On the Construction of Families of type $\Pi_1$ Subfactors
Each Containing a Middle Subfactors

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1. Introduction and Preliminaries

In the next chapter we are going to use locally trivial subfactors to construct a set of middle subfactors. The important of locally trivial subfactors was indicated by S.Pop at [9]. The simplest locally trivial subfactors are those having only two orthogonal projections in their relative commutant. It is well known that these subfactors are isomorphic to Jones subfactors. Also it is easy to show that these kind of subfactors do not possess middle subfactors. For the locally trivial subfactors that their relative commutant have dimension larger than two we believe that the above result is still valid ie there are no middle subfactors. Suppose we are given a pair $L \subset M$ of subfactors that are limiting algebras of a tower of commuting squares.

Then using the results in [1] we can show that if the inclusion graphs of the corresponding finite $C^*$ algebras are A graphs, then the subfactors $L \subset M$ do not have middle subfactors. One of the problems in the index theory, is to find the set of all the values for the indices of hyperfinite irreducible subfactors. Using the above constructions, we are going to show that the above set contains the interval $(37.0037, \infty)$.

2. Main Results

For a given pair of subfactors $N \subset M$. With $[M:N] = \lambda^{-1} < \infty$. Let $e$ be a projection in $M$ that induces the expectation $N$ onto $Q = (e)\cap N$. Let $P_1, P_2 = 1 - P_1$ be a partition of unity in $Q$. Using standard arguments as in [5], there exists an isomorphism $\Phi$, taking $NP_1$ onto $NP_2$. Let $I(P_1)$ be the set of all the elements of the form $I(P_1) = (x + \Phi(x); x \in NP_1)$. Then it is well known that $L(P_1)$ is a locally trivial subfactor of $N$. Also by (Lemma 2.2.1) [5], $[N : LP_1] = 1 = tr(P_1) + 1 = tr(P_2)$. Where $tr$ is a unique normalized trace on $M$. Suppose $P_1$, does not communicate with $e_0 = e$. Set $y = P_1eP_2 \neq 0$. Note that the relative commutant of $L(P_1)$, inside $M$ is spanned by the projection $P_1$. Since $y$, does not communicate with $P_1$, we have that the algebra, $H(P_1) = \langle L(P_1) \rangle$, $Y > 0$ is a middle subfactor which is strictly larger than $L(P_1)$. If under certain conditions $H(P_1)$ becomes strictly smaller than $M$, then $H(P_1)$ becomes a proper middle subfactor. In this case by the above arguments it is easy to see that the inclusions $I(P_1) \supseteq L(P_1)$ and $H(P_1) \subseteq M$ are irreducible inclusions of subfactors. Let us denote $r_1 = \{H(P_1); L(P_1)\}$ and $r_2 = \{M; H(P_1)\}$. Let $IR$, be the set of all irreducible subfactors of finite index. Let us denote by $IR$, the set of indices of all subfactors in $IR$. Then by (Proposition 2.1.15) [5] $r_1, r_2$ is in $IR$. It is easy to check that the set of the elements $f$ of the form $f = \sum_{j \in I} u_jg_jw_j$, with $g = E_Q(P_1), z_j \in Q, u_j = P_1x_jP_1, w_j = P_2y_jP_1, x_j, y_j \in N$ where $J$ is a set of indices is dense in $(H(P_1))(P)$. Let $e_j$ be a projection in $Q$, such that $E_Q(e_j) = \lambda$. Then $e_j$ induces the expectation of $Q$ onto the subfactor $Q_1 = (\lambda)\cap Q$. Next for a number $r, r \in IR$, construct an irreducible subfactor $Q_j \subseteq Q_j$, such that $Q_j \cap Q_j = r$. We can define the projection $e_j$, Using (corollary 