Summability of a Jacobi Series by Lower Triangular Matrix Method

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Abstract

The Jacobi polynomial \( P_n^{(\alpha, \beta)}(x) \) which is obtained from Jacobi differential equation is an orthogonal polynomial over the interval \([-1, 1]\) with respect to weight function \((1-x)^\alpha (1+x)^\beta\), \(\alpha > -1, \beta > -1\). Here Jacobi series has been taken and established a theorem on lower triangular matrix summability of a Jacobi series.

Keywords: summability, jacobi series, triangular matrix


1. Introduction

The Nörlund summability \((N, p_n)\) on Jacobi series has been studied by a number of researchers like Gupta [4], Choudhary [3], Thorpe [16], Pandey and Beohar [10], Prasad and Saxena [11], Beohar and Sharma [1], Pandey [9], Tripathi et al. [18] and Chandra [14]. After quite a good amount of work in the ordinary Nörlund summability of Jacobi series at the point \(x = 1\), Khare and Tripathi [5] discussed the generalized Nörlund summability \((N, p_n, q)\) of Jacobi series. The \((N, p, q)\) summability reduces to the \((N, p_n)\) summability for \(q_n = 1\) \(\forall n\). The Cesàro Summability of Jacobi series has been studied by Szili & Weisz [15]. The Cesàro Summability, Nörlund Summability, generalized Nörlund Summability are special cases of The matrix Summability method. In this paper a more general result than those Gupta [4], Choudhary [3], Khare and Tripathi [5] has been obtained so that their results come out as particular cases.

2. Definitions and Notations

Let \( f(x) \) be defined in closed interval \([-1, 1]\) such that the function

\[
(1-x)^\alpha (1+x)^\beta f(x) \in L[-1, 1]; \quad \alpha > -1, \beta > -1.
\]

The Jacobi series corresponding to this function is

\[
\sum_{n=0}^{\infty} a_n P_n^{(\alpha, \beta)}(x)
\]

(2.1)

where

\[
a_n = \frac{(2n + \alpha + \beta + 1) \Gamma(n+1) \Gamma(n+\alpha + \beta + 1)}{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}
\]

(2.2)

and \( P_n^{(\alpha, \beta)}(x) \) are Jacobi polynomials. Let \( T = (a_{n,k}) \) be an infinite lower triangular matrix method \( T \) satisfying the Silverman- Töeplitz [17] conditions of regularity i.e.

\[
\sum_{k=0}^{n} a_{n,k} \to 1 \text{ as } n \to \infty,
\]

(2.3)

\[
a_{n,k} = 0, \text{ for } k > n \text{ and } \sum_{k=0}^{n} \left| a_{n,k} \right| \leq M, \text{ where } M \text{ is a finite positive constant.}
\]

Let \( \sum_{n=0}^{\infty} u_n \) be an infinite series whose \( n^{th} \) partial sum is given by

\[
s_n = \sum_{v=0}^{n} u_v.
\]

The sequence - to - sequence transformation

\[
t_n = \sum_{k=0}^{n} a_{n,k} s_k
\]

defines the sequence \( \{t_n\} \) of matrix means of sequence \( \{s_n\} \), generated by the sequence of coefficient \( (a_{n,k}) \).

If

\[
t_n \to s \text{ as } n \to \infty,
\]

then the series \( \sum_{n=0}^{\infty} u_n \) or sequence \( \{s_n\} \) is said to be summable by matrix method to \( s \). It is denoted by

\[
t_n \to s(T) \text{ as } n \to \infty \text{ (Zygumund [19]).}
\]

We use the following notations:

\[
F(\varphi) = \left| f(\cos \varphi) - A \left( \sin \frac{\varphi}{2} \right)^{2\alpha+1} \left( \cos \frac{\varphi}{2} \right)^{2\beta+1} \right|
\]

(2.2)
A being fixed constant.

\[ \Psi(t) = \int_0^t |F(\phi)| d\phi \]

\[ \tau = \text{Integral part of } \frac{1}{\phi} = \left[ \frac{1}{\phi} \right] \]

3. Main Theorem

The purpose of this paper is to establish a theorem under a very general condition so that it generalizes all the known results for Nörlund summability \((N,p_n)\) of Jacobi series in this direction. In fact, we prove the following:

**Theorem:** Let \( T = (a_{n,k}) \) be an infinite lower triangular regular matrix such that the element \( a_{n,k} \) is positive, monotonic increasing in \( k \) for \( 0 \leq k \leq n \),

\[ A_n \equiv \sum_{k=0}^{n} a_{n,k}, \quad A_{n,n} = 1 \forall n \]

and

\[ n^{\alpha + \frac{1}{2}} A_n \left[ \frac{1}{\phi} \right] = O(1), \quad 0 < \delta < \pi \quad \text{as} \quad n \to \infty. \]

