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Some Common Fixed Point Theorems for Two Pairs of Weak Compatible Mappings of Type (A) in G_b-metric Space

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Abstract In this paper, we prove a common fixed point theorem for two pairs of weak compatible mappings of type (A) in G_b-metric space. Further our result is verified with the help of example.

Keywords: *common fixed point, G_b-metric spaces, weak compatible mappings of type (A)*

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1. Introduction and Preliminaries

Metric fixed point theory is one of the most important and fundamental area of analysis. Due to this a flood of research work have been generated from this area. As a part of this study generalisation of metric space becomes one of the most interesting topic in which many researchers have devoted and continued working. Since the introduction of metric space by Frachet, there is a lot of generalisation of this space. Some of them which can be mentioned are 2-metric space, *D*-metric space, cone metric space, fuzzy metric space, Menger space, probabilistic metric space, partial metric space, quasi metric space, *b*-metric space, multiplicative metric space, modular metric space, cyclic metric space, *S*-metric space, *b*-cone metric space etc.

In a recent paper, Aghajani *et.al.* [1] introduced a new generalisation of metric space. They used the concepts of both *G*-metric [2] and *b*-metric [3,4,5] and generated a new definition and named it as G_b-metric space. They also pointed out that the class of G_b-metric space is effectively larger than that of *G*-metric space and *G*-metric space becomes a particular case of G_b-metric space. They claimed that every G_b-metric space is topologically equivalent to a *b*-metric space. For more results on G_b-metric space one can study the research papers in [6-10] and references there in.

Definition 1.1 [2]

Let X be a nonempty set and $G: X^3 \to R^+$ be a function satisfying the following properties:

1. *$G(x, y, z) = 0$ if and only if $x = y = z$;*
2. *$0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;*
3. *$G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;*
4. *$G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);*
5. *$G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).*

Then the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Following definition was given by I. A. Bakhtin [3]

Definition 1.2 [3]

Let X be a (nonempty) set and $b \geq 1$a given real number. A function $d: X \times X \to R^+$ (nonnegative real numbers) is called a b-metric provided that, for all $x, y, z \in X$, the following conditions are satisfied:

1. *$d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$;*
3. *$d(x, z) \leq b[d(x, y) + d(y, z)]$*

The pair (X, d) is called a b-metric space with parameter b.

Definition 1.3 [1]

Let X be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $G: X \times X \times X \to R^+$ satisfies:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,
(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with x, y,
(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.
(G_b4) $G(x, y, z) = G(p[x, y, z])$, where p is a permutation of x, y, z (symmetry),
(G_b5) $G(x, y, z) \leq b(G(x, a, a) + G(a, y, z))$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized b-metric and pair (X, G) is called a generalized b-metric space or G_b-metric space.

It should be noted that, the class of G_b-metric spaces is effectively larger than that of *G*-metric spaces. Following example given by Aghajani [1] shows that a G_b-metric on *X* need not be a *G*-metric on *X*.

Example 1.4 [1]

Let (X, G) be a G-metric space, and $G_*(x, y, z) = G(x, y, z)^p$, where $p > 1$ is a real number. Note that G_* is a G_b-metric with $b = 2^{p-1}$.

Also in the above example, (X, G_*) is not necessarily a G-metric space. For example, let X = R and G-metric G *be defined by* $G(x, y, z) = 1/3(|x-y| + |y-z| + |x-z|)$, *for all* $x, y, z \in R$. *Then* $G(x, y, z)^2 = 1/9(|x-y|+|y-z|+|x-z|)^2$ *is a* G_b*-metric on* R *with* $b = 2^{2-1} = 2$, *but it is not a G-metric on R. To see this, let* $x = 3$, $y = 5$, $z = 7$, $a = 7/2$ *we get,* $G_*(3, 5, 7) = 64/9$, $G_*(3, 7/2, 7/2) = 1/9$, $G_*(7/2, 5, 7) = 49/9$, *so* $G_*(3, 5, 7) = 64/9 \leq 50/9 = G_*(3, 7/2, 7/2) + G_*(7/2, 5, 7)$.

Following definitions and properties are given in Aghajani *et. al.* [1].

Definition 1.5 [1]

A G_b*-metric G is said to be symmetric if* $G(x, y, y) = G(y, x, x)$ *for all* $x, y \in X$.

