A Fixed Point Approach to Hyers-Ulam-Rassias Stability of Nonlinear Differential Equations

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Abstract In this paper we use the fixed point approach to obtain sufficient conditions for Hyers-Ulam-Rassias stability of nonlinear differential. Some illustrative examples are given.

Keywords: hyers-ulam-rassias stability, fixed point, nonlinear differential equations


1. Introduction

The objective of this article is to investigate the Hyers-Ulam-Rassias Stability for the nonlinear differential equation

\[ y''(t) + 2f(t)y' + y + g(t, y) = 0, \quad t \in \mathbb{R}^+ \]  

(1)

and the perturbed nonlinear differential equation of second order

\[ y''(t) + 2f(t)y' + y + g(t, y) = h(t) \]  

(2)

by fixed point method under assumptions: \( f(t) > 0 \), \( g(t, y) \) are continuous, and that

\[ \frac{t}{2} \int_0^t |f(s)|ds \to \infty \quad \text{as} \quad t \to \infty, \]  

(3)

\[ \int_0^t \alpha - 2 \int_0^u f(u)du \leq \frac{(t-s)ds}{2} \leq \alpha \]  

(4)

where \( \alpha < 1 \), \( t \geq 0 \).

Suppose that there is \( L > 0 \) such that if \(|x|, |y| \leq L\), then

\[ |g(t, x) - g(t, y)| \leq Ld(t)|x - y|, \quad t \geq 0, \]  

(5)

where \( d(t) > 0 \), \( d(t) \to 0 \) as \( t \to \infty \), and \( g(t, 0) = 0 \).

Furthermore, we assume that there is a positive constant \( A \) such that \( A < L \), and \( h(t): [0, \infty) \to \mathbb{R} \) with

\[ \frac{t}{\alpha} \int_0^t h(s)ds \leq A, \quad t \geq 0 \]  

(6)

In 1940, Ulam [1] posed the stability problem of functional equations. In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when Hyers [2] gave a partial solution to Ulam's problem. During the last two decades very important contributions to the stability problems of functional equations were given by many mathematicians (see [3-11]). More than twenty years ago, a generalization of Ulam's problem was proposed by replacing functional equations with differential equations: the differential equation \( F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0 \) has the Hyers-Ulam stability if for given \( \varepsilon > 0 \) and \( y \) a function such that

\[ |y(t) - y_0(t)| \leq \varepsilon \]

there exists a solution \( y_0 \) of the differential equation such that

\[ \lim_{\varepsilon \to 0} K(\varepsilon) = 0, \]

where

\[ K(\varepsilon) = \frac{\varepsilon}{\varepsilon} \]  

The first step in the direction of investigating the Hyers-Ulam stability of differential equations was taken by Obloza (see [12,13]). Thereafter, Alsina and Ger [14] have studied the Hyers-Ulam stability of the linear differential equation \( y'(t) = y(t) \). The Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied in the papers ([15,16]) by using the method of integral factors. The results given in [17,18,19] have been generalized by Popa and Rasa [20,21] for the linear differential equations of nth order with constant coefficients. In addition to above-mentioned studies, several authors have studied the Hyers-Ulam stability for differential equations of first and second order (see 22-26). The Hyers-Ulam-Rassias Stability by Fixed Point Technique for Half-linear Differential Equations with Unbounded Delay has been established by Qarawani [27]. Burton in [28] has used fixed point theory to establish Liapunov stability for...
functional differential equations. Some researchers have used the fixed point approach to investigate the Hyers-Ulam stability for differential equations [e.g. [29,30]].

Definition 1 Let

$$S = \{ \phi : R^+ \to R \mid \phi(0) = y_0, \|\phi\| \leq L \}$$

on $R^+$, $\phi \in C$, where $R^+ = [0, \infty)$. We say that equation (1.2) (or (1.1) with $h(t) = 0$) has the Hyers-Ulam-Rassias (HUR) stability with respect to $\phi$ if there exists a constant $k > 0$ with the following property: For each $y(t) \in S$, if

$$|y'(t) + 2f(t)y' + y + g(t, y) - h(t)| \leq \phi(t),$$

then there exists some $y_0(t)$ of the equation (4) such that

$$|y(t) - y_0(t)| \leq k \phi(t).$$

Theorem 1 The Contraction Mapping Principle.

