Common Fixed Points of a Countable Family of I-Nonexpansive Multivalued Mappings in Banach Spaces

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Received April 24, 2014; Revised August 01, 2014; Accepted August 12, 2014

Abstract In this paper, we introduce a modified Ishikawa iteration for a countable family of multi-valued mappings. We use the best approximation operator to obtain weak and strong convergence theorems in a Banach space. We apply the main results to the problem of finding a common fixed point of a countable family of I-Nonexpansive multi-valued mappings.

Keywords: I-Nonexpansive multi-valued mapping, fixed point, weak convergence, strong convergence, Banach space, Ishikawa iteration


1. Introduction

Let D be a nonempty and convex subset of a Banach spaces E. The set D is called proximinal if for each \( x \in E \), there exists an element \( y \in D \) such that \( d(x,y) = d(x,D) \), where \( d(x,D) = \inf \{ ||x-z|| : z \in D \} \). Let CB(D), CCB(D), K(D) and P (D) denote the families of nonempty closed bounded subsets, nonempty closed convex bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D, respectively. The Hausdorff metric on CB(D) is defined by

\[
H(A,B) = \max \left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\}
\]

for \( A,B \in \text{CB}(D) \). A single-valued map \( T: D \rightarrow D \) is called nonexpansive if \( ||Tx-Ty|| \leq ||x-y|| \) for all \( x,y \in D \). A multi-valued mapping \( T: D \rightarrow \text{CB}(D) \) is called nonexpansive if \( H(Tx,Ty) \leq ||x-y|| \) for all \( x,y \in D \). An element \( p \in D \) is called a fixed point of \( T: D \rightarrow D \) (respectively, \( T: D \rightarrow \text{CB}(D) \)) if \( p \in Tp \) (respectively, \( p \in Tp \)). The set of fixed points of \( T \) is denoted by \( F(T) \). The mapping \( T: D \rightarrow \text{CB}(D) \) is called quasi-nonexpansive [1] if \( F(T) \neq \emptyset \) and \( H(Tx,Tp) \leq ||x-p|| \) for all \( x \in D \) and all \( p \in F(T) \). It is clear that every nonexpansive multi-valued mapping \( T \) with \( F(T) \neq \emptyset \) is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive (see [2]). It is known that if \( T \) is a quasi-nonexpansive multi-valued mapping, then \( F(T) \) is closed.

Throughout this paper, we denote the weak convergence and the strong convergence by \( \rightharpoonup \) and \( \rightarrow \), respectively. The mapping \( T: D \rightarrow \text{CB}(D) \) is called hemicompact if, for any sequence \( \{x_n\} \) in D such that \( d(x_n, Tx_n) \rightarrow 0 \) as \( n \rightarrow \infty \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \rightarrow p \in D \). We note that if \( D \) is compact, then every multi-valued mapping \( T: D \rightarrow \text{CB}(D) \) is hemicompact.

A Banach space \( E \) is said to satisfy Opial's condition [3] if for each \( x \in E \) and a sequence \( \{x_n\} \) in \( E \) such that \( x_n \rightarrow x \), the following condition holds for all \( x \neq y \) :

\[
\liminf_{n \rightarrow \infty} ||x_n - x|| < \liminf_{n \rightarrow \infty} ||x_n - y||
\]

The mapping \( T: D \rightarrow \text{CB}(D) \) is called demi-closed if for every sequence \( \{x_n\} \subset D \) and any \( y \in Tx_n \) such that \( x_n \rightarrow x \) and \( y_n \rightarrow y \), we have \( x \in D \) and \( y \in Tx \).

Remark 1.1 ([4]). If the space \( E \) satisfies Opial's condition, then \( I-T \) is demi-closed at 0, where \( T: D \rightarrow \text{K}(D) \) is a nonexpansive multi-valued mapping.

For a single-valued case, in 1953, Mann [5] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping \( T \) in a real Hilbert space \( H \):

\[
x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \forall n \in \mathbb{N}, \tag{1.1}
\]

where the initial point \( x_1 \) is taken in \( D \) arbitrarily and \( \{\alpha_n\} \) is a sequence in \((0,1)\).