Definition 1.6 [1]

Let (X, G) *be a* G_b*-metric space then for* $x_0 \in X$, $r > 0$, *the* G_b*-ball with centre* x_0 *and radius r is* $B_G(x_0, r) = \{y \in X | G(x_0, y, y) < r\}$.

For example, let $X = R$ and consider the G_b-metric G defined by

$$G(x,y,z) = 1/9(|x-y| + |y-z| + |x-z|)^2$$

for all $x, y, z \in R$. Then

$$B_G(3,4) = \{y \in X : G(3,y,y) < 4\}$$
$$= \{y \in X : 1/9(|y-3| + |y-3|)^2 < 4\}$$
$$= \{y \in X : |y-3|^2 < 9\}$$
$$= (0,6).$$

By some straight forward calculations, we can establish the following.

Proposition 1.7 [1]

Let X be a G_b*-metric space, then for each* $x, y, z, a \in X$ *it follows that*:

(1) if $G(x, y, z) = 0$ then $x = y = z$,
(2) $G(x, y, z) \leq b(G(x, x, y) + G(x, x, z))$,
(3) $G(x, y, y) \leq 2bG(y, x, x)$,
(4) $G(x, y, z) \leq b(G(x, a, z) + G(a, y, z))$

Definition 1.8 [1]

Let X be a G_b*-metric space, we define* $d_G(x, y) = G(x, y, y) + G(x, x, y)$, *it is easy to see that* d_G *defines a b-metric on X, which we call it b-metric associated with G.*

Proposition 1.9 [1]

Let X be a G_b*-metric space, then for any* $x_0 \in X$ *and* $r > 0$, *if* $y \in B_G(x_0, r)$ *then there exists a* $\delta > 0$ *such that* $B_G(y, \delta) \subseteq B_G(x_0, r)$.

Definition 1.10 [1]

Let X be a G_b*-metric space. A sequence* $\{x_n\}$ *in X is said to be*:

1. G_b*-Cauchy sequence if, for each* $\varepsilon > 0$, *there exists a positive integer* n_0 *such that, for all* $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;
2. G_b*-convergent to a point* $x \in X$ *if, for each* $\varepsilon > 0$, *there exists a positive integer* n_0 *such that, for all* $m, n \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

Proposition 1.11 [1] Let (X, G) be a G_b-metric space, then the following are equivalent:

1. *the sequence* $\{x_n\}$ *is* G_b*-Cauchy.*
2. *for any* $\varepsilon > 0$, *there exists* $n_0 \in N$ *such that* $G(x_n, x_m, x_m) < \varepsilon$, *for all* $m, n \geq n_0$.

Proposition 1.12 [1] *Let (X, G) be a* G_b*-metric space, then following are equivalent*:

1. $\{x_n\}$ is G_b-convergent to x.
2. $G(x_n, x_n, x) \to 0$ as $n \to \infty$.
3. $G(x_n, x, x) \to 0$ as $n \to \infty$.

Definition 1.13 [1] *A* G_b*-metric space X is called* G_b*-complete if every* G_b*-Cauchy sequence is* G_b*-convergent in X.*

Definition 1.14 [1] *Let (X, G) and (X, G') be two* G_b*-metric spaces. Then a function* $f : X \to X'$ *is* G_b*-continuous at a point* $x \in X$ *if and only if it is* G_b*-sequentially continuous at x, that is, whenever* $\{x_n\}$ *is* G_b*-convergent to x,* $\{f(x_n)\}$ *is* G'_b*-convergent to* $f(x)$.

Lemma 1.15 [1] *Let (X, G) be a* G_b*-metric space with* $b \geq 1$, *and suppose that* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *are* G_b*-convergent to x, y and z respectively. Then we have*

$$1/b^3 G(x,y,z) \leq \inf G(x_n, y_n, z_n)$$
$$\leq \sup G(x_n, y_n, z_n) \leq b^3 G(x,y,z).$$

In particular, if $x = y = z$, then we have $\lim_{n\to\infty} G(x_n,y_n,z_n) = \varepsilon$.

Jungck [11] introduced the concept of compatible mappings in metric spaces. Jungck, Murthy and Cho [12] introduced the concept of compatible mappings of type (A) on metric spaces and proved some common fixed point theorems for compatible mappings of type (A). In 1995, Pathak, Kang and Beak [13] introduced the concept of weak compatible mapping of type (A) and proved some common fixed point theorems for weak compatible mappings of type (A) on Menger spaces. Readers can see about various forms of compatible mappings in the research papers in [14-33] and references therein. We state the following definitions in the setting of G_b-metric space.

