Derivation of Continuous Linear Multistep Methods Using Hermite Polynomials as Basis Functions

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Abstract This paper concerns the derivation of continuous linear multistep methods for solving first-order initial value problems (IVPs) of ordinary differential equations (ODEs) with step number \( k = 3 \) using Hermite polynomials as basis functions. Adams-Bashforth, Adams-Moulton and optimal order methods are derived through collocation and interpolation technique. The derived methods are applied to solve two first order initial value problems of ordinary differential equations. The result obtained by the optimal order method compared favourably with those of the standard existing methods of Adams-Bashforth and Adams-Moulton.

Keywords: linear multistep method, hermite polynomial, collocation, interpolation, optimal order scheme, ordinary differential equation, initial value problem


1. Introduction

Linear multistep methods (LMMs) are very popular for solving initial value problems (IVPs) of ordinary differential equations (ODEs). They are also applied to solve higher order ODEs. LMMs are not self-starting hence, need starting values from single-step methods like Euler’s method and Runge-Kutta family of methods.

The general \( k \)-step LMM is as given in Lambert (1973)

\[
\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1}
\]

where \( \alpha_j \) and \( \beta_j \) are uniquely determined and \( \alpha_0 + \beta_0 \neq 0, \alpha_0 = 1 \).

The LMM in Equation (1) generates discrete schemes which are used to solve first-order ODEs. Other researchers have introduced the continuous LMM using the continuous collocation and interpolation approach leading to the development of the continuous LMMs of the form

\[
y(x) = \sum_{j=0}^{k} \alpha_j (x) y_{n+j} + h \sum_{j=0}^{k} \beta_j (x) f_{n+j} \tag{2}
\]

where \( \alpha_j \) and \( \beta_j \) are expressed as continuous functions of \( x \) and are at least differentiable once [8].

According to Okunuga and Ehigie [17], the existing methods of deriving the LMMs in discrete form include the interpolation approach, numerical integration, Taylor series expansion and through the determination of the order of LMM. Continuous collocation and interpolation technique is now widely used for the derivation of LMMs, block methods and hybrid methods.

Several continuous LMMs have been derived using different techniques and approaches: Alabi [5] derived continuous solvers of IVPs using Chebyshev polynomial in a multistep collocation technique; Okunuga and Ehigie [17] derived two-step continuous and discrete LMMs using power series as basis function; Mohammed [15] derived a linear multistep method with continuous coefficients and used it to obtain multiple finite difference methods which were directly applied to solve first-order ODEs; Odekunle et al [16] developed a continuous linear multistep method using interpolation and collocation for the solution of first-order ODE with constant stepsize; Adesanya et al [2,3] considered the method of collocation of the differential system and interpolation of the approximate solution to generate a continuous LMM, which is solved for the independent solution to yield a continuous block method; James et al [11,12] proposed a continuous block method for the solution of second order IVPs with constant stepsize, the method was developed by interpolation and collocation of power series approximate solution; Anake [6] developed a new class of continuous implicit hybrid one-step methods capable of solving IVPs of general second order ODEs using the collocation and interpolation techniques of the power series approximate solution; James et al [11,12] adopted the method of collocation and interpolation of power series approximate solution to generate a continuous LMM; Ehigie et al [9] proposed a two-step continuous multistep method of hybrid type for the direct integration of second order ODEs in a multistep collocation technique; Akinfenwa et al [4] developed a four step continuous block hybrid method with four non-step points for the direct solution of first-order IVPs; Adesanya et al [2,3] adopted the method
of collocation of the differential system and interpolation of the approximate solution at grid and off grid points to yield a continuous LMM with constant stepsize and James et al [11,12] developed a continuous block method using the approach of collocation of the differential system and interpolation of the power series approximate solution.

The introduction of continuous collocation schemes is of great importance as better global error can be estimated and approximations can be equally obtained. Also, the gap between the discrete collocation methods and the conventional multistep methods is bridged. In this study, we will develop continuous multistep collocation methods for the solution of first-order IVPs of ODEs using the probabilists’ Hermite polynomials as the basis function. The corresponding discrete schemes shall also be obtained.

