(1 - 2\nu)} \left[ L_1 Q_1 r^{m_1-1} + L_2 Q_2 r^{m_2-1} + L_3 r^{m_2-m_3} \right] \quad (12a) \]

\[ \sigma_\theta(r) = \frac{E_0 r^{m_1}}{(1 + \nu)(1 - 2\nu)} \left[ L_1 G_1 r^{m_1-1} + L_2 G_2 r^{m_2-1} + L_3 r^{m_2-m_3} \right] \quad (12b) \]

Mechanical boundary conditions are defined in cylindrical vessel as:

\[ \sigma_r |_{r=a} = \sigma_r |_{r=b} = p_0 \quad (13) \]

where \( p_a \) and \( p_b \) is inside and outside uniform pressure, respectively. \( L_1 \) to \( L_4 \) and \( Q_1 \) to \( Q_4 \) are introduced in Appendix B. By applying (13), \( L_1, L_2, L_3, L_4 \) are determined. These parameters are depended to physical and mechanical properties of cylindrical vessel.

### 2.1.2. Yield Behaviour of Vessel

Based on Prantle-Reuss flow rule, the total strain increment is split into elastic and plastic strain increments, as:

\[ de^{e}_i = de^{p}_i + de^{\theta}_i \quad i = r, \theta, z \quad (14) \]

where \( z \) denotes the longitudinal direction. Superscripts \( e \) and \( p \) indicate to quantities in the elastic and plastic zones, respectively. According to Prantle-Reuss flow rule, the stress-strain increments relations is, as [19]:

\[ \frac{de^p_r}{\sigma_r} = \frac{1}{2} \left( \frac{\sigma_\theta + \sigma_z}{\sigma_\theta + \sigma_z} \right) \]

\[ \frac{de^p_\theta}{\sigma_\theta} = \frac{1}{2} \left( \frac{\sigma_r + \sigma_z}{\sigma_r + \sigma_z} \right) \]

\[ \frac{de^p_z}{\sigma_z} = \frac{1}{2} \left( \frac{\sigma_\theta + \sigma_r}{\sigma_\theta + \sigma_r} \right) \quad (15) \]

Imposing the plane strain condition, leads to determination of longitudinal stress:

\[ \sigma_z = \frac{1}{2} (\sigma_\theta + \sigma_r) \quad (16) \]

According to the Von-Mises yield criterion [16]:

\[ 2Y^2(r) = (\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 \quad (17) \]

where \( Y(r) \) is the yield stress as defined in (1d). By substitution (16) into (17), Von-Mises yield criterion is defined, as follows:

\[ Y(r) = \sqrt{\frac{3}{2}}(\sigma_\theta - \sigma_r) \quad (18) \]

where \( r_y \) is the specific yield onset radius. In order to determine the yield onset radius and elastic limit pressure, non-dimensional parameter \( \phi \) is defined as:

\[ \phi = \frac{\sqrt{3}}{2} \frac{a}{r} \frac{m_4}{\sigma_0} \left( \frac{r}{r_y} - \frac{r}{r_y} \right) \quad (19) \]

\[ \bar{\sigma}_i = \frac{\sigma_i}{\sigma_0 m_4} i = r, \theta \quad (20) \]

In above equation, \( \bar{\sigma}_i \) are dimensionless stresses in radial and circumferential directions. To predict the yield onset, \( \phi \) assigns values, as follows:

\[ \begin{align*}
\phi &< 1 \rightarrow \text{Elastic zone} \\
\phi &\geq 1 \rightarrow \text{Yield onset} \\
\phi &> 1 \rightarrow \text{Plastic zone} 
\end{align*} \quad (21) \]

By substituting (18) into (3) and solving, the radial and circumferential stresses are obtained as:

\[ \sigma_r(r) = \frac{2}{\sqrt{3}} \frac{r^{m_4} Y_0}{m_4} + E_1 \quad (22a) \]

\[ \sigma_\theta(r) = \frac{2}{\sqrt{3}} \frac{r^{m_4} Y_0}{m_4} + E_1 \quad (22b) \]

where \( E_1 \) is the arbitrary constant depended to boundary conditions. Based on incompressibility assumption in plastic deformation, the summation of plastic strain components are set to zero [20]:

\[ \varepsilon_r^{p} + \varepsilon_\theta^{p} + \varepsilon_z^{p} = 0 \quad (23) \]