Let $(S, \rho)$ be a complete metric space and let $P : S \to S$. If there is a constant $\alpha < 1$ such that for each pair $\phi_1, \phi_2 \in S$ we have $\rho(P\phi_1, P\phi_2) \leq \alpha \rho(\phi_1, \phi_2)$, then there is one and only one point $\phi \in S$ with $P\phi = \phi$.

2. Main Results On Hyers-Ulam-Rassias Stability

Theorem 2 Suppose that $y(t) \in S$ satisfies the inequality (1) with small initial condition $y(0) = y_0$. Let $\varphi(t) : [0, \infty) \to (0, \infty)$ be a continuous function such that

$$\int_0^t \varphi(s)e^{-\int_s^t f(u)du}(t-s)ds \leq C\varphi(t),$$

$$\forall t \geq 0.$$

If (3)-(6) hold, then the solution of (1) is stable in the sense of Hyers-Ulam-Rassias.

Proof. Let $C$ be the space of all continuous functions from $R^+ \to R$ and define the set $S$ by

$$S = \left\{ \phi : R^+ \to R \mid \phi(0) = y_0, \|\phi\| \leq L, \text{ on } R^+, \phi \in C \right\}.$$

Then, equipped with the supremum metric $\|\cdot\|_\infty$, is a complete metric space. Now suppose that (3) holds. For $L$ and $\alpha$, find appropriate constants $\delta, a$ and $B$ such that

$$(1 + a)\delta + \frac{La}{2} + \frac{L\alpha}{2} \leq L.$$

Multiplying both sides of (1) by $e^{-\int_0^t f(s)ds}$, and then integrating once with respect to $t$ yields

$$\int_0^t \frac{2f(s)ds}{e^t} = y'(0) - y(0)e^{-\int_0^t f(s)ds}$$

and

$$\int_0^t \frac{2f(u)du}{e^t} + \frac{2f(u)du}{e^t} - g(s, y(s))e^{-\int_0^t f(s)ds} \leq \frac{2f(u)du}{e^t} - 2f(s)ds\right.$$
Since \( d(t) \to 0 \), as \( t \to \infty \), we can choose a number \( B \) sufficiently small such that \( 0 < d(t) \leq B \), on \( R^+ \) and with \( LB < 1 \).

Then from (4) we obtain

\[
\|P\phi\| \leq (1 + a)\delta + \frac{La}{2} + \frac{LBa}{2}
\]

which implies that \( \|P\phi\| \leq L \).

To see that \( P \) is a contraction under the supremum metric, let \( \phi, \eta \in S \), then

\[
\|P\phi(t) - (P\eta)(t)\| \\
\leq \int_0^t \left| \phi(s) - \eta(s) \right| e^{-\int_0^s f(u) \, du} \, ds \\
+ \int_0^t \left| g(s, \phi(s)) - g(s, \eta(s)) \right| e^{-\int_0^s f(u) \, du} \, ds \\
\leq \int_0^t e^{-\int_0^s f(u) \, du} \|\phi - \eta\| \, ds \\
+ LB \int_0^t e^{-\int_0^s f(u) \, du} \|\phi - \eta\| \, ds
\]

From this and in view of (4) and (11) we get the estimate

\[
\|P\phi(t) - (P\eta)(t)\| \leq \alpha \|\phi - \eta\|, \quad \text{with } \alpha < 1.
\]

Thus, by the contraction mapping principle, \( P \) has a unique fixed point, say \( y_0 \) in \( S \) which solves (1) and is bounded.

Next we show that the solution \( y_0 \) is stable in the sense of Hyers-Ulam-Rassias. From the inequality (7) we get

\[
- \phi(t) \leq y'(t) + 2f(t)y' + g(t, y) \leq \phi(t)
\]

Multiplying the inequality (12) by \( e^{-\int_0^t f(u) \, du} \), we obtain

\[
\left( \int_0^t f(u) \, du \right) \phi(t) \leq e^{-\int_0^t f(u) \, du} \left( y'(t) + 2f(t)y' + g(t, y) \right) \leq \phi(t)e^{\int_0^t f(u) \, du}
\]

Or equivalently, we have

\[
\left( \int_0^t f(u) \, du \right) \phi(t) \leq e^{-\int_0^t f(u) \, du} \left( y'(t) + 2f(t)y' + g(t, y) \right) \leq \phi(t)e^{\int_0^t f(u) \, du}
\]