However, we note that Mann's iteration process (1.1) has only weak convergence, in general; for instance, see [6,7,8].

Since 1953, Mann's iteration has extensively been studied by many authors (see, for examples, [9-18]). However, the studying of multivalued nonexpansive mappings is harder than that of single-valued nonexpansive mappings in both Hilbert spaces and Banach spaces.

The result of fixed points for multi-valued contractions and nonexpansive mappings by using the Hausdorff
metric was initiated by Markin [19]. Later, different iterative processes have been used to approximate fixed points of multi-valued nonexpansive mappings (see also [1, 20-26]).

In 2009, Song and Wang [26] proved strong and weak convergence theorems for Mann’s iteration of a multi-valued nonexpansive mapping $T$ in a Banach space. They studied strong convergence of the modified Mann iteration which is independent of the implicit anchor-like continuous path $z_t = tu + (1-t)T z_t$.

Let $D$ be a nonempty and closed subset of a Banach space $E$, $\{T_n\} \subset \mathcal{CB}(D)$ be a family of nonexpansive multi-valued mappings. Then we prove weak and strong convergence of its iterates to a fixed point of $T$.

(A) Choose $x_0 \in D$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n, \forall n \geq 0,$$

where $y_n \in T x_n$ such that $\| y_{n+1} - y_n \| \leq H(T x_{n+1}, T x_n) + \gamma_n$.

(B) For fixed $u \in D$, the sequence of modified Mann iteration is defined by $x_0 \in D$,

$$x_{n+1} = \beta_n u + \alpha_n x_n + (1-\alpha_n - \beta_n) y_n + u, \forall n \geq 0,$$

where $y_n \in T x_n$ such that $\| y_{n+1} - y_n \| \leq H(T x_{n+1}, T x_n) + \gamma_n$.

Very recently, Shahzad and Zegeye [2] obtained the strong convergence theorems for a quasi-nonexpansive multi-valued mapping. They relaxed the compactness of the domain of $T$ and constructed an iterative scheme which removes the restriction of $T$ namely $T_p = \{p\}$ for any $p \in \text{Fix}(T)$. The results provided an affirmative answer to some questions raised in [21]. In fact, they introduced iterations as follows:

Let $D$ be a nonempty and convex subset of a Banach space $E$, let $T : D \rightarrow \text{CB}(D)$ and let $(\alpha_n)$, $(\alpha'_n) \subset (0,1]$.

(C) The sequence of Ishikawa’s iteration is defined by $x_0 \in D$,

$$y_n = \alpha'_n z_n + (1-\alpha'_n) x_n,$$

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n) x_n, \forall n \geq 0,$$

where $z_n \in T x_n$ and $z_n \in T y_n$.

(D) Let $T : D \rightarrow \text{P}(D)$ and $P_{T x} = \{y \in T x : \|x-y\| = d(x,T x)\}$, where $PT$ is the best approximation operator. The sequence of Ishikawa’s iteration [30] is defined by $x_0 \in D$,

$$y_n = \alpha'_n z_n + (1-\alpha'_n) x_n,$$

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n) x_n, \forall n \geq 0,$$

where $z_n \in \text{Pr} x_n$ and $z_n \in \text{Pr} y_n$.

It is remarked that Hussain and Khan [27], in 2003, employed the best approximation operator $PT$ to study fixed points of $\ast$-nonexpansive multi-valued mapping $T$ and strong convergence of its iterates to a fixed point of $T$ defined on a closed and convex subset of a real Hilbert space.

Let $D$ be a nonempty, closed and convex subset of a Banach space $E$. Let $(T_n)_{n=1}^\infty$ be a family of multi-valued mappings from $D$ into $2^D$ and let $P_{T_n} x = \{y_n \in T_n x : \|x-y_n\| = d(x,T_n x)\}$, $n \geq 1$. Let $(\alpha_n)$ be a sequence in $(0,1)$.

(E) The sequence of the modified Ishikawa’s iteration is defined by $x_1 \in D$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) \text{Pr}_n y_n, \forall n \geq 1,$$

In this paper, we modify Mann’s iteration by using the best approximation operator $P_{T_n}$, $n \geq 1$ to find common fixed points of a countable family of nonexpansive multi-valued mappings $\{T_n\}_{n=1}^\infty, n \geq 1$. Then we prove weak and strong convergence theorems for a countable family of multi-valued mappings in Banach spaces. Finally, we apply our main result to the problem of finding a common fixed point of a family of nonexpansive multi-valued mappings.