If

\[ \int_{1-t}^{1} |f(u) - A| du = o \left( \frac{t}{\xi(t) \log t} \right) \quad \text{as} \quad t \to 0 \quad (3.1) \]

then the Jacobi series \((2.1)\) is summable \((T)\) to the sum \( A \) at \( x = 1 \) provided \( \xi(t) \) is positive monotonic non-decreasing function of \( t \) such that

\[ \sum_{\alpha} \alpha A_{n,k} = O \left( \frac{1}{2^{a+1}} \frac{1}{n} \right) \quad (3.2) \]

\(-\frac{1}{2} \leq \alpha \leq \frac{1}{2}, \quad \beta > -\frac{1}{2}\) and the antipole condition

\[ \int_{0}^{1} t^{\beta - \frac{1}{2}} |f(-\cos t) - A| dt = o(1) \quad \text{as} \quad n \to \infty \quad (3.3) \]

is satisfied.

4. Lemmas

The following lemmas are required for the proof of the theorem:

**Lemma 4.1.** (Szegö, [13]): If \( \alpha > -1, \beta > -1 \) then as \( n \to \infty \)

\[ P_n^{(\alpha,\beta)}(\cos \phi) = \begin{cases} 
O(n^\alpha) & \text{for } 0 \leq \phi < \frac{1}{n} \\
O(n^\beta) & \text{for } \pi - \frac{1}{n} \leq \phi \leq \pi \\
n^{-\frac{1}{2}} k(\phi) \left[ \frac{\cos(Np + \nu)}{\pi \sin \phi} \right] & \text{for } \frac{1}{n} \leq \phi < \pi - \frac{1}{n} 
\end{cases} \quad (4.1.1) \]

\[ \int_{1-t}^{1} |f(u) - A| du = o \left( \frac{t}{\xi(t) \log t} \right) \quad \text{as} \quad t \to 0 \quad (4.1.2) \]

\[ \int_{1-t}^{1} |f(u) - A| du = o \left( \frac{t}{\xi(t) \log t} \right) \quad \text{as} \quad t \to 0 \quad (4.1.3) \]

where

\[ k(\phi) = \pi^{-\frac{1}{2}} \left( \sin \frac{\phi}{2} \right)^{-\alpha} \left( \cos \frac{\phi}{2} \right)^{-\frac{1}{2}} \]

\[ N = n + \alpha + \beta + 1 \]

\[ \rho = -\left( \alpha + \frac{1}{2} \right) \pi \].

**Lemma 4.2.** (Gupta, [4]): The antipole condition \((3.3)\) includes

\[ \int_{1-t}^{1} |f(\cos \phi) - A| d\phi < \infty, \quad (4.2.1) \]

\[ b \text{ fixed, and} \]

\[ \int_{\delta}^{\pi} \left( \cos \frac{\phi}{2} \right)^{-\frac{1}{2}} |f(\cos \phi) - A| d\phi < \infty. \quad (4.2.2) \]

**Lemma 4.3** Condition \((3.1)\) is equivalent to

\[ \int_{0}^{1} |F(\phi)| d\phi = o \left( \frac{t^{2\alpha+2}}{\xi(t) \log t} \right), \quad \text{as} \quad t \to 0. \quad (4.3.1) \]

**Proof:**

\[ \int_{0}^{1} |F(\phi)| d\phi = \int_{0}^{1} |f(\cos \phi) - A| \sum_{a} A_{n,k} \left[ \frac{1}{\phi} \right] dt \]

\[ \leq \int_{0}^{1} |f(\cos \phi) - A| \sum_{a} A_{n,k} \left[ \frac{1}{\phi} \right] dt \]

\[ \leq t^{2\alpha+1} \int_{0}^{1} |f(\cos \phi) - A| d\phi \]

\[ = t^{2\alpha+1} O \left( \frac{t}{\xi(t) \log t} \right) \]

Conversely

\[ \int_{0}^{1} |F(\phi)| d\phi \leq t^{2\alpha+1} \int_{0}^{1} |f(\cos \phi) - A| d\phi \]

\[ = t^{2\alpha+1} \int_{0}^{1} |f(\cos \phi) - A| d\phi \]

\[ \leq t^{2\alpha+1} O \left( \frac{t}{\xi(t) \log t} \right) \]

\[ = t^{2\alpha+1} \int_{0}^{1} |f(\cos \phi) - A| d\phi \]

\[ = t^{2\alpha+1} O \left( \frac{t}{\xi(t) \log t} \right) \]