Integrate the last inequality from 0 to \( t \), and then multiply the obtained inequality by \( e^{-\int_0^t f(u) \, du} \) to get

\[
\phi(t) \leq e^{-\int_0^t f(u) \, du} \left( y'(t) + 2f(t)y' + g(t, y) \right) \leq \phi(t)e^{\int_0^t f(u) \, du}
\]

Integrating again with respect to \( t \), we have
\[ t - \int_{s}^{t} f(u) du \leq y(t) - y(0) \]
\[ y(t) = \frac{e^{-2s}}{s} \int_{0}^{s} f(u) du \]
\[ y'(t) = \frac{2e^{-2}}{s} \int_{0}^{s} f(u) du \]
\[ y(t) \leq C \phi + \alpha \int_{0}^{t} y(s) ds \]

Hence from (8), (20) we infer that \( P_y \leq C \phi \). To show that \( y_0 \) is stable we estimate the difference
\[ \|y(t) - y_0(t)\| \leq \|P_y - y_0\| \leq C \phi + \alpha \int_{0}^{t} y(s) ds \]

Thus
\[ \|y(t) - y_0(t)\| \leq \frac{C \phi}{1 - \alpha} \]

which means that (7) holds true (with \( h(t) = 0 \)) for all \( t \geq 0 \).

Example 1 Consider the differential equation
\[ y^{\prime\prime}(t) + (4 + 2 \sin t) y^{\prime} + y + \frac{\sin y}{(1 + t)^2} = 0. \]

Let us estimate the integrals
\[ \int_{0}^{t} f(s) ds = \int_{0}^{t} (2 + \sin t) ds \geq \int_{0}^{t} ds \geq t \to \infty, \]

and for all \( t > 0 \) we obtain
\[ \int_{0}^{t} e^{-2s} (t-s) ds \leq \frac{1}{4}, \]

Since \( g(t, y(t)) = \frac{\sin y}{(1 + t)^2} \), then
\[ |g(t, x) - g(t, y)| \leq \frac{1}{(1 + t)^2} |x - y|. \]

Therefore, we take \( d(t) = \frac{1}{(1 + t)^2} \), which tends to zero as \( t \to \infty \).

Now, if we set \( \phi(t) = e^{t} \), then we have
\[ \int_{0}^{t} e^{-2s} (t-s) ds \]
\[ = \int_{0}^{t} (4 + 2 \sin u) du \]
\[ \geq \int_{0}^{t} (1 - e^{-2t} - 2te^{-2t}) < \frac{1}{9} \leq C \phi(t), \]

with \( C \geq \frac{1}{9}, \forall t \geq 0 \).

Let us take \( L = 1, \alpha = \frac{1}{2}, B = 0.1 \). Then for the corresponding coefficients by (1.3), we can choose small positive constants \( a, \delta \) such that
\[ (1 + a) \delta + \frac{La}{2} + \frac{LaB}{2} \leq L \]
and so
\[ (1 + a) \delta \leq \frac{29}{40} \]

Thus, all the conditions of Theorem (3.1) are satisfied, hence the Eq. (3.6) is HUR stable for \( t \geq 0 \).

Theorem 3 Suppose that \( y(t) \in S \) satisfies the inequality (7) with small initial condition \( y(0) = y_0 \). Let \( \phi(t) : [0, \infty) \to (0, \infty) \) be a continuous function such that
\[ \int_{0}^{t} f(s) ds \leq C \phi(t), \forall t \geq 0. \] (13)

If (3)-(7) hold, then the solution of (2) is stable in the sense of Hyers-Ulam-Rassias.

Proof. Define \( S = \{ \phi : R^+ \to R | \phi(0) = y_0, ||\phi|| \leq L, \}

on \( R^+, \phi \in C \} \) where \( ||\phi|| \) is the supremum metric.

Then \( (S, ||\phi||) \) is a complete metric space.

Now suppose that (3) holds. For \( L, A \) and \( \alpha \) we find constants \( \delta, a \) and \( B \) so that
\[ (1 + a) \delta + \frac{La}{2} + \frac{LaB}{2} + A \leq L. \]

Applying the same approach used in Theorem 1 we define \( P : S \to S \) by
Then from (4) we obtain
\[
\|P\phi\| \leq (1 + a)\delta + \frac{La}{2} \int t_0 + \frac{LaB}{2} + A
\]
which implies that \(\|P\phi\| \leq L\).