2. Methods

In Awoyemi [7] and Onumanyi et al [18], some continuous LMM of the type in Equation (2) were developed using the collocation function of the form:

\[ y(x) = \sum_{j=0}^{k} a_j x^j. \]  

Awoyemi et al [8] proposed a similar function of the type in Equation (3)

\[ y(x) = \sum_{j=0}^{k} \alpha_j(T_j(x)-x_k) \]  

to develop LMM for the solution of third-order IVPs. Adeniyi and Alabi [1] used Chebyshev polynomial function of the form:

\[ y(x) = \sum_{j=0}^{M} \alpha_j T_j \left( \frac{x-x_k}{h} \right), \]

where \( T_j(x) \) are some Chebyshev function to develop continuous LMM.

In this paper, we propose the Probabilists’ Hermite polynomial of the form [13]:

\[ y(x) = \sum_{j=0}^{k} \alpha_j H_j(x-x_k) \]

where \( H_j(x) \) are probabilists’ Hermite polynomials generated by the formula:

\[ H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \left( x - \frac{d}{dx} \right)^n 1 \]

and whose recursive relation is

\[ H_{n+1}(x) = xH_n(x) - H_n'(x) \]

to develop continuous LMMs for the solution of first-order IVPs of ODEs of the form:

\[ y' = f(x,y(x)), y(x_0) = y_0. \]  

The first six probabilists’ Hermite polynomials are:

\[ H_0 = 1, H_1 = x, H_2 = x^2 - 1, \]
\[ H_3 = x^3 - 3x, H_4 = x^4 - 6x^2 + 3, H_5 = x^5 - 10x^3 + 15x. \]

2.1. Derivation of the Linear Multistep Methods

We wish to approximate the exact solution \( y(x) \) to the IVP in Equation (5) by a polynomial of degree \( n \) of the form:

\[ y(x) = \sum_{j=0}^{n} a_j H_j(x-x_k), x_k \leq x \leq x_{k+p} \]  

which satisfies the equations

\[ y'(x) = f(x,y(x)), x_k \leq x \leq x_{k+p} \]

\[ y(x_k) = y_k. \]

The Adams-Bashforth Method

To derive the three-step Adams-Bashforth method, we set \( n = 3 \) in Equation (6), yielding

\[ y(x) = a_0 + a_1 (x-x_k) + a_2 \left( (x-x_k)^2 - 1 \right) + a_3 \left( (x-x_k)^3 - 3(x-x_k) \right]. \]

Differentiating once gives the equation:

\[ y'(x) = a_1 + 2a_2 (x-x_k) + 3a_3 \left( (x-x_k)^2 - 1 \right). \]

Interpolating Equation (8) at \( x = x_{k+2} \) and collocating Equation (9) at \( x = x_k, x_{k+1}, x_{k+2} \) yields

\[ y(x_{k+2}) = a_0 + a_1 (x_{k+2} - x_k) + a_2 \left( (x_{k+2} - x_k)^2 - 1 \right) + a_3 \left( (x_{k+2} - x_k)^3 - 3(x_{k+2} - x_k) \right] = y_{k+2}, \]

\[ y'(x_{k+2}) = a_1 + 2a_2 \left( x_{k+2} - x_k \right) + 3a_3 \left( (x_{k+2} - x_k)^2 - 1 \right] = f_{k+2}. \]

In matrix form, we have:

\[
\begin{pmatrix}
1 & 2h & (4h^2-1) \\
0 & h & -3h \\
0 & 2h^2 & (3h^3-3h) \\
0 & 4h^2 & (12h^3-3h)
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{pmatrix} =
\begin{pmatrix}
y_{k+2} \\
hf_{k+1} \\
hf_{k+2}
\end{pmatrix}.
\]