Definition 1.16 *Let (X, G) be a* G_b*-metric space. A pair* $\{f, g\}$ *is said to be compatible mappings if* $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ *for some t in X.*

Definition 1.17 *Let (X, G) be a* G_b*-metric space. A pair* $\{f, g\}$ *is said to be compatible mappings of type (A) if* $\lim_{n\to\infty} G(fgx_n, ggx_n, ggx_n) = 0$ *and* $\lim_{n\to\infty} G(gfx_n, ffx_n, ffx_n) = 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ *for some t in X.*

Definition 1.18 *Let (X, G) be a* G_b*-metric space. A pair* $\{f, g\}$ *is said to be weak compatible mappings of type (A) if* $\lim_{n\to\infty} G(fgx_n, ggx_n, ggx_n) = 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ *for some t in X.*

The following propositions are easy to prove and hence we omit their proofs.

Proposition 1.19 *Let* $f, g : (X, G) \to (X, G)$ *be mappings. If f and g are weak compatible mappings of type (A) and* $ft = gt$ *for some t in X, then* $fgt = ggt$.

Proposition 1.20 *Let* $f, g : (X, G) \to (X, G)$ *be mappings. If f and g are weak compatible mappings of type (A) and* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ *for some t in X. Then we have* $\lim_{n\to\infty} gfx_n = ft$, *if f is continuous.*

The aim of this paper is to prove a common fixed point theorem for two pairs of weak compatible mappings of type (A) in G_b-metric space.

2. Main Results

Our first result is the following common fixed point theorem.

Theorem 2.1 Suppose that f, g, M and T are self-mappings on a complete G_b-metric space (X, G) such that $f(X) \subseteq T(X)$, $g(X) \subseteq M(X)$. If

$$G(fx, gy, gy) \leqslant \frac{q}{b^6} max \begin{Bmatrix} G(Mx,\ Ty,\ Ty), G(fx,\ Mx,\ Mx), \\ G(gy,\ Ty,\ Ty), \\ \frac{1}{2}\big(G(Mx,\ gy,\ gy) + G(fx,\ Ty,\ Ty)\big) \end{Bmatrix}$$

holds for each $x, y \in X$ with $0 < q < 1$ and $b \geqslant \frac{3}{2}$, then f, g, M and T have a unique common fixed point in X provided that M and T are continuous and pairs $\{f, M\}$ and $\{g, T\}$ are compatible.

Proof. Let $x_0 \in X$. As $f(X) \subseteq T(X)$, there exists $x_1 \in X$ such that $fx_0 = Tx_1$. Since $gx_1 \in M(X)$, we can choose $x_2 \in X$ such that $gx_1 = Mx_2$. In general, x_{2n+1} and $x_{2n+2}$are chosen in X such that $fx_{2n} = Tx_{2n+1}$ and $gx_{2n+1} = Mx_{2n+2}$. Define a sequence y_n in X such that $y_{2n} = fx_{2n} = Tx_{2n+1}$, and $y_{2n+1} = gx_{2n+1} = Mx_{2n+2}$, for all $n \geq 0$. Now, we show that y_n is a Cauchy sequence. Consider

