Total Domination Subdivision Number in Strong Product Graph

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Abstract  A set D of vertices in a graph G(V,E) is called a total dominating set if every vertex v∈V is adjacent to an element of D. The domination subdivision number of a graph G is the minimum number of edges that must be subdivided in order to increase the domination number of a graph. In this paper, we determine the total domination number for strong product graph and establish bounds on the total domination subdivision number for strong product graph.

Keywords: total dominating set, strong product graph, total domination number


1. Introduction

Let G=(V,E) be a simple graph on the vertex set V. In a graph G, a set D⊆V is a dominating set of G if every vertex in V−D is adjacent to some vertex in D. The domination number of a graph G is the minimum size of a dominating set in G, denoted by γ(G). A thorough study of fundamental domination appears in [2]. The concept of total domination in graphs was introduced by Cokayne, Dawes and Hedetemini [1]. A set of vertices in a graph G(V,E) is called a total dominating set if every vertex v∈V is adjacent to an element of S. The total domination number of a graph G is denoted by tSd G. In [2] the authors proved the total domination number for several families of graphs were determined in [3]. Nasrin Soltankhah showed that for any m,n ≥ 3, tSd G ≤ 3 [7]. The behaviour of several graph parameters in product graphs has become an interesting topic of research [6]. G. Yero and J. A. Rodríguez-Vel‘ázquez [11] proved that for any m,n ≥ 2, γ(Pm Pn) = \{m \over 3 \} \{n \over 3 \}. In this paper is to establish a bound of this type on tSd Pm Pn.

2. Main Result

In this section, we first determine the value of the total domination number of Pm Pn for m ≤ 4. Since P1 Pn = Pn, we have:

Proposition 2.1. For any n ≥ 2, we have

\[ tSd Pm Pn = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n}{2} + 1 & \text{if } n \equiv 1, 2, 3 \pmod{4} \end{cases} \]
Lemma 2.2. We have $\gamma_t(P_2 \boxtimes P_n) = \begin{cases} 2 & \text{for } n = 2, 3, 4 \\ 3 & \text{for } n = 5. \end{cases}$

Proof: To obtain totally dominate the vertices $(u_2, v_1)$ and $(u_2, v_2)$, we need two vertices $(u_1, v_1)$ and $(u_1, v_2)$. Therefore, $\gamma_t(P_2 \boxtimes P_2) = 2$. Last column of $P_2 \boxtimes P_3$ is totally dominated by $P_2 \boxtimes P_2$. Hence, $\gamma_t(P_2 \boxtimes P_3) = 2$.

Let us consider $P_2 \boxtimes P_4$ as block $B$. The last three columns of $P_2 \boxtimes P_3$ is block $B$. The first column of $P_2 \boxtimes P_4$ can be totally dominated by $B$. Hence, $\gamma_t(P_2 \boxtimes P_4) = 2$. In $P_2 \boxtimes P_3$, to totally dominate a vertex $(u_1, v_1)$, we need one vertex among $\{(u_2, v_3), (u_2, v_4), (u_1, v_4)\}$. Hence, $\gamma_t(P_2 \boxtimes P_3) = 3$.

The first three columns of $P_2 \boxtimes P_3$ is block $B$ and also the last column of $P_2 \boxtimes P_3$ is totally dominated by the fourth column. This completes the proof.

Proposition 2.3. For any $n \geq 6$, we have

$$\gamma_t(P_2 \boxtimes P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Proof:

Figure 1. $P_2 \boxtimes P_n$

Let $S$ be a total dominating set of $P_2 \boxtimes P_n$. Since $\gamma_t(P_2 \boxtimes P_4) = 2$. Suppose that $C_1, C_2, C_3$ and $C_4$ are four consecutive columns of $P_2 \boxtimes P_n$. To totally dominate the vertices $(u_1, v_{j+1})$ and $(u_1, v_{j+2})$, we need one vertex among $\{(u_1, v_1), (u_1, v_{j+1}), (u_2, v_{j+1})\}$ and one more vertex among $\{(u_1, v_1), (u_1, v_{j+3}), (u_2, v_{j+2})\}$. Now, to describe the total dominating set $S$, we consider block $B = P_2 \boxtimes P_4$ and $\cap B = \{(u_1, v_1), (u_1, v_3)\}$. If $n \equiv 0 \pmod{4}$, then $P_2 \boxtimes P_n$ can be partitioned with $\frac{n}{4}$ number of blocks $B$. If $n \equiv 1 \pmod{4}$, then $P_2 \boxtimes P_n$ can be partitioned with $\frac{n-5}{4}$ number of blocks $B$, plus a block $B' = P_2 \boxtimes P_3$ and $S \cap B' = \{(u_1, v_1), (u_1, v_3), (u_1, v_4)\}$. If $n \equiv 2 \pmod{4}$, then $P_2 \boxtimes P_n$ can be partitioned with $\frac{n-2}{4}$ number of blocks $B$, plus a block $B' = P_2 \boxtimes P_2$ and $S \cap B' = \{(u_1, v_1), (u_1, v_2)\}$. If $n \equiv 3 \pmod{4}$, then $P_2 \boxtimes P_n$ can be partitioned with $\frac{n-3}{4}$ number of blocks $B$, plus a block $B' = P_2 \boxtimes P_3$ and $S \cap B' = \{(u_1, v_1), (u_1, v_2)\}$.

