Approximate Controllability of Fractional Stochastic Perturbed Control Systems Driven by Mixed Fractional Brownian Motion

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Abstract In this paper, the approximate controllability of nonlinear Fractional order $0 < \alpha < 1$ Riemann-Liouville type stochastic perturbed control systems driven by mixed fractional Brownian motion in a real separable Hilbert spaces has been studied by using Krasnoselskii’s fixed point theorem, stochastic analysis theory, fractional calculus and some sufficient conditions.

Keywords: approximate controllability, mixed fractional brownian motion, fixed point theorem, perturbed control systems, mild solution, control function


1. Introduction

The aim main of this paper is to study the principle concepts of the approximate controllability for complicated classes of fractional order $0 < \alpha < 1$ Riemann-Liouville type stochastic perturbed control systems driven by mixed fractional Brownian motion. The following form is the system under our consideration,

\[ L^\alpha \frac{d^\alpha}{dt^\alpha} x(t) = -(A + \Delta A)x(t) + Bu(t) + F(t, x(t), \int_0^t h(t,s,x(s))ds) + G(t, x(t), \int_0^t g(s,x(s))dW(s)) + \sigma(t) \frac{dW^H(t)}{dt}, \]

where $L^\alpha$ is a continuous at $t \in [0,T]$ with the norm $\|x\|_{C^1_\alpha} = \left( \sup_{t \in [0,T]} (t^{1-\alpha} E|\|x(t)\|_B)^2 \right)^{1/2}$. $x_0$ is $F_0$-measurable $X$-valued random variable independent of $W$ and $W^H$ which defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a Hilbert space $K$. Let $Q$ be a positive, self-adjoint and trace class operator on $K$ and let $L_0(K; X)$ be the space of all $Q$-Hilbert-Schmidt operators acting between $K$ and $X$ equipped with the Hilbert-Schmidt norm $\| \cdot \|_{L_2}$. $W^H = \{ W^H(t), t \in [0,T] \}$ is a Q-fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The functions $F: [0,T] \times X_\beta \times X_\beta \times X_\beta \rightarrow X$, $h: [0,T] \times X_\beta \times X_\beta \rightarrow X_\beta$, $G: [0,T] \times X_\beta \times X_\beta \rightarrow L_2(K;X)$ and $\sigma: [0,T] \rightarrow L_2^0(Y;X)$ are continuous functions.

Approximate controllability of stochastic control system driven by fractional Brownian motion has been interested by many authors; Sakthivel [19] study for the approximate controllability of impulsive stochastic systems with fractional Brownian motion. Guendouzi and Idrissi, [7] established and discussed the approximate controllability result of a class of dynamic control systems described by nonlinear fractional stochastic functional
differential equations in Hilbert space driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Ahmed [2] investigate the approximate controllability problem for the class of impulsive neutral stochastic functional differential equations with finite delay and fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ in a Hilbert space. Abid, Hasan and Quez [11] studied the approximate controllability of fractional stochastic integro-differential equations which is derived by mixed type of fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ and wiener process in real separable Hilbert space.

In this paper we will study the approximate controllability of nonlinear stochastic system. More precisely, we shall formulate and prove sufficient conditions for the Approximate controllability of Fractional order differential equations in Hilbert space driven by mixed fractional Brownian motion in a real separable Hilbert spaces.

The rest of this paper is organized as follows, in section 2, we will introduced some concepts, definitions and some lemmas of semigroup theory and fractional stochastic calculus which are useful for us here. In section 3, we will prove our main result.

### 2. Preliminaries

In this section, we introduce some notations and preliminary results, which we needed to establish our results.