$$\begin{aligned} G(y_{2n}, y_{2n+1}, y_{2n+1}) &= G(fx_{2n}, gx_{2n+1}, gx_{2n+1}) \\ &\leq \frac{q}{b^6} \max \begin{Bmatrix} G(Mx_{2n}, Tx_{2n+1}, Tx_{2n+1}), \\ G(fx_{2n}, Mx_{2n}, Mx_{2n}), \\ G(gx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \\ 1/2\begin{pmatrix} G(Mx_{2n}, gx_{2n+1}, gx_{2n+1}) \\ +G(fx_{2n}, Tx_{2n+1}, Tx_{2n+1}) \end{pmatrix} \end{Bmatrix} \\ &= \frac{q}{b^6} \max \begin{Bmatrix} G(y_{2n-1}, y_{2n}, y_{2n}), \\ G(y_{2n}, y_{2n-1}, y_{2n-1}), \\ G(y_{2n+1}, y_{2n}, y_{2n}), \\ 1/2\begin{pmatrix} G(y_{2n-1}, y_{2n+1}, y_{2n+1}) \\ +G(y_{2n}, y_{2n}, y_{2n}) \end{pmatrix} \end{Bmatrix} \\ &\leq \frac{q}{b^6} \max \begin{Bmatrix} G(y_{2n-1}, y_{2n}, y_{2n}), \\ 2bG(y_{2n-1}, y_{2n}, y_{2n}), G(y_{2n+1}, y_{2n}, y_{2n}), \\ 1/2G(y_{2n-1}, y_{2n+1}, y_{2n+1}) \end{Bmatrix} \\ &\leq \frac{q}{b^6} \max \begin{Bmatrix} G(y_{2n-1}, y_{2n}, y_{2n}), \\ 2bG(y_{2n-1}, y_{2n}, y_{2n}), \\ G(y_{2n+1}, y_{2n}, y_{2n}), \\ b/2\begin{pmatrix} G(y_{2n-1}, y_{2n}, y_{2n}) \\ +G(y_{2n}, y_{2n+1}, y_{2n+1}) \end{pmatrix} \end{Bmatrix} \\ &\leq \frac{q}{b^6} \max \begin{Bmatrix} G(y_{2n-1}, y_{2n}, y_{2n}), \\ 2bG(y_{2n-1}, y_{2n}, y_{2n}), \\ 2bG(y_{2n}, y_{2n+1}, y_{2n+1}), \\ b/2\begin{pmatrix} G(y_{2n-1}, y_{2n}, y_{2n}) \\ +G(y_{2n}, y_{2n+1}, y_{2n+1}) \end{pmatrix} \end{Bmatrix}. \end{aligned}$$

Now, since $G(y_{2n-1}, y_{2n}, y_{2n}) \leq 2bG(y_{2n-1}, y_{2n}, y_{2n})$ and $G(y_{2n}, y_{2n+1}, y_{2n+1}) \leq 2bG(y_{2n}, y_{2n+1}, y_{2n+1})$ we have

$$G(y_{2n}, y_{2n+1}, y_{2n+1}) \leq \max \begin{Bmatrix} 2bG(y_{2n}, y_{2n+1}, y_{2n+1}), \\ 2bG(y_{2n-1}, y_{2n}, y_{2n}) \end{Bmatrix}.$$

If max = $2bG(y_{2n}, y_{2n+1}, y_{2n+1})$, we obtain

$$\begin{aligned} G(y_{2n}, y_{2n+1}, y_{2n+1}) &\leq 2q/b^5 G(y_{2n}, y_{2n+1}, y_{2n+1}) \\ &< G(y_{2n}, y_{2n+1}, y_{2n+1}) \end{aligned}$$

which is a contradiction.

So, max = $2bG(y_{2n-1}, y_{2n}, y_{2n})$ and we have

$$G(y_{2n}, y_{2n+1}, y_{2n+1}) \leq q/b^6 \big(2bG(y_{2n-1}, y_{2n}, y_{2n})\big)$$

i.e., $G(y_{2n}, y_{2n+1}, y_{2n+1}) \leq 2q/b^5 G(y_{2n-1}, y_{2n}, y_{2n})$.

Let $\lambda = 2q/b^5$. Since $b \geq 3/2$ we have that $0 < \lambda < 1$.

Now,

$$\begin{aligned} G(y_{2n}, y_{2n+1}, y_{2n+1}) &\leq \lambda G(y_{2n-1}, y_{2n}, y_{2n}) \\ &\leq \lambda^2 G(y_{2n-2}, y_{2n-1}, y_{2n-1}) \end{aligned}$$

and so on.

Hence, for all $n \geq 2$, we obtain

$$G(y_{n-1}, y_n, y_n) \leq \ldots \leq \lambda^{n-1} G(y_0, y_1, y_1). \tag{2}$$

Using (G_b5), and (2) for all $n > m$, we have

$$\begin{aligned} &G(y_m, y_n, y_n) \\ &\leq b\big(G(y_m, y_{m+1}, y_{m+1}) + G(y_{m+1}, y_n, y_n)\big) \\ &\leq bG(y_m, y_{m+1}, y_{m+1}) + b^2 G(y_{m+1}, y_{m+2}, y_{m+2}) \\ &\quad + b^2 G(y_{m+2}, y_n, y_n) \\ &\leq \ldots \\ &\leq b\begin{pmatrix} G(y_m, y_{m+1}, y_{m+1}) + b^2 G(y_{m+1}, y_{m+2}, y_{m+2}) \\ + \cdot ??+ b^{n-m} G(y_{n-1}, y_n, y_n) \end{pmatrix} \\ &\leq b\big(\lambda^m + b\lambda^{m+1} + \cdot ??+ b^{n-m-1}\lambda^{n-1}\big) G(y_0, y_1, y_1) \\ &\leq bG(y_0, y_1, y_1)\big(\lambda^m + b\lambda^{m+1} + \ldots\big) \\ &\leq \frac{b\lambda^m}{1 - b\lambda} G(y_0, y_1, y_1). \end{aligned}$$