This completes the proof.

Proposition 2.4. For $n \geq 3$, the total domination number of $P_2 \boxtimes P_2$ and $P_2 \boxtimes P_3$ are same.

Proof: Last two rows of $P_2 \boxtimes P_n$ is considered as blocks $B = P_2 \boxtimes P_2$ and the first row of $P_3 \boxtimes P_n$ is totally dominated by $B$, which completes the proof.

Observation 2.5. For $n \geq 1$, we have $P_2 \boxtimes P_n \subseteq P_n \boxtimes P_2$.

Proposition 2.6. For any $n \geq 4$, we have

Proof:
number of blocks $B$, plus a block $B' = P_4 \boxtimes P_1$ and $S \cap B' = \{(u_2, v_1), (u_3, v_1)\}$. If $n = 2(\mod 3)$, then $P_4 \boxtimes P_n$ can be partitioned with \( \frac{n-2}{3} \) number of blocks $B$, plus a block $B' = P_4 \boxtimes P_3$ and $S \cap B' = \{(u_2, v_1), (u_3, v_1)\}$.

This completes the proof.

**Theorem 2.7.** We have

\[
\gamma_t(P_m \boxtimes P_n) = \begin{cases} \left\lfloor \frac{m}{2} \right\rfloor \frac{n}{3} & \text{if } m = 0(\mod 4) \\ \left\lfloor \frac{m+1}{2} \right\rfloor \frac{n}{3} & \text{if } m = 1, 2, 3(\mod 4). \end{cases}
\]

**Proof:**

Let us consider $P_m \boxtimes P_1$ as block. Now to describe our total dominating set $S$, we consider block $B = P_m \boxtimes P_1$. If $m = 0(\mod 4)$, then $P_m \boxtimes P_1$ can be partitioned with \( \frac{n}{3} \) number of blocks $B$. By Proposition 2.1 and Observation 2.5, we have

\[
\gamma_t(P_m \boxtimes P_1) = \left\lfloor \frac{m}{2} \right\rfloor \frac{n}{3}.
\]

Let $xuv u v$ be a total dominating set of $P_m \boxtimes P_n$. By Proposition 2.1 and Observation 2.5, we have

\[
\gamma_t(P_m \boxtimes P_n) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor.
\]

This completes the proof.

3. **Subdivision Number for the Strong Product Graph**

**Proposition 2.8.** For $P_2 \boxtimes P_2$, we have $SD\gamma_t(P_2 \boxtimes P_2) = 2$.

**Proof:** Let $S$ be a total dominating set of $P_2 \boxtimes P_2$ and $S = \{(u_1, v_1), (u_1, v_2)\}$. Let $(P_2 \boxtimes P_2)'$ be obtained from $P_2 \boxtimes P_2$ by subdividing an edge $(u_1, v_1)\{(u_2, v_1)\}$ and adding a new vertex called $x$. Now, there is no change in total domination number, i.e., $\gamma_t(P_2 \boxtimes P_2) = \gamma_{t'}(P_2 \boxtimes P_2)$.

Let $(P_2 \boxtimes P_2)'$ be obtained from $P_2 \boxtimes P$ by subdividing the edges $(u_1, v_1)\{(u_2, v_1), (u_3, v_1)\}$ and adding new vertices respectively called $x$ and $y$. So, we need three vertices for totally domination. Therefore, $S' = \{(u_1, v_1), (u_1, v_2), (u_3, v_2)\}$.

Thus, $\gamma_t(P_2 \boxtimes P_2)' = 3$. By Lemma 2.2, we obtain that the total domination number of $(P_2 \boxtimes P_2)'$ is greater than the total domination number of $P_2 \boxtimes P_2$. This completes the proof.