## 2. Preliminaries

In this section, we give some characterizations and properties of the metric projection in a real Hilbert space.

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $D$ be a closed and convex subset of $H$. If, for any point $x \in H$, there exists a unique nearest point in $D$, denoted by $P_T x$, such that

$$\|x - P_T x\| \leq \|x - y\|, \forall y \in D$$

then $PD$ is called the metric projection of $H$ onto $D$. We know that $PD$ is a nonexpansive mapping of $H$ onto $D$.

**Lemma 2.1** ([28]). Let $D$ be a closed and convex subset of a real Hilbert space $H$ and $PD$ be the metric projection from $H$ onto $D$. Then, for any $x \in H$ and $z \in PD x$ if and only if the following holds:

$$\|x - z, y - z\| \leq 0, \forall y \in D$$

Using the proof line in Lemma 3.1.3 of [28], we obtain the following result.

**Proposition 2.2.** Let $D$ be a closed and convex subset of a real Hilbert space $H$. Let $T : D \rightarrow \text{CCB}(D)$ be a multi-valued mapping and $PT$ the best approximation operator. Then, for any $x \in D$ and $z \in PT x$ if and only if the following holds:

$$\langle x - z, y - z \rangle \leq 0, \forall y \in Tx$$

**Lemma 2.3** ([28]). Let $H$ be a real Hilbert space. Then the following equations hold:

1. $\|x - y\|^2 = \|x\|^2 - \|y\|^2 + 2 \langle x - y, y \rangle, \forall x, y \in H$;
2. $\|x + (1-t)y\|^2 = \|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \forall x, y \in H$.

for all $t \in [0,1]$ and $x, y \in H$.

We next show that $PT$ is nonexpansive under some suitable conditions imposed on $T$.

**Remark 2.4.** Let $D$ be a closed and convex subset of a real Hilbert space $H$. Let $T : D \rightarrow \text{CCB}(D)$ be a multi-valued mapping. If $T = T_p$, $\forall x \in D$, then $PT$ is a nonexpansive multi-valued mapping.

In fact, let $x, y \in D$. For each $a \in PT x$, we have

$$d(a, P_T y) \leq \|a - b\|, \forall b \in P_T y \quad (2.1)$$

From Proposition 2.2, we have

$$\langle x - y, a - b \rangle = \langle x - a, a - b \rangle + \langle y - b, b - a \rangle \geq 0$$
It follows that
\[ \|a - b\|^2 = (x - a, a - b) + (a - b, x - y), a - b) \] (2.2)
\[ \leq (x - y, a - b) \leq \|x - y\| \|a - b\| \]
This implies that
\[ \|a - b\| \leq \|x - y\| \] (2.3)
From (2.1) and (2.3), we obtain
\[ d(a, P_T y) \leq \|x - y\| \]
for every \( a \in P_X \). Hence \( \sup p_n \in P_T a, P_T y \leq \|x - y\| \).
Similarly, we can show that \( \sup p_n \in P_T a, P_T x, b \leq \|x - y\| \). Therefore \( H(P_T x, P_T y) \leq \|x - y\| \).

It is clear that if a nonexpansive multi-valued mapping \( T \) satisfies the condition that \( T x = T y, \forall x, y \in D \), then \( T \) is nonexpansive. The following example shows that if \( T \) is a nonexpansive multi-valued mapping satisfying the property that \( T x = T y, \) for all \( x \in D \).

**3. Strong and Weak Convergence of the Modified Ishikawa Iteration in Banach Spaces**

In this section, we first prove a strong convergence theorem for a countable family of multi-valued mappings under the SC-condition and Condition (A) and then prove a weak convergence theorem under the SC-condition in Banach spaces.

