On Weak Solutions of Systems of Strongly Nonlinear Parabolic Variational Inequalities

A.T. El-Dessouky*

Mathematics Department, Helwan University, Faculty of Science, Cairo, Egypt
*Corresponding author: adeltohamy60@gmail.com

Abstract In this paper we prove the existence of weak solutions for systems of variational inequalities of strongly nonlinear parabolic operators: 
\[ u_t^{(i)} + A^{(i)}(u)(x,t) + g^{(i)}(x,t;u^1,...,u^d) \in Q = \Omega \times (0,T), \]
where \[ A^{(i)}(u)(x,t) = \sum_{|\alpha| \leq m}(1)\partial^{\alpha} A_{\alpha}^{(i)}(x,t;D(u^1(x,t),...,u^d(x,t))), \]
\[ \ell = 1,2,...,d. \]

Keywords: strongly nonlinear parabolic operators-Systems of variational inequalities


1. Introduction

Consider the parabolic initial-boundary value problem
\[ u_t + A(u)(x,t) + g(x,t;u) = f(x,t), \]
in \[ Q = \Omega \times (0,T), \]
\[ u(0) = 0, \]
in \[ \Omega \]
\[ D^\alpha u = 0 \text{ on } \partial \Omega \times (0,T) \text{ for } |\alpha| \leq m - 1 \]
where
\[ A(u)(x,t) = \sum_{|\alpha| \leq m}(1)\partial^{\alpha} A_{\alpha}(x,t) \]
and \[ Du = (D^\alpha u)_{|\alpha| \leq m}. \]
If the coefficients \[ A_{\alpha} \] satisfy a polynomial growth conditions of order \( p \) in \( u \) and its space derivatives but \( g \) obeys no growth in \( u \), but merely a polynomial growth conditions of order \( p - 1 \) in \( u \) and its time derivative, the existence of weak solutions problems of the type (1) has been obtained by many authors (cf [1,4] and [5]). In [2] Browder and Brézis extended the above results to the corresponding class of variational inequalities. Their proof based on a type of compactness result. Our result can be viewed as a generalization to systems of variational inequalities for the work of [4] and [5]. Our proof relies on deriving a-priori bound for the time derivative of the solution in \( L^2(Q) \).

2. Prerequisites

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with a smooth (uniform \( C^m \)) boundary, \( 1 < p < \infty \) and \( m \) a positive integer. Denote by \( V = V^1 \times ... \times V^d \)
the Sobolev space
\[ \|v\|_{m,p} = \|v\|_X + \|D^\alpha v\|_{X^*}, \]
\[ \|v\|_p = \frac{1}{p} \int_Q \sum_{i=1}^d \sum_{|\alpha| \leq m} \partial^{\alpha} A_{\alpha}^{(i)}(x,t) u(x,t)^p dx. \]

For the Galerkin method, construct a sequence \( (w_i^{(i)})_{i=1}^\infty \subset \{C_0^0(\Omega)\}^d \) such that \( \bigcup_{i=1}^\infty \mathbb{Z}_n^{(i)} \) is dense in \( \{W_0^{1,p}(\Omega)\}^d \), \( j \geq p + m + N \). Denote by \( Y_n = C(\Omega) \times \mathbb{Z}_n^{(1)} \times ... \times \mathbb{Z}_n^{(d)} \).
Since \( \{W_0^{1,p}(\Omega)\}^d \) is continuously embedded in \( \{C_0^0(\Omega)\}^d \), which is a Banach space with the norm
\[ \|v\|_{C_0^0(\Omega)}^d = \sum_{i=1}^d \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} |\partial^{\alpha} u^{(i)}(x)|, \]
then for any \( v \in \{W_0^{1,p}(\Omega)\}^d \) there exists a sequence \( (v_k) \subset U \) such that \( v_k \to v \) in \( \{W_0^{1,p}(\Omega)\}^d \). Moreover, since the closure of \( \bigcup_{n=1}^\infty Y_n \) in \( \{C_0^0(\Omega)\}^d \), the set \( \mathbb{Z}_n^{(1)} \times ... \times \mathbb{Z}_n^{(d)} \) contains \( \{C_0^0(\Omega)\}^d \), then for any \( f \in \mathbb{X} \) there exists a sequence \( (f_k) \subset \mathbb{X} \) such that \( f_k \to f \) in \( \mathbb{X} \) in the weak sense [5].