Solving the system of equations by Gaussian elimination we obtain:

\[ a_0 = y_{k+2} - \frac{h}{3} (f_{k+2} + 4f_{k+1} + f_k) \]
\[ + \frac{1}{4h} (-f_{k+2} + 4f_{k+1} - 3f_k), \]

\[ a_1 = \frac{1}{h} (y_{k+2} - 4y_k + 6y_{k-1} - 4y_{k-2} + y_{k-3}), \]
\[ a_2 = \frac{1}{2h^2} (y_{k+2} - 8y_k + 12y_{k-1} - 8y_{k-2} + y_{k-3}), \]
\[ a_3 = \frac{1}{6h^3} (y_{k+2} - 12y_k + 24y_{k-1} - 24y_{k-2} + 8y_{k-3}). \]
\[ a_1 = f_k + \frac{1}{2h^2}(f_{k+2} - 2f_{k+1} + f_k), \]
\[ a_2 = \frac{1}{4h}(-f_{k+2} + 4f_{k+1} - 3f_k), \]
\[ a_3 = \frac{1}{6h^2}(f_{k+2} - 2f_{k+1} + f_k). \]

Substituting for \( a_j, j = 0, 1, 2, 3 \) in Equation (8) yields the continuous method:
\[ y(x) = f_k + a_1(x - x_k) + a_2\left[(x - x_k)^2 - 1\right] \]
\[ + a_3\left[(x - x_k)^3 - 3(x - x_k)\right] \]
\[ + a_4\left[(x - x_k)^4 - 6(x - x_k)^2 + 3\right] , \tag{10} \]
\[ y'(x) = a_1 + 2a_2(x - x_k) + 3a_3\left[(x - x_k)^2 - 1\right] \]
\[ + 4a_4\left[(x - x_k)^3 - 12(x - x_k)\right] . \tag{11} \]

The equations can be written in matrix form:
\[
\begin{bmatrix}
1 & 2h & (4h^2 - 1) & (8h^3 - 6h) & (16h^4 - 24h^2 + 3) \\
0 & 0 & -3h & 0 & a_0 \\
0 & h & 2h^2 & (3h^3 - 3h) & (4h^4 - 12h^2) \\
0 & h & 4h^2 & (12h^3 - 3h) & (32h^4 - 24h^2) \\
0 & h & 6h^2 & (27h^3 - 3h) & (108h^4 - 36h^2) \\
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} =
\begin{bmatrix} y_{k+2} \\ f_k \\ h f_{k+1} \\ h f_{k+2} \\ h f_{k+3} \end{bmatrix}.
\]

Solving the system of equations by Gaussian elimination, we have that:
\[ a_0 = y_{k+2} + \frac{h}{12}(2f_{k+3} - 9f_k + 18f_{k+1} - 11f_k) \]
\[ + \frac{1}{8h^3}(f_{k+3} - 3f_{k+2} + 3f_k + f_k - f_k) \]
\[ + \frac{h}{3}(f_{k+2} + 4f_{k+1} + f_k) . \]
\[ a_1 = f_k + \frac{h}{24}(2f_{k+3} - 9f_k + 18f_{k+1} - 11f_k) \]
\[ + \frac{1}{4h^3}(f_{k+3} - 3f_{k+2} + 3f_k + f_k - f_k) , \]
\[ a_2 = \frac{1}{6h^2}(f_{k+3} - 3f_{k+2} + 3f_k + f_k - f_k) , \]
\[ a_3 = \frac{1}{6h^2}(f_{k+3} - 3f_{k+2} + 3f_k + f_k - f_k) . \]

Interpolating Equation (12) at \( x = x_k+2 \) and collocating Equation (13) at \( x = x_k, x_{k+1}, x_{k+2}, x_{k+3} \) gives to the equations:
\[ y(x_{k+2}) = a_0 + a_1(x_{k+2} - x_k) + a_2\left[(x_{k+2} - x_k)^2 - 1\right] \]
\[ + a_3\left[(x_{k+2} - x_k)^3 - 3(x_{k+2} - x_k)\right] \]
\[ + a_4\left[(x_{k+2} - x_k)^4 - 6(x_{k+2} - x_k)^2 + 3\right] , \tag{12} \]
\[ y'(x_k) = a_1 - 3a_2 = f_k , \]
\[ y'(x_{k+1}) = a_1 + 2a_2(x_{k+1} - x_k) + 3a_3\left[(x_{k+1} - x_k)^2 - 1\right] \]
\[ + 4a_4\left[(x_{k+1} - x_k)^3 - 12(x_{k+1} - x_k)\right] = f_{k+1} , \tag{14} \]
\[ y'(x_{k+2}) = a_1 + 2a_2(x_{k+2} - x_k) \]
\[ + 3a_3\left[(x_{k+2} - x_k)^2 - 1\right] + a_4\left[(x_{k+2} - x_k)^3 - 12(x_{k+2} - x_k)\right] = f_{k+2} , \]
\[ y'(x_{k+3}) = a_1 + 2a_2(x_{k+3} - x_k) + 3a_3\left[(x_{k+3} - x_k)^2 - 1\right] \]
\[ + a_4\left[(x_{k+3} - x_k)^3 - 12(x_{k+3} - x_k)\right] = f_{k+3} . \]