Therefore, based on (4), (5) and with help of (14), the summation of total strain is obtained in terms of radial displacements, as follow:

\[ \frac{du}{dr} = \frac{u}{r} = \left[ \frac{(1 + \nu)(1 - 2\nu)}{E_0} \right] \frac{2}{\sqrt{3}} \frac{m_4}{m_4} + \frac{2}{E_0} \frac{(1 + \nu)(1 - 2\nu)}{r^{m_4}} + 2(1 + \nu) \alpha_0 C_1 r^{m_2 - m_3} \quad (24) \]

By solving the above equation, radial displacement is obtained in the following form:

\[ u(r) = \frac{F_1}{r} + F_2 r^{m_2 - m_3 + 1} + F_3 r^{m_2 + 1} + \frac{F_4 r^{m_4 - m_3 + 1} + F_5 r^{1 - m_1}}{r} \quad (25) \]
$F_1$, $F_2$, $F_3$ and $F_4$ are introduced in Appendix C.

By applying the thermal Dirichlet boundary conditions in (8), mechanical boundary conditions in (13) and with help of the continuity relations in yield onset location of (26), distribution of the stresses in elastic and plastic zones are calculated.

\[ u^p(r^p) = u^P(r^p) \]  \hspace{1cm} (26a)

\[ \sigma^p_r(r^p) = \sigma^P_r(r^p), \sigma^p_\theta(r^p) = \sigma^P_\theta(r^p) \]  \hspace{1cm} (26b)

where $r_p$ is representative for plastic onset radius.

3. Numerical Results

Consider a thick cylindrical vessel with inner radius of $a=0.5\text{m}$ and outer radius of $b=1\text{m}$, $\nu=0.3$. The thermal boundary conditions are taken as:

\[ q_a=0, \quad q_b=0, \quad k_a=0(\text{°C})^{-1}, k_b=1(\text{°C})^{-1} \]

\[ k_a=0(\text{°C})^{-1}, k_b=1(\text{°C})^{-1}, \Delta T=T(a)-T(b) \]

The quantities of $q_i$ parameters of steel (material in base) with respect to thermal gradient are given in Table 1 in order to determine the Young modulus and thermal expansion coefficient in different temperatures and to substitute them in (1a) and (1b). According to the datum in Table 1, some computational results are provided to sketch the stress components, radial displacement, parameter of $\phi$ and the growth pattern of yielded zone.

Table 1. Parameters of Yield Modulus and Thermal Expansion for Stainless Steel [8]

<table>
<thead>
<tr>
<th>$q_i$</th>
<th>$E_i(\text{Pa})$</th>
<th>$\alpha_i(\text{°C}^{-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>201.04×10^6</td>
<td>12.33×10^{-5}</td>
</tr>
<tr>
<td>$q_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$q_3$</td>
<td>3.079×10^{-4}</td>
<td>8.806×10^{-5}</td>
</tr>
<tr>
<td>$q_4$</td>
<td>-6.534×10^{-7}</td>
<td>0</td>
</tr>
<tr>
<td>$q_5$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 1. Variation of $\phi$ in the radial direction respect to variation of temperature gradient, $(m_1,m_2,m_3,m_4)=(2.2,-2.9,-2.2,-4.0)$

Figure 1 shows the variation of parameter of $\phi$ with respect to temperature variation along the radial direction. The vessel is subjected to internal pressure of $p=8×10^8\text{Pa}$. As illustrated, the temperature gradient influences the plastic onset radius and causes the plastic deformation launches to grow at different distance from inner surface.

Dimensionless stress variation is demonstrated in Figure 2 in two states, one with temperature gradient and the other one without temperature gradient. As Figure 2 shows, the temperature gradient influences the distribution of stresses slightly. Based on the data in Figure 2, Figure 3 is presented to show the effects of gradient temperature for $(m_1,m_2,m_3,m_4)=(-4.23,8.0,8.0,-6.0)$ on consequent dimensionless radial displacement, $\bar{u}=u/a^4\mu Y_0$. It is evident that an increase in the thermal gradient leads to an increase in the radial displacement.