On taking limit as $m, n \to \infty$, we have $G(y_m, y_n, y_n) \to 0$ as $b\lambda < 1$. Therefore $\{y_n\}$ is a Cauchy sequence. Since X is a complete G_b-metric space, there is some y in X such that

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Mx_{2n+2} = y.$$

We show that y is a common fixed point of f, g, M and T. Since M is continuous, therefore

$$\lim_{n\to\infty} M^2 x_{2n+2} = My \text{ and } \lim_{n\to\infty} Mfx_{2n} = My.$$

Since the pair $\{f, M\}$ is weak compatible type (A), $\lim_{n\to\infty} G(fMx_{2n}, Mfx_{2n}, Mfx_{2n}) = 0$. So by proposition 1.20, we have

$$\lim_{n\to\infty} fMx_{2n} = My.$$

Putting $x = Mx_{2n}$ and $y = x_{2n+1}$ in (1) we obtain

$$G(fMx_{2n}, gx_{2n+1}, gx_{2n+1}) \le \frac{q}{b^6}\max\left\{\begin{array}{l} G\left(M^2x_{2n}, Tx_{2n+1}, Tx_{2n+1}\right), \\ G\left(fMx_{2n}, M^2x_{2n}, M^2x_{2n}\right), \\ G(gx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \\ 1/2\left(\begin{array}{l} G\left(M^2x_{2n}, gx_{2n+1}, gx_{2n+1}\right) \\ +G(fMx_{2n}, Tx_{2n+1}, Tx_{2n+1}) \end{array}\right) \end{array}\right\}. \tag{3}$$

Taking the upper limit as $n \to \infty$ in (3) and using Lemma 1.15, we get

$$\begin{aligned} &G(My, y, y)/b^3 \\ &\le \limsup_{n\to\infty} G(fMx_{2n}, gx_{2n+1}, gx_{2n+1}) \\ &\le \frac{q}{b^6}\max\left\{\begin{array}{l} \limsup_{n\to\infty} G\left(M^2x_{2n}, Tx_{2n+1}, Tx_{2n+1}\right), \\ \limsup_{n\to\infty} G\left(fMx_{2n}, M^2x_{2n}, M^2x_{2n}\right), \\ \limsup_{n\to\infty} G(gx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \\ 1/2\left(\begin{array}{l} \limsup_{n\to\infty} G\left(M^2x_{2n}, gx_{2n+1}, gx_{2n+1}\right) \\ + \limsup_{n\to\infty} G(fMx_{2n}, Tx_{2n+1}, Tx_{2n+1})) \end{array}\right) \end{array}\right\} \\ &\le \frac{q}{b^6}\max\left\{\begin{array}{l} b^3G(My, y, y), 0, 0, \\ b^3\left(\begin{array}{l} G(My, y, y) \\ +G(My, y, y) \end{array}\right)/2 \end{array}\right\} \\ &= \frac{q}{b^6}\left(b^3G(My, y, y)\right) \\ &= \frac{q}{b^3}G(My, y, y). \end{aligned}$$

Consequently, $G(My, y, y) \le qG(My, y, y)$. As $0 < q < 1$, so $My = y$. Using continuity of T, we obtain $\lim_{n\to\infty} T^2x_{2n+1} = Ty$ and $\lim_{n\to\infty} Tgx_{2n+1} = Ty$.