1.8) [13], there exits a projection $e_j$ in $Q_j$, such that $e_j$ induces the expectation of $Q_j$ onto The subfactor $Q_j$, with $Q_j = (\lambda)\cap Q_j$. This process will induce the following tunnel, $M \supseteq N \supseteq Q \supseteq Q \supseteq Q$. Let us set $P_1 = q e_j e_j$ with $q$ a projection in $Q_j$. Now we can check that the following set of elements, $f$ of the form, $f = \sum_{j \in J} x_j z_j y_j$, with $x_j, z_j, y_j$ as in the above and $J$ a set of indices will be a dense subset of $(H(P_1))(P)$. In particular assuming now that $H(P_1) = M$ implies that the above set of elements are dense in $M_P$. Furthermore as we mentioned in the above for any number $r \in IR$, $e_j \in Q_j$, can be chosen such that $tr(e_j) = r$. For example suppose...
tr(e_i) = tr(e_0) = .5. Then it is easy to see that there exists a unitary $V e(e_i) = \sum_i e_i V^* e_i$, with $V$ a type $\sigma_1$ Von Neumann algebra, such that $V e_i V^* = e_i - \epsilon_i$. Then we can express the $f$ from the above as $f = (1 - \lambda^2) \{ e_1 e_0 g e_1 e_2 e_3 + e_1 e_0 g e_2 e_3 V e(e_3) \}$. For a given real number $\lambda$, let $[\lambda]$, be the largest integer which is smaller or equal to $[\lambda]$; set $\lambda' = \lambda - [\lambda]$. Then there exist unitary operators $f_k$, such that $f = (1 - \lambda') \{ e_1 e_0 g e_1 e_2 e_3 + e_1 e_0 g e_2 e_3 V e(e_3) \}$. The we have, $f_k = V_k f k + f k^*$. Using our definition of $f_k = e_i$, this implies, $f = (1 - \lambda') \{ e_1 e_0 g e_1 e_2 e_3 + e_1 e_0 g e_2 e_3 V e(e_3) \}$. Let $U = \sum_{i=1}^{n} e_i$. Then $U$ is a unitary in $L$. Set $g = \sum_{i=1}^{n} e_i f_{2i-1}$. Then we have, $f = (1 - \lambda') \{ e_1 e_0 g e_1 e_2 e_3 + e_1 e_0 g e_2 e_3 V e(e_3) \}$. Next for each $i < n$, there exists a unitary $m_i \in L$, such that $m_i e_i m_i^* = f_{2i-1}$. Hence $g = \sum_{i=1}^{n} m_i e_i m_i^*$. Next since for $i \neq j$, $m_i e_i m_i^* = m_j e_j m_j^* = 0$. For $i \neq j$, let us define $y = e_i m_i^* e_j$. Then it is easy to check $tr( yy^*) = 0$. This implies that $y = 0$. Another useful relation that we will need later is the following equality, $e_0^* g e_0 = n e_0$, that can be checked easily. Let us define the operator $h$, with $h = (\sum m_i) e_1 (\sum m_i^*) = g e_0^* g$, with $1 \leq i, k \leq n$ and $q = \sum_i m_i$. Then using the above relations we can see that $h$ is a projection, $tr(h) = tr(g)$, and $U h U^*$ is orthogonal to $h$. Hence we get, $tr(h) = tr(U h U^*) = .5$. This will implies that $f$ can be expressed as, $f = (1 - \lambda') \{ e_1 e_0 g e_1 e_2 e_3 + e_1 e_0 g e_2 e_3 V e(e_3) \}$. Suppose $\lambda' = 2n+1$. Then we have the following partition of unity, $e = f_1, f_2, \ldots, f_{2n+1}$. Where the above projections have equal traces. Furthermore their exists a unitary $U_2 \in L_{q+2n+1}$ such that $f$ can be expressed as, $f = (1 - \lambda') \{ e_1 e_0 g e_1 e_2 e_3 + e_1 e_0 g e_2 e_3 V e(e_3) \}$. Then we have the following partition of unity by the following projections, $e = f_1, f_2, \ldots, f_{2n+1}, f_{n+1}, f^*$. Where $\sigma_1$ and $\sigma_2$ can take values 0 or 1, depending if the corresponding projections $f_{n+1}$ and $f^*$ are or are not equal to zero. Furthermore for $k \neq 0$, all non zero projections $f_k$'s, have equal traces and $tr(f_0) < \lambda$. Hence generally $f$ can be expressed as, $f = (1 - \lambda') \{ e_1 e_0 g e_1 e_2 e_3 + e_1 e_0 g e_2 e_3 V e(e_3) \}$. Now let us set the following notations. 