We introduce the following hypotheses for \( A^{(i)}(u) + g^{(i)}(x,t,u) \).

On Weak Solutions of Systems of Strongly Nonlinear Parabolic Variational Inequalities
A1) $A_{\alpha}^{(t)}: \mathbb{Q} \times \mathbb{R}^d \to \mathbb{R}$ is continuous in $\xi \in \mathbb{R}^d$ for almost all $(x,t) \in \mathbb{Q}$ and $c_1 > 0$ and a fixed function $K_1 \in L^p(Q)_d$

$$
\left| A_{\alpha}^{(t)}(x,t;\xi) \right| \leq c_1 \left| \xi \right|^{p-1} + K_1(x,t),
$$

for all $\alpha$, all $(x,t) \in \mathbb{Q}$, $\ell = 1, 2, \ldots, d$ and all $\xi \in \mathbb{R}^d$.

A2) For all $(x,t) \in \mathbb{Q}$ and two distinct $\xi, \xi' \in \mathbb{R}^d$

$$
\sum_{i=1}^d \sum_{j=0}^m \left| \partial_j A_{\alpha}^{(t)}(x,t;\xi) - \partial_j A_{\alpha}^{(t)}(x,t;\xi') \right| (\xi_i^{(t)} - \xi'^{(t)}_i) > 0.
$$

A3) There exists a constant $c_2 > 0$ and a fixed function $K_2 \in L^\infty(Q)_d$ such that for all $(x,t) \in \mathbb{Q}$ and all $\xi \in \mathbb{R}^d$

$$
\sum_{i=1}^d \sum_{j=0}^m \left| \partial_j A_{\alpha}^{(t)}(x,t;\xi) \right| (\xi_i^{(t)})^{(t)} \geq c_2 \left| \xi \right|^p - K_2(x,t).
$$

G) $g^{(t)}: \mathbb{Q} \times \mathbb{R}^d \to \mathbb{R}$ is continuous in $\xi \in \mathbb{R}^d$ for almost all $(x,t) \in \mathbb{Q}$ and measurable in $(x,t)$ for all $r, r' \in [0, T]$. Moreover, each $g^{(t)}$ is nondecreasing in $r$ for fixed $(x,t) \in \mathbb{Q}$ and each $g^{(t)}(x,t;0) = 0$, for all $(x,t) \in \mathbb{Q}$.

3. Formulation of the Problem

Write

$$
G_0^{(t)}(x,t,r) = \int_0^r g^{(t)}(x,t,s)ds.
$$

By G) each $G_0^{(t)}$ as a function of $r$ is convex, nonnegative and once differentiable.

For $u \in X$, set

$$
\Gamma^{(t)}(u) = \int \Omega G_0^{(t)}(x,t,u)dxdt.
$$

Let $K$ be a closed convex subset of $V$ containing the origin. Define a proper lower semicontinuous Gateaux differentiable function $\phi^{(t)}: X \to (-\infty, \infty]$:

$$
\phi^{(t)}(u) = \begin{cases} 
\Gamma^{(t)}(u) & \text{if } u(t) \in K \text{ a.e.} \ 
\infty & \text{otherwise.}
\end{cases}
$$

**Definition:** A function $u_0 = (u_0^{(t)}, \ldots, u_n^{(t)}) \in \mathbb{Y}_n$ is called a Galerkin solution of the associated variational inequalities for (1) if

$$
\begin{align*}
\int_0^T \left[ \frac{\partial u_n^{(t)}}{\partial t}, v^{(t)} - u_n^{(t)} \right] dt + \int_0^T \left[ T(u_n), v^{(t)} - u_n^{(t)} \right] dt \\
+ \phi^{(t)}(v) - \phi^{(t)}(u_n) \\
\geq \int_0^T \left[ I_n^{(t)}, v^{(t)} - u_n^{(t)} \right] dt, \\
v \in \mathbb{Y}_n, \ell = 1, 2, \ldots, d.
\end{align*}
$$

where

$$
\left[ T(u), z^{(t)} \right] = \int \Omega \sum_{i=1}^d \sum_{j=0}^m A_{\alpha}^{(t)}(x,t,Du) D^jz^{(t)}dx.
$$

The existence of a Galerkin solution and its main property, in view of our hypotheses, will be given by the following lemma [3].