Since g and T are weak compatible type (A), $\lim_{n\to\infty} G(gTx_n, Tgx_n, Tgx_n) = 0$. So, by proposition 1.20, we have $\lim_{n\to\infty} gTx_{2n} = Ty$. Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (1), we obtain

$$G(fx_{2n}, gTx_{2n+1}, gTx_{2n+1}) \le \frac{q}{b^6}\max\left\{\begin{array}{l} G\left(Mx_{2n}, T^2x_{2n+1}, T^2x_{2n+1}\right), \\ G(fx_{2n}, Mx_{2n}, Mx_{2n}), \\ G\left(gTx_{2n+1}, T^2x_{2n+1}, T^2x_{2n+1}\right), \\ 1/2\left(\begin{array}{l} G(Mx_{2n}, gTx_{2n+1}, gTx_{2n+1}) \\ +G\left(fx_{2n}, T^2x_{2n+1}, T^2x_{2n+1}\right) \end{array}\right) \end{array}\right\}. \tag{4}$$

Taking upper limit as $n \to \infty$ in (4) and using Lemma 1.15, we obtain

$$\begin{aligned} &G(y, Ty, Ty)/b^3 \\ &\le \limsup_{n\to\infty} G(fx_{2n}, gTx_{2n+1}, gTx_{2n+1}) \\ &\le \frac{q}{b^6}\max\left\{b^3\left(\begin{array}{l} G(y, Ty, Ty), 0, 0, \\ b^3/2G(y, Ty, Ty) + G(y, Ty, Ty) \end{array}\right)\right\} \\ &= \frac{q}{b^3}G(y, Ty, Ty) \end{aligned}$$

which implies that $Ty = y$. Also, we can apply condition (1) to obtain

$$G(fy, gx_{2n+1}, gx_{2n+1}) \le \frac{q}{b^6}\max\left\{\begin{array}{l} G(My, Tx_{2n+1}, Tx_{2n+1}), \\ G(fy, My, My), \\ G(gx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \\ 1/2\left(\begin{array}{l} G(My, gx_{2n+1}, gx_{2n+1}) \\ +G(fy, Tx_{2n+1}, Tx_{2n+1}) \end{array}\right) \end{array}\right\}. \tag{5}$$

Taking upper limit $n \to \infty$ in (5), and using $My = Ty = y$, we have

$$\begin{aligned} &G(fy, y, y)/b^3 \\ &\le \frac{q}{b^6}\max\left\{\begin{array}{l} b^3G(My, y, y), b^3G(fy, My, My), \\ b^3G(y, y, y), b^3\left(\begin{array}{l} G(My, y, y) \\ +G(fy, y, y) \end{array}\right)/2 \end{array}\right\} \\ &= \frac{q}{b^3}G(fy, y, y), \end{aligned}$$

which implies that $G(fy, y, y) = 0$ and $fy = y$ as $0 < q < 1$. Finally, from condition (1), and the fact $My = Ty = fy = y$, we have

$$\begin{aligned} &G(y, gy, gy) = G(fy, gy, gy) \\ &\le \frac{q}{b^6}\max\left\{\begin{array}{l} G(My, Ty, Ty), G(fy, My, My), G(gy, Ty, Ty), \\ 1/2\left(G(My, gy, gy) + G(fy, Ty, Ty)\right) \end{array}\right\} \\ &\le \frac{q}{b^3}G(y, gy, gy) \le qG(y, gy, gy), \end{aligned}$$

which implies that $G(y, gy, gy) = 0$ and $gy = y$. Hence $My = Ty = fy = gy = y$. If there exists another common fixed point x in X for f, g, M and T, then

$$G(x,y,y)=G(fx,gy,gy)$$
$$\leq \frac{q}{b^6}\max\left\{\begin{array}{l}G(Mx,Ty,Ty),G(fx,Mx,Mx),G(gy,Ty,Ty),\\1/2\big(G(Mx,gy,gy)+G(fx,Ty,Ty)\big)\end{array}\right\}$$
$$=\frac{q}{b^6}\max\left\{\begin{array}{l}G(x,y,y),G(x,x,x),G(y,y,y),\\1/2\big(G(x,y,y)+G(x,y,y)\big)\end{array}\right\}$$
$$=\frac{q}{b^6}G(x,y,y)\leq qG(x,y,y),$$

which further implies that $G(x, y, y) = 0$ and hence, $x = y$. Thus, y is a unique common fixed point of f, g, M and T.