Figure 2. Variation of radial and circumferential stresses along radial direction with//without temperature gradient for $(m_1,m_2,m_3,m_4)=(-4.23,8.0,8.0,-6.0)$

Figure 3. Variation of radial displacement along radial direction with//without temperature gradient

Variation of elastic limit pressure for different temperature gradient is depicted in Figure 4. As shown, yield state commences in two points of the vessel simultaneously, whereas we found two yield pressure values in two radii of pressure vessel.

In addition, we showed the distribution of plastic zone in the sectors of vessel for various quantities of thermal gradient in Figure 5. By imposing internal pressure, the middle surface is located in plastic zone whilst the inner and outer surfaces are in the elastic zone. Applying the
thermal gradient above 325°C, the outer surface is completely exposed to the plastic zone. It can be observed that increasing the temperature gradient causes the plastic region to be thickened.

Consequently, as illustrated in Figure 8, the distribution of radial and circumferential stresses is evaluated. It is obvious that raising the temperature gradient leads to the increase in the values of stress components.


table

<table>
<thead>
<tr>
<th>T(°C)</th>
<th>Present Work</th>
<th>Reference[5]</th>
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</thead>
<tbody>
<tr>
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<td>0.7761</td>
</tr>
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</tr>
<tr>
<td>600</td>
<td>0.7654</td>
<td>0.7411</td>
</tr>
<tr>
<td>700</td>
<td>0.7578</td>
<td>0.7358</td>
</tr>
</tbody>
</table>

4. Conclusion

In this paper, we implemented the thermo-elasto plastic analysis of thick-walled cylindrical vessel. For this purpose, the governing equilibrium equation of vessel was solved based on plane strain condition as an axisymmetric problem in polar coordinate system. By imposing Prantle-
Reuss relations and using Von-Mises criteria, one dimensionless normalized parameter of $\varphi$ was obtained. When $\varphi$ got less than one, the pressure vessel remained in the elastic zone, and for those values equal or larger than one, the yield condition commenced. The results showed the possibility of the formation and growth of the plastic zones along with the radial direction from the outside, inside or middle surface of the FGM pressure vessel that depended on FG index parameters and the temperature gradient. By specific internal pressure and temperature gradient, the yielding started in two radii of vessel simultaneously. Besides, for a specific internal pressure, the direction of plastic zone spread could change by varying the thermal gradient. As a suggestion to extend the research in this area, we propose studying the strain hardening behavior of steel in the plastic zone. Likewise, studying the elastic-plastic behavior of cylindrical or spherical vessel when subjected to transient gradient temperature can be interesting as a research subject in future.

Appendix A

\[ D_1 = (h_b q_a - h_a q_b) / \left[ \begin{array}{c} h_a (m_3 k_a b^{-m_3}) - h_b b^{-m_3} \\ -h_b b^{-m_3} - h_a a^{-m_3} \end{array} \right] \]

\[ D_2 = \left[ \begin{array}{c} q_a (m_3 k_b b^{-m_3}) - h_b b^{-m_3} \\ -q_b (m_3 k_a a^{-m_3}) - h_a a^{-m_3} \end{array} \right] \]

Appendix B

\[ Q_1 = (1 - \nu) n_1 + \nu \]

\[ Q_2 = (1 - \nu) n_2 + \nu \]

\[ Q_3 = L_3 \frac{(m_2 - m_3)(1 - \nu)}{1 + \nu} - D_2 \alpha_0 \]

\[ Q_4 = L_3 \frac{(m_2 - m_3)(1 - \nu)}{1 + \nu} - D_1 \alpha_0 \]

\[ G_1 = (n_1 - 1) \nu + 1 \]

\[ G_2 = (n_2 - 1) \nu + 1 \]

\[ G_3 = L_4 (1 + \nu m_2) - D_2 \alpha_0 \]

\[ G_4 = L_4 (1 + \nu m_2) - D_1 \alpha_0 \]

Appendix C

\[ F_1 = \frac{2(1 + \nu) \alpha_0 D_1}{m_2 - m_3 + 2} \]

\[ F_2 = \frac{2(1 + \nu) \alpha_0 D_2}{m_2 + 2} \]

\[ F_3 = \frac{2(1 + \nu)(1 - 2\nu)(2 + m_4) \nu_0}{\sqrt{3 m_4} E_0 (m_4 - m_1 + 2)} \]

\[ F_4 = \frac{2 E_0 (1 + \nu)(1 - 2\nu)}{E_0 (2 - m_1)} \]

References