**Definition (2.1), [5]:**

Let $H$ be a constant belonging to $(0, 1)$. A one dimensional fractional Brownian motion $B^H = \{B^H_t, t \geq 0\}$ of Hurst index $H$ is a continuous and centered Gaussian process with covariance function

$$E(B^H_t B^H_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) , \text{ for } t, s \geq 0. \quad (1)$$

- If $H = \frac{1}{2}$, then the increments of $B^H$ are non-correlated, and consequently independent. So $B^H$ is a Wiener Process which we denote further by $B$.
- If $H \in (\frac{1}{2}, 1)$ then the increments are positively correlated.
- If $H \in (0, \frac{1}{2})$ then the increments are negative correlated. $B^H$ has the integral representation

$$ B^H_t = \int_0^t K_H(t,s) dB(s) \quad (2) $$

where, $B$ is a wiener process and the kernel $K_H(t,s)$ defined as

$$ K_H(t,s) = \frac{1}{\Gamma(1-2H)} \int_0^{\min(t,s)} \left( \frac{u}{\beta(2-2H,\beta)} \right)^{\frac{1}{2}(1-2H)} \left( t - s \right)^{-\frac{1}{2}} du \quad (3)$$

$$ \frac{\partial K}{\partial t} (t,s) = \frac{1}{\Gamma(1-2H)} \int_0^{\min(t,s)} \left( \frac{u}{\beta(2-2H,\beta)} \right)^{\frac{1}{2}(1-2H)} \left( t - s \right)^{-\frac{1}{2}} du \quad (4)$$

$cH = \left[ \frac{H(2H-1)}{\beta(2-2H,\beta)} \right]^\frac{1}{2}$, $t > s$ and $\beta$ is a beta function.

In the case $H = \frac{1}{2}$, we shall use Ito Isometry theorem

**Lemma (2.1), “Ito isometry theorem”, [11]:**

Let $V[0,T]$ be the class of functions such that $f:[0,T] \times \Omega \to \mathbb{R}$, $f$ is measurable, $\mathcal{F}_t$-adapted and $E \left[ \int_0^T (f(t,\omega))^2 dt \right] \leq \infty$. Then for every $f \in V[0,T]$, we have

$$ E \left[ \int_0^T f(t,\omega) dB(t) \right]^2 = E \left[ \int_0^T (f(t,\omega))^2 dt \right] \quad (5)$$

where $B$ is a wiener process.

Now, we denote by $\xi$ the set of step functions on $[0, T]$. If $\Phi \in \xi$, then we can write it the form as:

$$ \Phi(t) = \sum_{k=1}^n a_k 1_{[t_k,t_{k+1})}(t) , \text{ where } t \in [0,T]. \quad (6)$$

The integral of a step function $\Phi \in \xi$ with respect to one dimensional fractional Brownian motion is defined

$$ \int_0^T \Phi(t) dB^H_t = \sum_{k=1}^n a_k (B^H_{t_{k+1}} - B^H_{t_k}) $$

where $a_k \in \mathbb{R}$, $0 = t_0 < t_2 < \ldots < t_{n+1} = T$.

Let $\mathcal{K}$ be the Hilbert space defined as the closure of $\xi$ with respect to the scalar product $\langle 1_{[0,1]}, 1_{[0,1]} \rangle = R^2(1) + \mathbb{E}(B^H_t B^H_s)$. The mapping $1_{[0,1]} \to \{B^H_t(t), \mathcal{F}(0,T)\}$ can be extended to an isometry between $\mathcal{K}$ and $\text{span}^{L^2(\Omega)} \{B^H_t, t \in [0,T]\}$, i.e. the mapping $\mathcal{K} \to L^2(\Omega, \mathcal{F}), \Phi \to \int_0^T \Phi(t) dB^H_t$ is isometry.

**Remark (2.2), [6]:**

- If $H = \frac{1}{2}$ and $\mathcal{K} = L^2((0,T])$, then by use Ito isometry, we have

$$ E \left( \int_0^T \Phi(t) dB^H_t \right)^2 = \int_0^T (\Phi(t))^2 dt \quad (6)$$

- If $H > \frac{1}{2}$, we have

$$ R_H(s,t) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}) , t, s \geq 0 \quad (7)$$

$$ \frac{\partial R_H}{\partial t} = H(|t|^{2H-1} - |t - s|^{2H-1}) \quad \frac{\partial R_H}{\partial s} = H(2H - 1) |t - s|^{2H-2} ds dt \quad (8)$$

**Lemma (2.2), [6]:**

For any functions $\Phi, \varphi \in L^2([0,T] \times \Omega)$, we have

i) $E \left( \int_0^T \Phi(t) dB^H_t \int_0^T \varphi(s) dB^H_s \right) = H(2H - 1) \times \int_0^T \int_0^T \Phi(t) \varphi(s) |t - s|^{2H-2} ds dt \quad (9)$

ii) $E \left( \int_0^T \Phi(t) dB^H_t \right)^2 = \int_0^T \int_0^T \Phi(t) \Phi(s) |t - s|^{2H-2} ds dt \quad (10)$