**Theorem 3.1.** Let \( D \) be a closed and convex subset of a uniformly convex Banach space \( E \) which satisfies Opial’s condition. Let \( \{T_n\} \) and \( \tau \) be two families of multivalued mappings from \( D \) into \( P(D) \) with \( D \) a weak convergence theorem under the SC-condition in \( E \), which satisfies Opial’s condition. Let \( \{T_n\} \) and \( \{T_m\} \) be mappings from \( D \) into \( P(D) \) with \( \tau \) a weak convergence theorem under the SC-condition and Condition (A) and then prove a strong convergence theorem for a countable family of multi-valued mappings \( \{T_n\} \) and \( \{T_m\} \).

Proof. Since \( \limsup_{n \to \infty} \|x_n - p\| \) exists and \( 0 < \liminf_{n \to \infty} \|x_n - p\| \leq \limsup_{n \to \infty} \|x_n - p\| < 1 \), \( \lim \|x_n - p\| = 0 \).

By the properties of \( g \), we can conclude that \( \lim_{n \to \infty} x_n = z_n \).

Since \( <T_n> \) satisfies the SC-condition, there exists \( C_n \in T_{\infty} \) such that
\[ \lim_{n \to \infty} \|x_n - z_n\| = 0 \] (3.3)
for every \( T_{\infty} \). Since \( <x_o> \) is bounded, there exists a subsequence \( <x_{o_k}> \) of \( <x_o> \) converging weakly to some \( q_1 \in D \).

It follows from (A2) and (3.3) that \( q_1 \in T_{\infty} \) for every \( T_{\infty} \). Next, we show that \( <x_o> \) converges weakly to \( q_1 \), take another subsequence \( <x_{o_k}> \) of \( <x_o> \) converging weakly to some \( q_2 \in D \). Again, as above we can conclude that \( q_2 \in T_{\infty} \) for every \( T_{\infty} \). Finally, we show that \( q_1 = q_2 \).

Assume \( q_1 \neq q_2 \). Then by Opial’s condition of \( E \), we have
\[ \lim_{n \to \infty} \|x_n - z\| = \lim_{k \to \infty} \|x_{o_k} - z\| = \lim_{k \to \infty} \|x_{o_k} - q_2\| < \lim_{k \to \infty} \|x_{o_k} - q_2\| \]
and Also
\[ \|x_{n+1} - P\| = \|x_n + (1 - \alpha_n) P_{T_n} y_n - P\| \]
\[ = \|x_n + (1 - \alpha_n) P_{T_n} y_n - (1 - \alpha_n) P\| \leq \alpha_n \|x_n - P\| + (1 - \alpha_n) \|P_{T_n} y_n - P\| \]
\[ \leq \alpha_n \|x_n - P\| + (1 - \alpha_n) \|T_n P_{T_n} y_n - P\| \]
\[ \leq \alpha_n \|x_n - P\| + (1 - \alpha_n) \|T_n P_{T_n} P_{T_n} y_n - P\| \]
\[ \leq \alpha_n \|x_n - P\| + (1 - \alpha_n) \|H(P_{T_n} x_n, P_{T_n} P_{T_n} y_n)\| \]
\[ \leq \alpha_n \|x_n - P\| + (1 - \alpha_n) \|z_n - P\| \]
\[ \leq \alpha_n \|x_n - P\| + (1 - \alpha_n) \|z_n - P\| \]
for every \( p \in F(T) \). Then \( <\|x_n - p\|> \) is a decreasing sequence and hence \( \lim_{n \to \infty} \|x_n - p\| \) exists for every \( p \in F(T) \). For \( p \in F(T) \), since \( <x_n> \) and \( <z_n> \) are bounded by Lemma 2.9, there exists a continuous, strictly increasing and convex function \( g: [0,1) \) with \( g(0) = 0 \) such that
\[ \beta_n (1 - \beta_n) g(\|x_n - z_n\|) \leq \|x_n - p\| - \|y_n - p\| \]
It follows that
\[ \beta_n (1 - \beta_n) g(\|x_n - z_n\|) \leq \|x_n - p\| - \|y_n - p\| \]
Since
\[ \lim_{n \to \infty} \|x_n - p\| \] exists and
\[ 0 < \liminf_{n \to \infty} \|x_n - p\| \leq \limsup_{n \to \infty} \|x_n - p\| < 1, \]
\[ \lim_{n \to \infty} g(\|x_n - z_n\|) = 0 \]
By the properties of \( g \), we can conclude that
\[ \lim_{n \to \infty} x_n = z_n \]
which is a contradiction. Therefore \( q_1 = q \). This shows that \( \langle x_n \rangle \) converges weakly to a fixed point of \( \tau \) for every \( T \in \tau \). This completes the proof.