**Lemma:** For every $n \in \mathbb{N}$ there exists a Galerkin solution $u_0 \in \mathbb{Y}_n \cap X$ such that

$$
\| u_0 \|_{L_2} \leq c(n \in \mathbb{N}).
$$

4. Existence Theorem

**Theorem.** Let the hypotheses $A_1$- $A_3$ and G) be satisfied. Let $f^{(t)} \in C^1(0,T; L^2(\Omega))$ be given. Then

(i) there exists $u \in \mathbb{Y}$ with $u(t) \in K$ a.e., $u(0) = 0$ such that

$$
< u^{(t)}, v^{(t)} - u^{(t)} > + < T(u), v^{(t)} - u^{(t)} > + \phi^{(t)}(v) - \phi^{(t)}(u) \geq 2\phi^{(t)}(u) > 0,
$$

for every $v \in C^1(0,T; L^2(\Omega))$ for which $\phi^{(t)}(v) < \infty$.

(ii) there exists $u \in \mathbb{Y} \cap K$ with $u(t) \in K$ a.e., $u(0) = 0$ such that

$$
< u^{(t)}, v^{(t)} - u^{(t)} > + < T(u), v^{(t)} - u^{(t)} > + \phi^{(t)}(v) - \phi^{(t)}(u) \geq 2\phi^{(t)}(u) > 0,
$$

for every $v \in C^1(0,T; L^2(\Omega))$ for which $\phi^{(t)}(v) < \infty$. 

Including the existence of a solution

$$
\xi(t) = (\xi^{(t)}(1), \ldots, \xi^{(d)}(d)) \in \mathbb{Y}_n.
$$
for every \( v \in C^1(0,T,\mathbb{R}^d) \) with \( u(t) \in \text{K.a.e.} \). Proof of (i): By the above lemma, there exist Galerkin solutions \( u_n \in Y_n \) of (2) such that 
\[
\left\| u_n(t) \right\|_2 \leq c.
\]

Set \( \nu=0 \) in (2) we get the uniform boundedness from above of the numerical sequence \( \{ \left( T(u_n), u_{n}^{(t)} \right) \}_{n \in \mathbb{N}} \). The proof will follow if we can show the following assertions for some subsequence of \( (u_n) \):
\[
\frac{\partial u_{n}^{(t)}}{\partial t} \to u_{l}^{(t)} \text{ weakly in } L^{2}(Q),
\]
\[
u_{n} \to u \text{ weakly in } X \text{ and strongly in } L^{p}\left(0,T,[W^{m-1,p}(\Omega)]^{d}\right),
\]
\[
<T(u_n),z^{(t)}> \to <T(u),z^{(t)}>, \forall z^{(t)} \in C_{0}^{\infty}(Q) \]
\[
\liminf_{n} <T(u_n),u_{n}^{(t)}> \geq <T(u),u_{l}^{(t)}>, \forall n \in \mathbb{N}
\]

and
\[
-\infty < \phi^{(l)}(u) \leq \liminf_{n} \phi^{(l)}(u_n) < \infty.
\]

To show (4): Given \( \epsilon > 0 \), any \( n \in \mathbb{N} \) and any \( w_n = (w_{n}^{(t)}, ... ,w_{n}^{(t)}) \in Y_n \), put
\[
w_{n}^{(t)} = \frac{u_{n}^{(t)} - v^{(t)}}{\epsilon}.
\]

Since \( v^{(t)} \) is arbitrary, \( w_{n}^{(t)} \) is absolutely for a given \( u_{n}^{(t)} \). Then (2) yields
\[
\left\{ \frac{\partial u_{n}^{(t)}}{\partial t}, w_{n}^{(t)} \right\} + \left\{ T(u_n), w_{n}^{(t)} \right\} = \frac{1}{\epsilon} \left[ \phi^{(t)}(u_n(t)) - \epsilon w_n(t) + \phi^{(t)}(u_n(t)) \right]
\]
\[
\leq \left( f_{n}^{(t)}(t), w_{n}^{(t)}(t) \right).
\]