From this Lemma above, we obtain

$$ E \left( \int_0^T \Phi(t) dB^H_t \right)^2 = H(2H - 1) \times \int_0^T \int_0^T \Phi(t) \Phi(s) |t - s|^{2H-2} ds dt \quad (9)$$

**Remark (2.2), [6]:**

The space $\mathcal{F}$ contains the set of functions $\Phi \in L^2([0,T])$, such that $\int_0^T \int_0^T \Phi(t) \Phi(s) |t - s|^{2H-2} ds dt < \infty$, which includes $L^2((0,T])$.

Now, let $\mathcal{H}$ be the Banach space of measurable functions on $[0, T], \Phi \in L^1([0,T])$, such that

$$ \|\Phi\|_{\mathcal{H}}^2 = H(2H - 1) \int_0^T \int_0^T \Phi(t) \Phi(s) |t - s|^{2H-2} ds dt < \infty \quad (10)$$

**Lemma (2.3), [10]:**
There exists $M > 1$ such that the semigroup of uniformly bounded operators in $X$, that is, $\left( \Phi \right)_{t \geq 0} = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \Phi_n$, where $\lambda_n \geq 0$ for $n = 1, 2, \ldots$, are non-negative real numbers with finite trace $\text{Tr} \ Q = \sum_{n=1}^{\infty} \lambda_n < \infty$. The infinite dimensional fractional Brownian motion on $Y$ can be defined by using covariance operator $Q$ as

$$W_{H}^{(t)} = W_{H}^{(0)} + \sum_{n=1}^{\infty} \sqrt{\lambda_n} \Phi_n H_{n}(t),$$

where $B_{H}^{(n)}(t)$ are one dimensional fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$.

In order to defined stochastic integral with respect to the Q-fractional Brownian motion. We introduce the space $L_{2}^{0}(Y, X)$ of all Q-Hilbert- Schmidt operators that is with the inner product $(\Phi, \varphi)_{L_{2}^{0}} = \sum_{n=1}^{\infty} (\Phi e_{n}, \varphi e_{n})$ is a separable Hilbert space.

**Lemma (2.4), [10]:**

Let $(\Phi(\cdot))_{t \in [0,T]}$ be a deterministic function with values in $L_{2}^{0}(Y, X)$ whose integral $H_{t}^{(s)}$ is defined by

$$L_{2}^{0}(Y, X) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \Phi_n H_{n}(s),$$

then the above sum in (11) is well defined as an $X$-valued random variable and we have

$$E \left\| \int_{0}^{t} \varphi(s) \ dW_{H}^{(s)} \right\|^{2} \leq 2H t^{2u-1} \int_{0}^{t} \left\| \varphi(s) \right\|_{L_{2}^{0}}^{2} \ ds \tag{12}$$

**Definition (2.2), [18]:**

The Riemann-Liouville derivative of order $\alpha > 0$ with lower limit zero for a function $f$ can be written as:

$$L^{D}_{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{0}^{t} \frac{f(s)}{(t-s)^{n-\alpha}} \ ds \tag{13}$$

where, $t > 0, n - 1 < \alpha < n$.

**Definition (2.4), [18]:**

The Laplace transform of the Riemann-Liouville fractional derivative of order $\alpha > 0$ gives as:

$$L(L^{D}_{\alpha} f(t)) = \lambda^{\alpha} L(f(t)(\lambda)) - \sum_{n=0}^{\alpha-1} \lambda^{k} \left[ L^{D}_{\alpha-k+1} f(t) \right]_{t=0}(\lambda) \tag{14}$$

where, $n - 1 < \alpha < n$.

**Lemma (2.6), [17]:**

Let $-\Delta$ be the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ on a Hilbert space $X$. If $-\Delta A$ is a bounded linear operator on $X$ then $-(A + \Delta A)$ is the infinitesimal generator of an analytic semigroup $\bar{T}(t)$, $t \geq 0$ on $X$.