**Lemma 4.4** If \((a_{n,k})\) is non-negative and non-decreasing with \(0 \leq k \leq n\), then, for \(0 \leq a < b \leq \infty, \ 0 \leq t \leq \pi\) and for any \(n\),
\[
\sum_{k=a}^{b} a_{n,k} e^{i(k+\rho)\varphi} = O(A_{n,r}) \quad (4.4.1)
\]

where \( \tau = \text{Integral part of } \frac{1}{r} = \lfloor \frac{1}{r} \rfloor \).

Lemma 4.4 may be proved by the following technique of Lemma 4.1 in Lal [6].

Lemma 4.5 Under the condition of the theorem on \((a_n,k)\), for large \(n\), uniformly in \(0 < \varphi \leq \pi\), \(0 \leq a \leq b \leq n\),

\[
\int (\alpha + \beta + 2) = = = = (4.5.1)
\]

where \(\rho = \frac{\alpha + \beta + 2}{2}, \gamma = -\left(\alpha + \frac{3}{2}\right)^2\).

Proof:

\[
\sum_{k=a}^{b} a_{n,k} \cos \{(k+\rho)\varphi - \gamma\} k^{\alpha + \frac{1}{2}} = O(n^{\alpha + \frac{1}{2}} A_{n,r})
\]

by Abel’s Lemma.

Lemma 4.6 Under the hypothesis of the theorem,

\[
\sum_{k=1}^{n} a_{n,k} k^{\alpha - \frac{1}{2}} = O(n^{\alpha - \frac{1}{2}}) \quad (4.6.1)
\]

Proof:

\[
\sum_{k=1}^{n} a_{n,k} k^{\alpha - \frac{1}{2}} = \sum_{k=1}^{[n/2]} a_{n,k} k^{\alpha - \frac{1}{2}} + \sum_{k=[n/2]+1}^{n} a_{n,k} k^{\alpha - \frac{1}{2}}
\]

\[
= O(a_n[n/2]) \sum_{k=1}^{[n/2]} k^{\alpha - \frac{1}{2}} + O(n^{\alpha - \frac{1}{2}}) \sum_{k=[n/2]+1}^{n} a_{n,k}
\]

\[
= O(a_n[n/2]) n^{\alpha - \frac{1}{2}} + O(n^{\alpha - \frac{1}{2}}),
\]

since \(\sum_{k=1}^{n} a_{n,k} = 1\). Also,

\[
1 \geq \sum_{k=[n/2]+1}^{n} a_{n,k} \geq \frac{n}{2} a_{n,[n/2]} \quad \text{and putting this in the}
\]

above gives the result

\[
\sum_{k=1}^{n} a_{n,k} k^{\alpha - \frac{1}{2}} = O(n^{\alpha - \frac{1}{2}}).
\]

Lemma 4.7 Let

\[
M_n(\varphi) = 2^{\alpha + \beta + 1} \sum_{k=0}^{n-1} a_{n,k} \lambda_k F_k^{\alpha + 1, \beta} (cos \varphi)
\]

where

\[
\lambda_n = \frac{2^{-\alpha + \beta + 1} \Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1)} \approx \frac{2^{-\alpha + \beta + 1} \Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)} n^{\alpha + 1}
\]

then for \(-\frac{1}{2} \leq \alpha < \frac{1}{2}\), \(\beta > -\frac{1}{2}\) and if \(a_{n,k}\) satisfies the hypothesis of the theorem,

\[
M_n(\varphi) = O(n^{2\alpha + 2}) \quad \text{for } 0 \leq \varphi < \frac{1}{n}
\]

\[
O(n^{\alpha + \beta + 1}) \quad \text{for } \pi - \frac{1}{n} \leq \varphi \leq \pi
\]

\[
= O\left(\frac{\alpha + \frac{1}{2}}{n} A_{n,r} \sin \frac{\varphi}{2} - \frac{3}{2}\right)
\]

\[
+ O\left(\cos \frac{\varphi}{2} - \frac{5}{2}\right)
\]

for \(0 \leq \varphi < \frac{1}{n}\) (4.7.3).