***Example* 2.2** Let $X = [0, 1]$ be endowed with G_b-metric $G*(x, y, z) = (|y + z - 2x| + |y - z|)^2$, where $b = 4$. Define f, g, M and T on X by

$$f(x)=(x/4)^8,$$
$$g(x)=(x/8)^4,$$
$$M(x)=(x/4)^4,$$
$$T(x)=(x/8)^2.$$

Obviously, $f(X) \subseteq T(X)$ and $g(X) \subseteq M(X)$. Furthermore, the pairs $\{f, M\}$ and $\{g, T\}$ are weak compatible mappings of type (A). For each $x, y \in X$, we have

$$G(fx,gy,gy)=(|gy-fx|+|fx-gy|)^2$$
$$=(2|fx-gy|)^2=4\Big((x/4)^8-(y/8)^4\Big)^2$$
$$=4\Big((x/4)^4+(y/8)^2\Big)^2.\Big((x/4)^4-(y/8)^2\Big)^2$$
$$\leq\Big(1/4^4+1/8^2\Big)^2 G(Mx,Ty,Ty)$$
$$=\frac{\frac{25}{4^4}}{4^4}G(Mx,Ty,Ty)$$

where $\frac{25}{4^4}\leq q \leq 1$ and $b = 4$. Thus, f, g, M and T satisfy all condition of Theorem 2.1. Moreover 0 is the unique common fixed point of f, g, M and T.

Corollary 2.3 Let (X, G) be a complete G_b-metric space and $f, g: X \to X$ two mappings such that

$$G(fx,gy,gy)$$
$$\leq\frac{q}{b^6}\max\left\{\begin{array}{l}G(x,y,y),G(fx,x,x),G(gy,y,y),\\1/2\big(G(x,gy,gy)+G(fx,y,y)\big)\end{array}\right\}$$

holds for all $x, y \in X$ with $0 < q < 1$ and $b \geq 3/2$. Then, there exists a unique point y in X such that $fy = gy = y$.

Proof. If we take $M = T = I_X$ (identity mapping on X), then theorem 2.1 gives that f and g have a unique common fixed point.

Note. If we take f and g as identity maps on X, then Theorem 2.1 gives that M and T have a unique common fixed point.

Corollary 2.4 Let (X, G) be a complete G_b-metric space and $f: X \to X$ mapping such that

$$G(fx,fy,fy)\leq\frac{q}{b^6}\max\left\{\begin{array}{l}G(x,y,y),G(fx,x,x),G(fy,y,y),\\1/2\big(G(x,fy,fy)+G(fx,y,y)\big)\end{array}\right\}$$

holds for all $x, y \in X$ with $0 < q < 1$ and $b \geq 3/2$. Then f has a unique fixed point in X.

Proof. Take M and T as identity maps on X and $f = g$ and then apply Theorem 2.1.

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References

[1] Aghajani A., Abbas M. and Roshan J. R., Common Fixed Point of Generalized Weak Contractive Mappings in Partially Ordered Gb-metric Spaces, Filomat28:6 (2014), 1087-1101.

[2] Mustafa Z, Sims B. A new approach to generalized metric spaces. J Nonlinear Convex Anal, 2006, 7: 289297.

[3] I. A. Bakhtin, *The contraction principle in quasimetric spaces*, Funct. Anal., Gos. Ped. Inst. Unianowsk, 30(1989), 26-37.

[4] Czerwik, S. Contraction mappings in b−metric spaces, Acta Math. Inf. Univ. Ostraviensis.1, (1993), 5-11.

[5] Czerwik, S. Nonlinear set-valued contraction mappings in *b*-metric spaces, Atti. Sem. Mat. Fis. Univ. Modena, 46(2), (1998), 263-276.

[6] Z. Mustafa, J.R. Roshan and V. Parvaneh, Coupled coincidence point results for (ψ, φ)-weakly contractive mappings in partially ordered G_b-metric spaces, Fixed Point Theory Appl., 2013:206 (2013).

[7] Z. Mustafa, J. R. Roshan and V. Parvaneh, Existence of a tripled coincidence point in ordered G_b-metric spaces and applications to a system of integral equations, Journal of Inequalities and Applications 2013, 2013:453.

[8] Shaban Sedghi, Nabi Shobkolaei, Jamal Rezaei Roshan and Wasfi Shatanawi, Coupled fixed point theorems in G_b-metric spaces, Matematiki Vesnik, 66, 2 (2014), 190-201 June 2014.

[9] Bulbul Khomdram, Yumnam Rohen and Thokchom Chhatrajit Singh, Coupled fixed point theorems in G_b-metric space satisfying some rational contractive conditions, SpringerPlus (2016), 5:1261.

[10] Bulbul Khomdram and Yumnam Rohen, Quadrupled common fixed point theorems in G_b-metric spaces, *IJPAM*, 109(2), (2016), 279-293.

[11] G. Jungck, Compatible mappings and common fixed points, *Int. J. Math. Math. Sci*., 9 (4) (1986), 771-779.