**Remark (2.3):**

Assume that $\bar{T}(t)$, $t \geq 0$ is a compact analytic semigroup of uniformly bounded operators in $X$, that is, there exists $M > 1$ such that $\|\bar{T}(t)x\| \leq M$.

**Definition (2.5):**

An $X_{\beta}$-valued process $x(t)$ is called a mild solution of the system (1) if $x(t) \in C_{t=0}^{1} [0, T]; L^{2}(\Omega, X_{\beta})$ and, for $t \in [0, T]$ satisfies the integral equation

$$x(t) = \bar{T}_{\alpha}(t)(t)^{H-1} x_{0} + \int_{0}^{t} \bar{T}_{\alpha}(t-s)(t)^{H-1} B\{u(s)ds \}
+ \int_{0}^{t} \bar{T}_{\alpha}(t-s)(t)^{H-1} \left\{ \int_{0}^{s} \bar{T}_{\alpha}(s-r)\left( \int_{0}^{r} g(s, r, x(r))dr \right)ds \right\} \, dW_{t}$$

where $\bar{T}_{\alpha}(t) = \int_{0}^{t} \lambda_{\alpha}^{t-s} W_{H}(s) \ ds \tag{15}$

**3. Main Result of the Approximately Controllable**

In this section, we formulate and prove the result on approximate controllability of nonlinear fractional stochastic perturbed control system driven by mixed fractional Brownian motion in (1). To establish our results, we introduce the following assumptions:

a) The operator $\bar{T}(t)$ is a compact for any $t > 0$.

b) The linear fractional order system of corresponding the system (3.41) which has following form:

$$L^{D}_{\alpha} x(t) = A x(t) + B\{u(t), t \in [0, T] \}
+ L^{D}_{\alpha-1} x(t)_{t=0} = x_{0}, \frac{1}{2} < \alpha < 1 \tag{16}$$

is an approximately controllable on $[0, T]$.  

c) The functions $F:[0, T] \times X_{\beta} \times X_{\beta} \rightarrow X, h:[0, T] \times X_{\beta} \rightarrow X_{\beta}$ are satisfying linear growth and Lipschitz conditions. This mean that, for any $x, y \in X_{\beta}$ there exists positive constants $K_{1}, K_{2} > 0$ and $K_{3}, K_{4} > 0$ such that

$$\left\| F(t, x(t), \int_{0}^{t} h(t, s, x(s))ds \right\| \leq K_{2}(1 + \|x\|_{\beta}^{2})$$

$$\left\| F(t, x(t), \int_{0}^{t} h(t, s, x(s))ds \right\|^{2} \leq K_{3} \|x - y\|_{\beta}^{2} \tag{17}$$

$$\left\| h(t, s, x) - h(t, s, y) \right\|_{\beta}^{2} \leq K_{4}(1 + \|x\|_{\beta}^{2})$$

where $K_{2}, K_{3}, K_{4} > 0$.  

The operator $\bar{T}_{\alpha}(t)$ is a compact operator in $X$ for $t > 0$.  


Also, F is a uniformly bounded. In other word, there exists $D_1 > 0$ such that
\[
\left\| F \left( t, x(t), \int_0^t h(t, s, x(s))ds \right) \right\|_X^2 < D_1, \text{ for } t \in [0, T]
\]
d) The function $\sigma : [0, T) \rightarrow L^2(Y; X)$ satisfies, for every $t \in [0, T]$
\[
\int_0^t \| \sigma(s) \|^2_{L^2} ds < \infty \text{ and there exists } C_1 > 0 \text{ such that sup}_t \| \sigma(s) \|^2_{L^2} \leq C_1.
\]
e) $G$ is $F_1$ adapted with respect to $t$, such that, for every $t \in [0, T]$, satisfy the following:
\[
i. \ E \left\| (Gx)(t) \right\|_{L^2}^2 \text{ Exists.}
\]
\[
ii. \int_0^t E \left\| (Gx)(t) \right\|_{L^2}^2 ds < \infty.
\]
\[
iii. \text{There exists } D_2 > 0, \text{ such that sup}_t \left\| (Gx)(t) \right\|_{L^2}^2 < D_2.
\]
where, $(Gx)(t) = G \left( t, x(t), \int_0^t g(s, x(s)) \right)$

**Definition (3.1):**

The system (1) is said to be approximately controllable on $[0, T]$ if and only if the reachable set $R(\theta, T)$ exists.