Proof: For \(0 \leq \varphi < \frac{1}{n}\)

\[
M_n(\varphi) = O(1) \sum_{k=0}^{n} a_{n,k} k^{2\alpha + 2} \quad \text{by (4.1.1)}
\]

\[
= O(n^{2\alpha + 2}) \sum_{k=0}^{n} a_{n,k}
\]

\[
= O(n^{2\alpha + 2}) A_{n,n}
\]

\[
= O(n^{2\alpha + 2})
\]

For \(\pi - \frac{1}{n} \leq \varphi \leq \pi\)

\[
M_n(\varphi) = O(1) \sum_{k=0}^{n} a_{n,k} k^{\alpha + 1} k^{\beta} \quad \text{by (4.1.2)}
\]

\[
= O(1) \sum_{k=0}^{n} a_{n,k} k^{\alpha + \beta + 1}
\]

\[
= O(n^{\alpha + \beta + 1}) \sum_{k=0}^{n} a_{n,k}
\]

\[
= O(n^{\alpha + \beta + 1}) A_{n,n}
\]

\[
= O(n^{\alpha + \beta + 1})
\]

For \(\frac{1}{n} \leq \varphi < \pi - \frac{1}{n}\)

\[
M_n(\varphi) = O(1) \sum_{k=0}^{n} a_{n,k} k^{\alpha + 1} k^{\beta} \quad \text{by (4.1.3)}
\]

\[
= O(1) \left[\frac{1}{n} \sum_{k=0}^{n} a_{n,k} k^{\alpha + 1} k^{\beta} \right]
\]

\[
= O\left(\frac{1}{n} \sum_{k=0}^{n} a_{n,k} k^{\alpha + \beta + 1} \Gamma(\alpha + 1)
\]

\[
\approx \frac{2^{-\alpha + \beta + 1} \Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1)} \approx \frac{2^{-\alpha + \beta + 1} \Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)} n^{\alpha + 1}
\]

by (4.1.3).
\[ t_n = \sum_{k=0}^{n} a_{n,k} S_k(1) \]

\[ t_n - A = \sum_{k=0}^{n} a_{n,k} \left( S_k(1) - A \right) = \int_{0}^{\pi} F(\phi) M_n(\phi) d\phi. \]

In order to prove the theorem, we have to show that

\[ I = \int_{0}^{\pi} F(\phi) M_n(\phi) d\phi = o(1) \text{ as } n \to \infty. \]

Let us denote

\[ I = \int_{0}^{\delta} \int_{1}^{\pi} + \int_{\frac{1}{n}}^{\delta} \int_{1}^{\pi} F(\phi) M_n(\phi) d\phi \]

\[ = I_1 + I_2 + I_3 + I_4 \text{ say}, \]

\[ \delta \text{ being a suitable constant.} \]

\[ I_1 = \int_{0}^{\frac{1}{n}} F(\phi) M_n(\phi) d\phi \]

In order of to estimate \( I_2 \), we employ the asymptotic relation given in 4.7.3), thus

\[ I_2 = O \left( \int_{0}^{\frac{1}{n}} \int_{1}^{\pi} F(\phi) M_n(\phi) d\phi \right) \]

\[ = O(1) \text{ as } n \to \infty. \]

In order to estimate \( I_2 \), we employ the asymptotic relation given in 4.7.3),

\[
\begin{aligned}
I_2 &= O \left( \frac{1}{n} \int_{0}^{\pi} \left| F(\phi) \right| \left( n^{2\alpha+2} - \frac{1}{n^{2\alpha+2}} \xi(n) \log n \right) d\phi \right) \\
&= O(1) \text{ as } n \to \infty.
\end{aligned}
\]

In order to estimate \( I_2 \), we employ the asymptotic relation given in 4.7.3),

\[
\begin{aligned}
I_2 &= \left( \frac{1}{n} \right) \int_{0}^{\pi} \int_{1}^{\pi} F(\phi) \left( n^{2\alpha+2} - \frac{1}{n^{2\alpha+2}} \xi(n) \log n \right) d\phi \\
&= O(1) \text{ as } n \to \infty.
\end{aligned}
\]

\[
\begin{aligned}
I_2 &= O \left( \frac{1}{n} \int_{0}^{\pi} \left| F(\phi) \right| \left( n^{2\alpha+2} - \frac{1}{n^{2\alpha+2}} \xi(n) \log n \right) d\phi \right) \\
&= O(1) \text{ as } n \to \infty.
\end{aligned}
\]