The Optimal Order Method
The optimal order scheme is an implicit multistep method similar to the Adams-Moulton method. To derive the three-step optimal order method, we shall consider the system of equations in Equation (14) except for \( y(x_{k+2}). \) Interpolating Equation (12) at \( x_{k+1} \) we have
\[ y(x_{k+1}) = a_0 + a_1(x_{k+1} - x_k) + a_2\left[(x_{k+1} - x_k)^2 - 1\right] \]
\[ + a_3\left[(x_{k+1} - x_k)^3 - 3(x_{k+1} - x_k)\right] \]
\[ + a_4\left[(x_{k+1} - x_k)^4 - 6(x_{k+1} - x_k)^2 + 3\right] = y_{k+1} . \]
The corresponding matrix of the equations is:

\[
\begin{pmatrix}
  1 & (h^2 - 1) & (h^3 - 3h) & (h^4 - 6h^2 + 3h) \\
  0 & h & -3h & 0 \\
  0 & 2h^2 & (3h^3 - 3h) & (4h^4 - 12h^2) \\
  0 & 4h^2 & (12h^3 - 3h) & (32h^4 - 24h^2) \\
  0 & 6h^2 & (27h^3 - 3h) & (108h^4 - 36h^2)
\end{pmatrix}
\begin{pmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3 \\
  a_4
\end{pmatrix}
= \begin{pmatrix}
  y_{k+1} \\
  a_1 \\
  a_2 \\
  a_3 \\
  a_4
\end{pmatrix}
\frac{h}{3} (f_{k+3} + 4f_{k+2} + f_{k+1}).
\]

Solving the system of equations give the same result as in Equation (15) above except for \(a_0\). Thus, we have \(a_0\) to be:

\[
a_0 = y_{k+1} + \frac{1}{12h} \left(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k\right) + \frac{1}{8h^3} \left(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k\right) - \frac{h}{24} \left(f_{k+3} - 5f_{k+2} + 9f_{k+1} + 9f_k\right).
\]

Substituting for \(a_j\), \(j = 0, 1, 2, 3, 4\) in Equation (12) yields the continuous three-step optimal order method:

\[
y(x) = y_{k+1} - \frac{h}{24} \left(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k\right) + f_k(x - x_k) + \frac{1}{12h} \left(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k\right) + \frac{1}{8h^3} \left(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k\right) - \frac{h}{24} \left(f_{k+3} - 5f_{k+2} + 9f_{k+1} + 9f_k\right).
\]

Evaluating Equation (18) at \(x = x_{k+3}\), we obtain the discrete form:

\[
y_{k+3} = y_{k+1} + \frac{h}{3} (f_{k+3} + 4f_{k+2} + f_{k+1}).
\]

3. Numerical Examples

In this section, we will apply the derived three-step methods of Adams- Bashforth, Adams-Moulton and the proposed optimal order to solve two IVPs of ODEs. Errors associated with the methods will also be obtained. The four-stage Runge-Kutta method is used to obtain the starting values, and the four-stage Adams-Bashforth method is used as a predictor to the implicit schemes. The result and errors obtained are tabulated for clarity.