**Lemma (3.2):**

Let $x(t) = (x(t), u(t), s(t))$ be a solution of the system (1).

**Lemma (3.3):**

There exists positive real constant $N_C$, such that for all $x \in C \left( [0, T]; L^2(\Omega, X) \right)$,
\[
E \left\| u^0(t, x) \right\|^2 \leq N_C
\]
Now, obtain Lemmas (2.5), (2.7) and the assumptions (a)-(e), we obtain
\[
E \|u^0(t,x)\|^2 \leq N_c
\]
where,
\[
N_c = \frac{12L_2^4M^4}{\theta^2(\Gamma(\alpha))^4}E\|x_0\|_B^2 + \frac{(T^{2a-2}E\|x_0\|_B^2)}{\theta^2(\Gamma(\alpha))^2}C_1
\]
\[
+ \frac{6L_4^2M^2T^{2a-2q_{0j}}}{\theta^2(\Gamma(\alpha))^2(1-2a\beta)(1-2a\beta)}\tilde{N}_{a,\beta}D_1
\]
\[
+ \frac{6L_4^2M^2T^{2a-2q_{0j}}}{\theta^2(\Gamma(\alpha))^2(1-2a\beta)(1-2a\beta)}\tilde{N}_{a,\beta}D_2
\]
\[
+ \frac{6L_4^2M^2T^{2a-2q_{0j}}}{\theta^2(\Gamma(\alpha))^2(1-2a\beta)(1-2a\beta)}\tilde{N}_{a,\beta}C_1
\]
\[
= \frac{12L_2^4M^4}{\theta^2(\Gamma(\alpha))^4}E\|x_0\|_B^2 + \frac{(T^{2a-2}E\|x_0\|_B^2)}{\theta^2(\Gamma(\alpha))^2}C_1
\]

Now, for any \(T > 0\), consider the operator \(\Psi_0\) on \(C_{1-a}(L^2(\Omega,\mathbb{R}))\) defined as follows:
\[
(\Psi_0x)(t) = t^{-a} x_0 + \int_0^t \int_0^t \int_0^t (t-s)^{-a} f(s,x(s),x(s),x(r))ds dr ds
\]
\[
+ \int_0^t (t-s)^{-a} \sigma(s) dW(s)
\]

Also, for any \(\delta > 0\), the subset \(B_\delta\) of \(C_{1-a}(L^2(\Omega,\mathbb{R}))\) is defined as \(B_\delta = \{x(t) \in C_{1-a}(L^2(\Omega,\mathbb{R})) : \|x(t)\|_{C_{1-a}} \leq \delta\}\).

**Lemma (3.4):**
For any \(\theta > 0\), there exists \(\delta > 0\) such that \(\Psi_0(B_\delta) \subset B_{\delta}\).

**Proof:**
To prove this, we take for each \(T > 0\), there exists \(x(t) \in B_{\delta}\) such that \(\|\Psi_0x(t)\|_{C_{1-a}} > \delta\), for \(t \in [0,T]\), \(t\) may depend on \(\delta\). However, on the other hand, we have
\[
\|\Psi_0x(t)\|_{C_{1-a}}^2 = 5T^{2a-2}\|\int_0^t \int_0^t \int_0^t (t-s)^{-a} f(s,x(s),x(s),x(r))ds dr ds + \int_0^t \int_0^t (t-s)^{-a} \sigma(s) dW(s)\|_{C_{1-a}}^2
\]

By dividing both sides of above inequality by \(\delta\) and taking the limit as \(\delta \to 0\), which is a contradiction. Thus, for each \(T > 0\), there exists positive number \(\delta\) such that \(\Psi_0(B_\delta) \subset B_{\delta}\).