In particular, since \( w_{n}^{(t)} \) is arbitrary we can write (9) in the form
\[
\left( \frac{\partial u_{n}^{(t)}}{\partial t} - \frac{u_{n}^{(t)}(t+\epsilon) - u_{n}^{(t)}(t)}{-\epsilon} \right) - T(u_n(t)) \frac{u_{n}(t+\epsilon) - u_{n}(t)}{-\epsilon} \leq \frac{\phi^{(t)}(u_n(t)) - \epsilon \frac{\partial u_{n}(t)}{\partial t} - \phi^{(t)}(u_n(t))}{-\epsilon}.
\]

\[\text{Allowing } \epsilon \to 0, \text{we get}
\]
\[
\left\{ \frac{\partial u_{n}^{(t)}}{\partial t}, f_{n}^{(t)}(t), \frac{u_{n}(t+\epsilon) - u_{n}(t)}{-\epsilon} \right\} \leq \left( \frac{\partial u_{n}^{(t)}}{\partial t}, f_{n}^{(t)}(t), \frac{u_{n}(t+\epsilon) - u_{n}(t)}{-\epsilon} \right).
\]

where \( \phi^{(t)}(u) \) is the Gateaux derivative of \( \phi(u) \) at \( u_{n}(t) \). Similarly, from (2), we get
\[
\left\{ \frac{\partial u_{n}^{(t)}}{\partial t} - \frac{u_{n}^{(t)}(t+\epsilon) - u_{n}^{(t)}(t)}{-\epsilon} \right\} - \frac{\partial u_{n}(t)}{\partial t} \left( u_{n}(t+\epsilon) - u_{n}(t) \right) \leq \left( \frac{\partial u_{n}^{(t)}}{\partial t}, f_{n}^{(t)}(t), \frac{u_{n}(t+\epsilon) - u_{n}(t)}{-\epsilon} \right).
\]

Therefore
\[
\| \frac{\partial u_{n}^{(t)}}{\partial t} \|_2 \leq \| f_{n}^{(t)}(t) \|_2 + \int_{0}^{T} \| \frac{\partial u_{n}(t)}{\partial t} \|_2 \| \frac{u_{n}(t+\epsilon) - u_{n}(t)}{-\epsilon} \|_2 dt + \frac{\| \phi^{(t)}(u_n(t)) \|_2}{2} \| \frac{\partial u_{n}(t)}{\partial t} \|_2.
\]

The above of the numerical sequence above of the numerical sequence
On the other hand, we get from (2)
\[
\left( \frac{\partial u_n(t)}{\partial t}, \frac{\partial u_n(t)}{\partial t} \right) + T(u_n(t), \frac{\partial u_n(t)}{\partial t}) \leq |\phi(t)(u_n(t))| \left( \left\| \frac{\partial u_n(t)}{\partial t} \right\|_2 + \left( \frac{f_n(t)}{2}, \frac{\partial u_n(t)}{\partial t} \right). \right.
\]

In particular,
\[
\left\| \frac{\partial u_n(t)}{\partial t} \right\|_2 \leq \left( \phi(t)(u_n(t)) \right) \left( \left\| \frac{\partial u_n(t)}{\partial t} \right\|_2 + \left( \frac{f_n(t)}{2}, \frac{\partial u_n(t)}{\partial t} \right). \right.
\]

From (12) and (13), we may apply Gronwall’s inequality to get the estimate
\[
\left( \left\| \frac{\partial u_n(t)}{\partial t} \right\|_2 \right)^2 \leq \text{const} \left( \| u_n(0) \|_2 \right)^2, \quad \forall n \in \mathbb{N}. \quad (14)
\]

Using A1 and G), taking Young’s inequality into account, we get
\[
\left( \left\| \frac{\partial u_n(t)}{\partial t} \right\|_2 \right)^2 \leq \text{const} \| u_n(t) \|_2^2, \quad \forall n \in \mathbb{N} \quad \text{and} \quad t \in [0, T]
\]
and (4) follows. Assertion (5) is a direct consequence of A2, G) and Aubin’s lemma. Assertion (8) follows from the lower semicontinuity of \( \phi(t) \).