**Proposition 2.9.** For $B = P_2 \boxtimes P_2$, we have $S \cap B = \{(u_1, v_1), (u_1, v_2)\}$.

**Proof:** To describe our total dominating set $S$, we consider block $B = P_2 \boxtimes P_2$ and $S \cap B = \{(u_1, v_1), (u_1, v_2)\}$. Since $SD\gamma_t(P_2 \boxtimes P_2) = 2$. Thus, we have $SD\gamma_t(P_2 \boxtimes P_2) = 2$.

**Proposition 2.10.** For $P_2 \boxtimes P_4$, we have $SD\gamma_t(P_2 \boxtimes P_4) = 1$.

**Proof:** Let $S$ be a total dominating set of $P_2 \boxtimes P_4$ and $S = \{(u_1, v_2), (u_1, v_3)\}$. Let $(P_2 \boxtimes P_4)'$ be obtained from $P_2 \boxtimes P_4$ by subdividing an edge $(u_2, v_1)(u_1, v_2)$ and adding a new vertex called $x$. To totally dominate $(u_2, v_1)$, we need one vertex among $\{(u_1, v_1), (u_1, v_2)\}$. Therefore, $S' = \{(u_1, v_1), (u_1, v_2), (u_3, v_3)\}$. Thus, $\gamma_t(P_2 \boxtimes P_4)' = 3$.

By Lemma 2.2, we obtain that the total domination number of $(P_2 \boxtimes P_4)'$ is greater than the total domination number of $P_2 \boxtimes P_4$. This completes the proof.

**Proposition 2.11.** For $P_2 \boxtimes P_3$, we have $SD\gamma_t(P_2 \boxtimes P_3) = 1$.

**Proof:** Let $S$ be a total dominating set of $P_2 \boxtimes P_3$ and $S = \{(u_1, v_2), (u_1, v_3)\}$. Let $(P_2 \boxtimes P_3)'$ be obtained from $P_2 \boxtimes P_3$ by subdividing an edge $(u_2, v_1)(u_1, v_2)$ and adding a new vertex called $x$. To totally dominate $(u_2, v_1)$, we need one vertex among $\{(u_1, v_1), (u_1, v_2)\}$. Therefore, $S' = \{(u_1, v_1), (u_1, v_2), (u_1, v_3)\}$. Thus, $\gamma_t(P_2 \boxtimes P_3)' = 4$. By Lemma 2.2, we obtain that the total
domination number of \((P_2 \boxtimes P_3)^r\) is greater than the total domination number of \(P_2 \boxtimes P_3\). This completes the proof.

**Theorem 2.8.** For \(n \geq 4\), we have \(\text{Sd} \gamma_1(P_2 \boxtimes P_n) = 1\).

**Proof:** To describe our total dominating set \(S\), we consider block \(B = P_2 \boxtimes P_n\) and \(S \cap B = \{(u_1, v_2), (u_1, v_3)\}\). Since \(\text{Sd} \gamma_1(P_2 \boxtimes P_4) = 1\) and by Proposition 2.3, we have \(\text{Sd} \gamma_1(P_2 \boxtimes P_n) = 1\).

**Theorem 2.9.** For \(n \geq 3\), subdivision number of \(P_2 \boxtimes P_n\) and \(P_3 \boxtimes P_n\) are same.

**Proof:** Last two rows of \(P_3 \boxtimes P_n\) is considered as blocks \(B = P_2 \boxtimes P_n\) and the first row of \(P_3 \boxtimes P_n\) is totally dominated by \(B\), which completes the proof.

**Theorem 2.10.** For \(n \geq 4\), we have \(\text{Sd} \gamma_1(P_4 \boxtimes P_n) = 1\).

**Proof:** To describe our total dominating set \(S\), we consider block \(B = P_4 \boxtimes P_3\) and \(S \cap B = \{(u_2, v_2), (u_3, v_2)\}\). By Theorem 2.9, we have \(\text{Sd} \gamma_1(P_4 \boxtimes P_3) = 1\). Thus, \(\text{Sd} \gamma_1(P_4 \boxtimes P_n) = 1\).

**Theorem 2.11.** For \(n \geq 4\), we have \(\text{Sd} \gamma_1(P_m \boxtimes P_n) = 1\).

**Proof:** To describe our total dominating set \(S\), we consider block \(B = P_2 \boxtimes P_n\). By Theorem 2.10, we have \(\text{Sd} \gamma_1(P_4 \boxtimes P_n) = 1\). Thus, \(\text{Sd} \gamma_1(P_m \boxtimes P_n) = 1\).

**References**