Now, for \( I_2 \), given \( c > 0 \) choose \( \delta \) such that if \( 0 < t \leq \delta \), then

\[ \Psi(t) = \int_{0}^{t} \left| F(\phi) \right| d\phi < c e^{\frac{2\alpha+2}{\xi(\frac{1}{n}) \log \frac{1}{n}}}.
\]
\[ I_{2.1} = O\left(n^{\frac{a+1}{2}}\right) \begin{pmatrix} \frac{1}{n} F(\phi) A_{n,\frac{1}{2}} & -x^{\frac{a-\frac{3}{2}}{2}} d\phi \\ \frac{1}{n} \Psi(\phi) & \frac{1}{n} \end{pmatrix} \]

\[
= O\left(n^{\frac{a+1}{2}}\right) e^{\frac{1}{n} \Psi(\phi) d\left(\frac{A_{n,\frac{1}{2}}}{\phi} + \frac{1}{n} \right)}
\]

We have, \[ I_{2.1.1} \]

\[
I_{2.1.1} = O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \Psi(\phi) d\left(\frac{A_{n,\frac{1}{2}}}{\phi} + \frac{1}{n} \right)}
\]

\[
< O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \Psi(\phi) d\left(\frac{A_{n,\frac{1}{2}}}{\phi} + \frac{1}{n} \right)}
\]

Again, for \[ I_{2.1.2} \]

\[
I_{2.1.2} = O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \Psi(\phi) d\left(\frac{A_{n,\frac{1}{2}}}{\phi} + \frac{1}{n} \right)}
\]

\[
< O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \Psi(\phi) d\left(\frac{A_{n,\frac{1}{2}}}{\phi} + \frac{1}{n} \right)}
\]

and using the change of variables \[ x = \frac{1}{\phi} \], we get

\[
= O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \frac{x^{-2a-2}}{\xi(x) \log x} d\left(\frac{x^{\frac{a+3}{2}} A_{n,[1]}(x)}{\phi} \right)}
\]

Collecting (5.3) – (5.8), we get

\[
I_2 = o(1) \text{ as } n \to \infty.
\]

Now, for \[ I_{2,1.2.2} \]

\[
I_{2,1.2.2} = O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \Psi(\phi) d\left(\frac{A_{n,\frac{1}{2}}}{\phi} + \frac{1}{n} \right)}
\]

\[
< O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \Psi(\phi) d\left(\frac{A_{n,\frac{1}{2}}}{\phi} + \frac{1}{n} \right)}
\]

Collecting (5.3) – (5.8), we get

\[
I_2 = o(1) \text{ as } n \to \infty.
\]

Considering \[ I_3 \], we have

\[
I_3 = \int_{\delta}^{\frac{1}{\delta}} F(\phi) A_{n,\frac{1}{2}} d\phi
\]

\[
= O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \frac{x^{-2a-2}}{\xi(x) \log x} d\left(\frac{x^{\frac{a+3}{2}} A_{n,[1]}(x)}{\phi} \right)}
\]

and

\[
I_{2,1.2.1} + I_{2,1.2.2}
\]

If \[ m \] is the integers with \[ m \leq \frac{1}{\delta} \leq m + 1 \], then

\[
I_{2,1.2.1} = O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \frac{x^{-2a-2}}{\xi(x) \log x} d\left(\frac{x^{\frac{a+3}{2}} A_{n,[1]}(x)}{\phi} \right)}
\]

\[
= O(n^{\frac{a+1}{2}}) e^{\frac{1}{n} \frac{x^{-2a-2}}{\xi(x) \log x} d\left(\frac{x^{\frac{a+3}{2}} A_{n,[1]}(x)}{\phi} \right)}
\]

by Abel’s Lemma

\[
\text{as } n \to \infty.
\]
6. Applications

The following particular cases are obtained:

(1) The result of Gupta [4] becomes particular case of our main theorem if,

$$a_{n,k} = \frac{p_{n-k}}{p_n} \text{ where } p_n = \sum_{k=0}^{n} p_k \neq 0 \text{ and } \xi(t) = 1 \forall t.$$

(2) The result of Chaudhary [3] becomes particular case of our theorem if,

$$a_{n,k} = \frac{p_{n-k}}{p_n} \text{ and } \xi(t) = \frac{P(t)}{t P(t) \log t} \forall t.$$

(3) The result of Khare and Tripathi [5] becomes particular case of our main theorem if,

$$a_{n,k} = \frac{p_{n-k} q_k}{R_n} \text{ where } R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \text{ and } \xi(t) = 1 \forall t.$$

7. Conclusion

Cesáro, Nörlund, generalized Nörlund Summability methods are the particular cases of matrix Summability method. In this paper matrix Summability method taken with a condition (3.1) on the Jacobi series (2.1) so that series (2.1) is summable at x=1 to sum A. The result of Gupta [4], Chaudhary [3] and Khare and Tripathi [5] are particular cases of my result.

References