$$n_1 = e_1 e_0 g e_2 e_3, \quad n_2 = e_1 e_0 g e_2 e_3, \quad n_3 = e_1 e_0 g e_2 e_3, \quad n_4 = e_1 e_0 g e_2 e_3, \quad G = N_{\sigma_1, \sigma_2}.$$ 

**Lemma 2** Keeping the same notations as in the above, and assuming that $\sigma_1$ and $\sigma_2$ both different from zero and without loss of generality, we have the following equalities. 

$$E_{n_{\sigma_1, \sigma_2}} \left( n_{\sigma_1, \sigma_2}^* n_{\sigma_1, \sigma_2} \right) = e_1 e_0 g e_2 e_3, \quad E_{G} \left( \sigma_1^* \sigma_2^* \right) = 0, \quad k = 1, 3, 4.$$ 

Proof Note that since $e_3$ commutes with all the above operators, drop-ping $e_3$ from all the operations does not makes any different from the final outcome. So in the following operations we ignore the existence of $e_3$. In particular we can identify $N_{\sigma_1}$ with $G = N_{\sigma_1}$. Since the proof of the above equalities are very similar, we only show some of the equalities. 

$$E_{G} \left( m_{n_1}^* m_{n_1} \right) = \lambda^2 e_1 e_0 g e_2 e_3 e_0 g e_0^* e_1 e_0 g e_2 e_3.$$ 

Hence we get, $E_{G} \left( m_{n_1} m_{n_1}^* \right) = \lambda^2 e_1 e_0 g e_2 e_3 e_0 g e_0^* e_1 e_0 g e_2 e_3$. 

$$E_{G} \left( m_{n_2}^* m_{n_2} \right) = \lambda^2 e_1 e_0 g e_2 e_3 e_0 g e_0^* e_1 e_0 g e_2 e_3.$$ 

Hence we get, $E_{G} \left( m_{n_2} m_{n_2}^* \right) = \lambda^2 e_1 e_0 g e_2 e_3 e_0 g e_0^* e_1 e_0 g e_2 e_3$.
This implies, $E_G(n_3 n_1) = \lambda^2 q e_3 U_3^* q e_3$. It equals $q e_3 U_3^* q e_3 / n$, which equals $U_3^* q e_3 U_3^* q e_3 / n$. But using the above relations, $(U_2^* q e_2 U_2^*) (q e_2 q) = 0$, which implies $E_G(n_3 n_1) = 0$. Now let us calculate $E_G(n_1 n_4)$. Let $n_1$ and $n_4$ be a pair of irreducible subfactors. Then \(H(P) = \langle L(P)\rangle\) is a proper irreducible subfactor of \(M\). In particular, letting $q$ vary in $Q$, we get that $\text{IRR}$ includes the interval \([\omega / (1/2\omega) (1-1/2\omega), \infty]\) = \([37.0037, \infty]\).

At this end note that by the above arguments there exists a function $\Phi$, acting on the above interval.

**Theorem 4** Keeping the same notations as in the above, suppose $\lambda = \omega$ and $(\lambda_1) = 2$. Let $N \supset M$, $[M:N] = \lambda^{-1}$ be a pair of irreducible subfactors and let define a projection $p = q e_3 e_3$, for some projection $q$ in $Q$. Then $H(P) = \langle L(p)\rangle$ is a proper irreducible subfactor of $M$. In particular, letting $q$ vary in $Q$, we get that $\text{IRR}$ includes the interval \([\omega / (1/2\omega) (1-1/2\omega), \infty]\) = \([37.0037, \infty]\).

At this end note that by the above arguments there exists a function $\Phi$, acting on the above interval.

**References**