Now, Let \(\Psi = \Psi_1 + \Psi_2\) where,
Proof:

Let \( t \in [0, T] \), and \( y_1, y_2 \in B_{\bar{g}} \), we have

\[
\begin{align*}
& t^{2-2\alpha} E \left\| \Psi_1 y_1(t) - \Psi_1 y_2(t) \right\| \leq \frac{N_{a,\beta}^{2} T^{-2\alpha \beta} K_1}{(2\alpha - 1)(1 - 2\alpha \beta)} \sup_{t \in [0, T]} \left\| y_1(t) - y_2(t) \right\| \leq \gamma \left\| y_1(t) - y_2(t) \right\| \leq \frac{\gamma T^{2-2\alpha \beta} K_1}{(2\alpha - 1)(1 - 2\alpha \beta)} \sup_{t \in [0, T]} \left\| y_1(t) - y_2(t) \right\|,
\end{align*}
\]

where \( \gamma = \frac{N_{a,\beta}^{2} T^{-2\alpha \beta} K_1}{(2\alpha - 1)(1 - 2\alpha \beta)} < 1 \), hence \( \Psi_1 \) is a contraction.

**Lemma (3.7):**

Assume that the assumptions (a) - (e) hold, then the operator \( \Psi_2 \) maps bounded sets into bounded sets in \( B_{\bar{g}} \).

**Proof:**

Let \( x, y \in B_{\bar{g}} \) and for any \( x, y \in B_{\bar{g}}, (\Psi_1 y)(t) + (\Psi_2 x)(t) \in B_{\bar{g}}, \) for \( t \in [0, T] \).

**Proof:**

Let \( x, y \in B_{\bar{g}} \) and \( y_1, y_2 \in B_{\bar{g}} \), we have

\[
\begin{align*}
& t^{2-2\alpha} E \left\| \Psi_1 y_1(t) - \Psi_1 y_2(t) \right\| \leq \frac{N_{a,\beta}^{2} T^{-2\alpha \beta} K_1}{(2\alpha - 1)(1 - 2\alpha \beta)} \sup_{t \in [0, T]} \left\| y_1(t) - y_2(t) \right\| \leq \gamma \left\| y_1(t) - y_2(t) \right\| \leq \frac{\gamma T^{2-2\alpha \beta} K_1}{(2\alpha - 1)(1 - 2\alpha \beta)} \sup_{t \in [0, T]} \left\| y_1(t) - y_2(t) \right\|,
\end{align*}
\]

where \( \gamma = \frac{N_{a,\beta}^{2} T^{-2\alpha \beta} K_1}{(2\alpha - 1)(1 - 2\alpha \beta)} < 1 \), hence \( \Psi_1 \) is a contraction.

**Lemma (3.6):**

Assume that the assumptions (a) - (e) hold, then for any \( \theta > 0 \), and for any \( x, y \in B_{\bar{g}}, (\Psi_1 y)(t) + (\Psi_2 x)(t) \in B_{\bar{g}}, \) for \( t \in [0, T] \),

\[
\begin{align*}
& t^{2-2\alpha} E \left\| \Psi_1 y_1(t) - \Psi_1 y_2(t) \right\| \leq \frac{N_{a,\beta}^{2} T^{-2\alpha \beta} K_1}{(2\alpha - 1)(1 - 2\alpha \beta)} \sup_{t \in [0, T]} \left\| y_1(t) - y_2(t) \right\| \leq \gamma \left\| y_1(t) - y_2(t) \right\| \leq \frac{\gamma T^{2-2\alpha \beta} K_1}{(2\alpha - 1)(1 - 2\alpha \beta)} \sup_{t \in [0, T]} \left\| y_1(t) - y_2(t) \right\|,
\end{align*}
\]

where \( \gamma = \frac{N_{a,\beta}^{2} T^{-2\alpha \beta} K_1}{(2\alpha - 1)(1 - 2\alpha \beta)} < 1 \), hence \( \Psi_1 \) is a contraction.
Applying Holder’s inequality and by using Ito isometry, Lemmas (2.5), (2.7), (3.3) and the assumptions (a)-(e), we obtain
\[ t^{2-2\alpha}E \left\| \Psi_x(t) \right\|_D^2 \leq D \]
where, \( D = \frac{M^2}{(\alpha)^2} E \left\| x_0 \right\|_D^2 + \frac{4}{2(2\alpha-1)(1-2\alpha)^2} \int_0^T (2\alpha-1)^{2\alpha} \|\mathbf{u}_0 \|^2 \| \mathbf{u}_0 \|^2 d\tau + 2T^{2-2\alpha} N_\alpha^2 \beta \left\| \mathbf{u}_0 \right\|_D^2 \left\| \mathbf{u}_0 \right\|_D^2 C_1 \]