[12] Jungck, G., Murthy, P. P. and Cho, Y. J., Compatible mappings of type (A) and common fixed points, *Math. Japonica*, 38, (1993), 381-390.

[13] Pathak, H. K., Kang, S. M. and Baek, J. H., Weak compatible mappings of type (A) and common fixed point in Menger spaces, *Comm. Korean Math. Soc*., 10 (1), (1995), 67-83.

[14] Yumnam Rohen and Bidya Devi, Relationship between various types of compatible mappings, *IJMSEA*, 4(I) (2010), 67-82.

[15] Yumnam Rohen, Fixed point theorems of compatible mappings of type(R) in metric spaces, *Int. J. Open Problems Compt. Math.*, 3(5), (2010) 138-143.

[16] L. Dwijendra and Yumnam Rohen, Common fixed point theorems of compatible mappings of type (A) in fuzzy metric spaces, *IJMSEA*, 5(VI) (2011), 307-316.

[17] M. Koireng and Yumnam Rohen, Common fixed point theorems of compatible mappings of type (P) in fuzzy metric spaces, *Int. J. of Math. Anal*., 6(4), (2012), 181-188.

[18] L. Ibeni and Yumnam Rohen, Related fixed point theorems for three set-valued mappings on three uniform spaces, *Int. J. of Math. Anal*., 6(9), (2012), 415-427

[19] M. Krishnanado, Yumnam Rohen and L. Shambhu, Common fixed point theorems of compatible mappings of type(R), *IJMSEA*, 6(I) (2012), 355-361.

[20] Yumnam Rohen, Th. Indubala, O. Budhichandra and Leenthoi N., Common fixed point theorem for compatible mappings of type (A), *IJMSEA*, 6(II) (2012), 323-333.

[21] M. Krishnanando, Yumnam Rohen, O. Budhichandra and Mairembam Bina, Weak compatible and compatible mappings of type (R) in cone metric space, *IJMSEA*, 6(II) (2012), 161-170.

[22] M. Koireng Meitei and Yumnam Rohen, Some fixed point theorems for compatible mappings in uniform space, *IJMSEA*, 6(IV) (2012), 107-115.

[23] M. Koireng Meitei Leenthoi N. and Yumnam Rohen, Common fixed points of compatible mappings of type (R), *General Mathematics Notes*, 10(1) (2012), 58-62.

[24] M. Popeshwar Singh and Yumnam Rohen, Common fixed points of compatible mappings, *Journal of Scientific Research,* 4(3) (2012), 603-608.

[25] L. Ibeni Devi and Yumnam Rohen, Compatible mappings of type (A-1) and type (A-2) in cone metric space, *The Journal of Mathematics and Computer Sciences* 2(6), (2012), 1612-1621.

[26] Yumnam Rohen and M. Ranjit Singh, Compatible mappings in cone metric space, for the book "*Weakly commuting mappings after thirty years in metric fixed point theory*" edited by S. Sessa, 2012 47-54.

[27] Shahidur Rahman, Yumnam Rohen and M. Popeshwar Singh, Generalised common fixed point theorems of A-compatible and S-compatible mappings, *American J. of Applied Mathematics and Statistics*, 1(2), (2013), 27-29.

[28] L. Ibeni Devi and Yumnam Rohen, Remarks on common fixed point theorems of Rajesh et. al., *IJMSEA*, 7(VI) (2013) 97-109.

[29] L. Bishwakumar and Yumnam Rohen, Common fixed point theorem for six self-maps in a cone metric space through semi compatibility and weak compatibility, *IJMSEA*, 7(VI) (2013) 121-133.

[30] Yumnam Rohen and L. Premila Devi, Some fixed point theorems of semicompatible mappings, *Journal of International Academy of Physical Sciences*, 18(2) (2014) 125-129.

[31] Thokchom Chhatrajit and Yumnam Rohen, A comparative study of relationship among various types of spaces, *International Journal of Applied Mathematics*, 28(1) (2015), 29-36.

[32] Yumnam Rohen and L. Premila Devi, Some fixed point theorems of semi compatible and occasionally weakly compatible mappings in menger space, *American Journal of Applied Mathematics and Statistics*, 3(1) (2015), 29-33.

[33] Laishram Shanjit, Yumnam Rohen and P. P. Murthy, Some fixed point theorems of R-weakly commuting mappings in multiplicative metric spaces, *Gazi University Journal of Science*, 29(4) (2016) 855-867.