Example 1

Consider the IVP

\[
y' = -y, 0 \leq x \leq 1, y(0) = 1.
\]

**Exact Solution:** \(y(x) = e^{-x}\).

**Example 2**

Consider the IVP

\[
y' = xy, 0 \leq x \leq 1, y(0) = 1.
\]

**Exact Solution:** \(y(x) = e^{\left(\frac{x^2}{2}\right)}\).

The error is defined as:

\[
Error = \left| y(x) - y_n(x) \right|,
\]

where \(y(x)\) is the exact solution and \(y_n(x)\) is the approximate solution.

| Table 1. Result of Example 1, with stepsize \(h = 0.1\) |
|---|---|---|---|---|
| \(x\)-value | Exact Solution \(y(x)\) | Adams-Bashforth Approximation \(y_n(x)\) | Adams-Moulton Approximation \(y_n(x)\) | Optimal Order Approximation \(y_n(x)\) |
| 0.1 | 0.9048374180 | 0.9048375000 | 0.9048375000 | 0.9048375000 |
| 0.2 | 0.8187307531 | 0.8187309014 | 0.8187309014 | 0.8187309014 |
| 0.3 | 0.7408182207 | 0.7408182288 | 0.7408182288 | 0.7408182288 |
| 0.4 | 0.6703204670 | 0.6703196483 | 0.6703196483 | 0.6703196483 |
| 0.5 | 0.606350697 | 0.606457288 | 0.6065299217 | 0.6065303868 |
| 0.6 | 0.5488116361 | 0.5487208666 | 0.5488105242 | 0.5488133400 |
| 0.7 | 0.4965853038 | 0.4964816829 | 0.4965838086 | 0.4965847443 |
| 0.8 | 0.4493289641 | 0.4492164248 | 0.4493270829 | 0.4493283868 |
| 0.9 | 0.4065696957 | 0.4065405809 | 0.4065674014 | 0.4065688105 |
| 1.0 | 0.3678794412 | 0.3677565415 | 0.3678768199 | 0.3678785973 |

| Table 2. Comparison of Absolute Error for Example 1 |
|---|---|---|---|
| \(x\)-value | Error in Adams-Bashforth Scheme | Error in Adams-Moulton Scheme | Error in Optimal Order Scheme |
| 0.1 | 8.196404 E-008 | 8.196404 E-008 | 8.196404 E-008 |
| 0.2 | 1.483283 E-007 | 1.483283 E-007 | 1.483283 E-007 |
| 0.3 | 3.240871 E-005 | 9.187851 E-007 | 3.826926 E-008 |
| 0.4 | 5.562367 E-005 | 3.977782 E-007 | 3.563031 E-008 |
| 0.5 | 7.939093 E-005 | 7.379661 E-007 | 2.728729 E-007 |
| 0.6 | 9.154951 E-005 | 1.111848 E-06 | 2.961309 E-007 |
| 0.7 | 1.036209 E-004 | 1.495200 E-06 | 5.594959 E-007 |
| 0.8 | 1.125393 E-004 | 1.881206 E-06 | 5.773206 E-007 |
| 0.9 | 1.188289 E-004 | 2.258378 E-06 | 8.492640 E-007 |
| 1.0 | 1.228997 E-004 | 2.621222 E-06 | 8.439141 E-007 |
Optimal Order Approximation $y_n(x)$

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<tr>
<th>$x$-value</th>
<th>Exact Solution $y(x)$</th>
<th>Adams-Bashforth Approximation $y_n(x)$</th>
<th>Adams-Moulton Approximation $y_n(x)$</th>
<th>Optimal Order Approximation $y_n(x)$</th>
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### Table 4. Comparison of Absolute Error for Example 2

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<th>Error in Adams-Moulton Scheme</th>
<th>Error in Optimal Order Scheme</th>
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</table>

### 4. Discussion

Three continuous and discrete LMMs are derived through the technique of collocation and interpolation using the probabilists’ Hermite polynomials as basis functions. The result has shown that continuous and discrete LMMs can be derived using any polynomial function and approach. Table 1-Table 4 presents the results by the three derived methods: Equation (11), (17) and (19). Specifically, Table 2 and Table 4 give the errors by these methods. From the results obtained, the Adams-Moulton method produced better results than the Adams-Bashforth method but the proposed optimal order method is the most accurate.

### 5. Conclusion

The approach and basis functions applied in deriving the LMMs in this paper are different from those of some other researchers, though, the obtained LMMs (continuous and discrete) are same. Also, the proposed optimal order scheme has shown superiority over the standard existing methods of Adams-Bashforth and Adams-Moulton of the same step number, in terms of accuracy.

### References