By taking the supremum over \( t \in [0,T] \) for both sides, we get
\[ \sup_{t \in [0,T]} t^{2-2\alpha}E \left\| \Psi_x(t) \right\|_D^2 \leq D. \]

Therefore, for each \( x \in B_\delta \), we get \( \left\| \Psi_x(t) \right\|_{C_{1-a}} \leq D \). Then \( \Psi_x \) maps bounded sets into bounded sets in \( B_\delta \).

**Lemma (3.8):**
Assume that the assumptions (a) – (e) hold, then \( \Psi_x \) is a continuous on \( B_\delta \).

**Proof:**
Let \( \{x_n\}_{n=1}^\infty \) be a sequence in \( B_\delta \) such that \( x_n \to x \) as \( n \to \infty \) in \( C_{1-a} \left( [0,T] \cup L^2(\Omega, X_\beta) \right) \). For each \( t \in [0,T] \), we have
\[ t^{2-2\alpha}E \left\| \Psi_x(t) \right\|_D^2 \leq 2t^{2-2\alpha}E \left\| \left( \frac{\mathbf{T}_a(t) - \mathbf{T}_a(t-t)}{1-2\alpha} \right)^a \right\|_D^2 \]

From Ito isometry and lemma (2.7), we obtain
\[ t^{2-2\alpha}E \left\| \Psi_x(t) \right\|_D^2 \leq 2L^2 \beta N_\alpha^2 \beta t^{2-2\alpha} \int_0^T \left( t^{2-2\alpha} + 2t^{1-2\alpha} \right) \left\| \left( \mathbf{G}(s)(x_0) - \mathbf{G}(s)(x) \right) \right\|_D^2 \]

Therefore, it follows from the continuity of \( G \) and \( u^0 \) that for each \( t \in [0,T] \), \( G(s,x)(s), \int_0^s g(v,x(v))dW_\nu \) \( \to (s,x(s),0) \), \( \mathbf{u}(s,x,v) \) and \( u^0(s,x) \to u^0(s,x) \), using the Lebesgue dominated convergence theorem that for all \( t \in [0,T] \), we conclude
\[ \left\| \mathbf{b}(x_0)(t) - \mathbf{b}(x)(t) \right\| \to 0 \]

**Lemma (3.9):**
If the assumptions (a) – (e) are hold, then for \( x \in B_\delta \), the set \( \{\Psi_x(t), t \in [0,T]\} \) is equicontinuous.

**Proof:**
Let \( t_1, t_2 \in [0,T] \) such that \( 0 < t_1 < t_2 \leq T \). Then, from the equation (24), we have
\[ E \left\| \Psi_x(t_2) - \Psi_x(t_1) \right\|_D^2 \leq 7E \left\| \mathbf{T}_a(t_2) - \mathbf{T}_a(t_1) \right\| \left\| x_0 \right\|_D^2 \]

Now, from Lemma (2.7), noting the fact that for every \( \epsilon > 0 \), there exists \( \tau > 0 \) such that, whenever \( t_2 - t_1 < \tau \), for every \( t_1, t_2 \in [0,T] \), \( \left\| \mathbf{T}_a(t_2) - \mathbf{T}_a(t_1) \right\|_D^2 < \epsilon \). Therefore, when \( t_2 - t_1 < \tau \), we have
\[ E \left\| \Psi_x(t_1) - \Psi_x(t_1) \right\|_D^2 \leq 7E \left\| \mathbf{T}_a(t_2) - \mathbf{T}_a(t_1) \right\| \left\| x_0 \right\|_D^2 \]

Finally, we have
\[ E \left\| \Psi_x(t_2) - \Psi_x(t_1) \right\|_D^2 \leq 7E \left\| \mathbf{T}_a(t_2) - \mathbf{T}_a(t_1) \right\| \left\| x_0 \right\|_D^2 \]

Therefore, \( \Psi_x \) is continuous on \( B_\delta \).
The right hand of the inequality above tends to 0 as $t_2 \to t_1$ and $\epsilon \to 0$. Hence for $x \in B_{\delta}$, the set $\{ \Psi_x \in [0, T] \}$ is equicontinuous.