To see that \(P\) is a contraction under the supremum metric, let \(\phi, \eta \in S\), then
\[
\left\| (P\phi)(t) - (P\eta)(t) \right\| \\
\leq \frac{t}{\int (t-s)\phi(s) - \eta(s)e} - \frac{2}{\int f(u)du} ds \\
+ \frac{t}{\int (t-s)g(s, \phi(s)) - g(s, \eta(s))e} - \frac{2}{\int f(u)du} ds \\
\leq \frac{t}{\int (t-s)e} - \frac{2}{\int f(u)du} ds \\
+ LB\frac{t}{\int (t-s)e} - \frac{2}{\int f(u)du} ds \\
\]

From this and using (4) and (11) we get the estimate
\[
\left\| (P\phi)(t) - (P\eta)(t) \right\| \leq \alpha \|\phi - \eta\|, \quad \text{with } \alpha < 1.
\]

Thus, by the contraction mapping principle, \(P\) has a unique fixed point, say \(y_0\) in \(S\) which solves (1) and is bounded.

Next we show that the solution \(y_0\) is stable in Hyers-Ulam-Rassias. From the inequality (7) we get
\[
-\varphi(t) \leq y''(t) + 2f(t)y' + y + g(t, y) - h(t) \leq \varphi(t) \quad (14)
\]

Multiplying the inequality (14) by \(e^0\), we obtain
\[
-\int y''(t) + 2f(t)y't + y + g(t, y) - h(t) dt \leq \int \varphi(t) dt \\
\leq \int y(t) - y(0) + \int (t-s)g(s, y(s))e^0 ds
\]

Integrating again with respect to \(t\), we have
\[
-\int t \varphi(s) - 2\int f(u)du ds \\
\leq y(t) - y(0) + \int (t-s)g(s, y(s))e^0 ds
\]

Or equivalently, we have
\[
-\int 2f(u)du \leq \left\{ \begin{array}{c} t \varphi(t) \\ y'(t) \\ 2f(u)du \end{array} \right\} \\
\int 2f(u)du \leq \left\{ \begin{array}{c} y(t) \\ y'(t) \\ 2f(u)du \end{array} \right\}
\]

Integrating the last inequality from 0 to \(t\), and then multiplying the obtained inequality by \(e^0\)

we get
\[
\int \varphi(s) - 2f(u)du ds \\
\leq y(t) - y(0) + \int (t-s)g(s, y(s))e^0 ds
\]

Integrating again with respect to \(s\), we have
\[
\int \varphi(s) - 2f(u)du ds \\
\leq y(t) - y(0) + \int (t-s)g(s, y(s))e^0 ds
\]
Now, to show that \( y_0 \) is stable we estimate the difference
\[
\left\| y(t) - y_0(t) \right\| \leq C\varphi + \alpha \left\| y - y_0 \right\|
\]
which completes the proof.

Example 2 Consider the nonlinear differential equation
\[
y''(t) + (4 + 2\sin t)y' + y + \frac{\sin y}{(1 + t)^2} = \frac{e^{-2t}\cos^2 t}{1 + t}
\]
One can similarly, as in Example 1 establish the validity of conditions (1.3)-(1.6). So, to establish the stability of this equation, it remains to estimate the integral
\[
t \int_{0}^{t} f(s) ds \leq \int_{0}^{t} (t-s)e^{-s} ds \leq \int_{0}^{t} e^{-s} ds \leq \frac{1}{1-s}
\]
Let us take \( L = 1, \alpha = \frac{1}{2}, A = \frac{1}{2e^2}, \) and \( B = 0.1 \).

Then for these coefficients by (3), we can choose small positive constants \( a, \delta \) such that
\[
(1+a)\delta + \frac{La}{2} + \frac{LaB}{2} + A \leq L
\]
From which it follows that
\[
(1+a)\delta \leq \frac{29}{40} - \frac{1}{2e^2} = \frac{53}{80}
\]
Hence the conditions of Theorem 2 are satisfied.

3. Conclusion

We have obtained two theorems which provide the sufficient conditions for the Hyers-Ulam-Rassias Stability of solutions of two nonlinear differential equations. To illustrate the results we provided two examples satisfying the assumptions of the two proved theorems.

References