To prove (6) and (7), it suffices to show
\[
\limsup_n \int_0^T T(u_n), u_n - u(t) \, dt \leq 0. \quad (15)
\]

Since for any \( v \in C^t(0, T, [C_0^\Omega(\Omega)]) \) we may find a subsequence \( (v_{n_k}) = (v_{n_1}, \ldots, v_{n_k}) \subset U_n \in Y_n \) such that \( v_{n_k} \rightharpoonup u \) weakly in Y, we get from (2)
\[
\int_0^T \left( \frac{\partial u_{n_k}(t)}{\partial t}, v_{n_k}(t) \right) \, dt + \int_0^T \left( T(u_{n_k}), v_{n_k} - u(t) \right) \, dt \\
\leq \int_0^T \left( \frac{\partial u_{n_k}(t)}{\partial t}, v(t) \right) \, dt + \int_0^T \left( T(u_{n_k}), v_{n_k} - u(t) \right) \, dt \\
+ \phi(t)(v_{n_k}) - \phi(t)(u_n) - \int_0^T \left( f_n(t), v_{n_k} - u(t) \right) \, dt.
\]

Letting \( n \to \infty \), keeping \( k \) fixed, we have
\[
< u_{n_k}(t), v_{n_k}(t) > + \limsup_n \int_0^T \left( T(u_{n_k}), v_{n_k} - u(t) \right) \, dt \\
\leq \limsup_n \int_0^T \left( T(u_{n_k}), v_{n_k} - u(t) \right) \, dt \\
+ \phi(t)(v_{n_k}) - \liminf_n \phi(t)(u_n) - \int_0^T \left( f_n(t), v_{n_k} - u(t) \right) \, dt.
\]

Since the left hand side of this inequality is independent of \( k \), allowing \( k \to \infty \) we get (15) and (i) of the theorem follows. To prove (ii) little arguments are needed. For this aim, define the truncated perturbation \( g_k^{(i)}(x, t, u) \) by
\[
g_k^{(i)}(x, t, u) = \begin{cases} g^{(i)}(x, t, u) & \text{if } g^{(i)}(x, t, u) > k \\
g^{(i)}(x, t, u) & \text{otherwise.} \end{cases}
\]

From (i), there exists \( u_k \in Y \) with \( u_k(t) \in K \). e. \( u_k(0) = 0 \) such that
\[
\int_0^T \left( \frac{\partial u_k(t)}{\partial t}, v(t) - u_k(t) \right) \, dt + \int_0^T \left( T(u_k), v(t) - u_k(t) \right) \, dt \\
+ \phi_k^{(i)}(v) - \phi_k^{(i)}(u_k) \geq 0\int_0^T \left( f^{(i)}, v(t) - u_k(t) \right) \, dt,
\]
for every \( v(t) \in C^t(0, T, [C_0^\Omega(\Omega))] \) for which
\[
\phi_k^{(i)}(v) < \infty
\]
where
\[
\phi_k^{(i)}(u_k) = \int_Q G_0^{(i)}(x, t; u_k(x, t)) \, dx dt
\]
and
\[
G_0^{(i)}(x, t; r) = \int_0^r g_0^{(i)}(x, t; s) \, ds.
\]

Using the subgradient inequality for \( G_0^{(i)}(x, t; r) \) as a function of \( r \), we have
\[
\int_0^T \left( \frac{\partial u_k(t)}{\partial t}, v(t) - u_k(t) \right) \, dt + \int_0^T \left( T(u_k), v(t) - u_k(t) \right) \, dt \\
+ \int_0^T \left( g_0^{(i)}(x, t; u_k), v(t) - u_k(t) \right) \, dx dt \\
\geq \int_0^T \left( f^{(i)}, v(t) - u_k(t) \right) \, dt.
\]

The rest of the proof is more or less word for word as in (i).

Example: As an example which can be handled by our result, consider the variational inequalities associated with the following system
\[
\frac{\partial u}{\partial t}(t) - \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D\alpha \left( D^\alpha u(t) \right) u(t) - D^\alpha u(t) \quad \ell = 1, 2, \ldots, d, \quad p \geq 2.
\]

References