**Lemma (3.10):**
If the assumption (a) is hold. Then for each $t \in [0, T]$, the set $\tilde{U}(t) = \{ \Psi_x, x \in B_{\delta} \}$ is relatively compact in $B_{\delta}$.

**Proof:**
Let $t \in (0,T]$ be a fixed and $0 < \tilde{a} < t$, for every $y > 0$, $x \in B_{\delta}$, we define
\[
\Psi_{2y}^+(x)(t) = \int_0^\infty \hat{a}^y \ar_{\Psi_x}(r) \hat{T}(t-s)^{y-1} \frac{1}{\hat{a}^y} x_0 \, ds + \int_0^{t-\hat{a}^y} \hat{a}^y \ar_{\Psi_x}(r) \hat{T}(t-s)^{y-1} \frac{1}{\hat{a}^y} Bu_0(t,x) \, drds
\]
and (3.6) with applying Krasnoselskii theorem, we conclude that the operator $\Psi_0$ has a fixed point, which gives rise to mild solution of system (1) with stochastic control function given in (20). This completes the proof.

**Theorem (3.2):**
If the assumptions (a) – (e) are satisfied. Then the stochastic control system (1) is approximately controllable on $[0,T]$.

**Proof:**
For every $\theta > 0$, let $x_0$ be a fixed point of the operator $\Psi_0$ in the space $C_{1-\alpha}$, which is a mild solution under the stochastic control function in (20) of the stochastic system (1). Then, we have
\[
x_0(T) = x_T - \epsilon R(\theta, T) \left[ (\hat{a}^y) \hat{T}(T)^{y-1} x_0 \right] - \int_0^T 0 R(\theta, T) \hat{a}(s) dW(s)
\]
\[
+ \int_0^T 0 R(\hat{a}, T) \hat{T}(T-s)^{y-1} \left[ \frac{1}{\hat{a}^y} G(s,x_0(s),0) h(s,r,x_0(r)) dr \right] ds
\]
By using the assumptions (c) the function $F$ is uniformly bounded, such that
\[
E \left[ F(t,x,\int_0^t h(t,s,x) \, ds) \right] \leq D_1 \text{, for any } 0 \leq s \leq T.
\]
Then, for all $(s,\alpha) \in [0, T] \times \Omega$, there is a subsequence of sequence $\{ F(s,x_0(s),\int_0^t h(s,r,x_0(r))dr) \}$ denoted by $\{ F(s,x_0(s),\int_0^t h(s,r,x_0(r))dr) \}$ which is weakly converging to say $F(s)$ in $X$. Also, there is a subsequence of sequence $\{ G(s,x_0(s),\int_0^t g(s,v,x_0(v))dW(v)) \}$ denoted by $\{ G(s,x_0(s),\int_0^t g(s,v,x_0(v))dW(v)) \}$ which is weakly converging to say $G(s)$ in $L^2(K, X)$. On the other hand, from the assumption (b), for all $0 \leq t < T$ the operator $R(\theta, T) \to 0$ Strongly as $\theta \to 0^+$ and $\| R(\theta, T) \| \leq 1$. Then
\[
E \left[ x_0(T) - x_0 \right]^2 \leq 8 E \left[ 0 R(\theta, x_0) \hat{T}(T)^{y-1} x_0 \right]^2 + 8 E \int_0^T 0 R(\theta, x_0) \hat{T}(T)^{y-1} x_0 \hat{T}(T)^{y-1} x_0 \, ds
\]
\[
+ 8 E \int_0^T 0 R(\hat{a}, x_0) \hat{T}(T)^{y-1} x_0 \hat{T}(T)^{y-1} x_0 \, ds
\]
\[
+ 8 E \int_0^T 0 R(\hat{a}, x_0) \hat{T}(T)^{y-1} x_0 \hat{T}(T)^{y-1} x_0 \, ds
\]
By using the Lebesgue dominated convergence theorem, we obtain  \( E\|x_\theta(T) - x_T\|^2 \to 0 \) as \( \theta \to 0^+ \). This gives the approximate controllability.

References