**Corollary 3.2.** Let \( D \) be a closed and convex subset of a uniformly convex Banach space \( E \) which satisfies Opial's condition. Let \( \{ T_n \} \) and \( \tau \) be two families of nonexpansive multivalued mappings from \( D \) into \( K(D) \) with \( F(\tau) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset \). Let \( \{ x_n \} \) be a sequence in \( (0, 1) \) such that 0 \( \lim \inf_{n \to \infty} a_n \leq \lim \sup_{n \to \infty} a_n < 1 \). Let \( \{ x_n \} \) be generated by (1.2). Assume that for each \( n \in \mathbb{N} \),

\[
H \left( P_{T_n} x, P_{T_n} p \right) \leq \| x - p \| \quad \forall x \in D, p \in F(\tau);
\]

(B1) the best approximation operator \( P_{T_n} \) is nonexpansive for every \( T \in \tau \);

(B2) \( P_{T_n} \) is a sequence in \( (0, 1) \) such that 0 \( \lim \inf_{n \to \infty} a_n \leq \lim \sup_{n \to \infty} a_n < 1 \). Let \( \{ x_n \} \) be generated by (1.2). Assume that for each \( n \in \mathbb{N} \),

\[
H \left( P_{T_n} x, P_{T_n} p \right) \leq \| x - p \| \quad \forall x \in D, p \in F(\tau);
\]

(B3) \( F(\tau) \) is closed.

If \( \{ T_n \} \) satisfies the SC-condition and Condition (A), then \( \langle x_n \rangle \) converges strongly to an element in \( F(\tau) \).

Proof. It follows from the proof of Theorem 3.1 that \( \lim_{n \to \infty} x_n - c_0 \) exists for every \( p \in F(\tau) \) and \( \lim_{n \to \infty} x_n - c_0 \) where \( c_0 = \lim_{n \to \infty} x_n \) for every \( T \in \tau \). Since \( \langle x_n \rangle \) satisfies the SC-condition, there exists \( c_0 \in T x_0 \) such that

\[
\lim_{n \to \infty} x_n - c_0 = 0
\]

for every \( T \in \tau \). This implies that

\[
\lim_{n \to \infty} d \left( x_n, T x_0 \right) \leq \lim_{n \to \infty} d \left( x_n, P_{T_n} x_n \right) \leq \lim_{n \to \infty} \left\| x_n - c_0 \right\| = 0
\]

for every \( T \in \tau \). Since that \( \langle x_n \rangle \) satisfies Condition (A), we have

\[
\lim_{n \to \infty} d \left( x_n, F(\tau) \right) = 0.
\]

It follows from (B3), there is a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) and a sequence \( \{ p_k \} \) such that

\[
\left\| x_{n_k} - p_k \right\| \leq \frac{1}{2^k}
\]

(3.4)

for all \( k \). From (3.1), we obtain

\[
\left\| x_{n_{k+1}} - p \right\| \leq \left\| x_{n_{k+1}} - x_{n_{k+2}} \right\| \leq \left\| x_{n_{k+2}} - p \right\|
\]

(3.5)

Next, we show that \( \{ p_k \} \) is a Cauchy sequence in \( D \). From (3.3) and (3.5), we have

\[
\left\| P_{T_{n+1}} - P_{T_n} \right\| \leq \left\| P_{T_{n+1}} - x_{n+1} \right\| + \left\| x_{n+1} - x_n \right\| \leq \frac{1}{2^n}
\]

(3.6)

This implies that \( \{ p_k \} \) is a Cauchy sequence in \( D \) and thus converges to \( q \in D \). Since \( P_{T_{n+1}} \) is nonexpansive for every \( T \in \tau \),

\[
d \left( p_k, T q \right) \leq d \left( p_k, P_{T_n} q \right) \leq H \left( P_{T_{n+1}} p_k, P_{T_n} q \right) \leq \left\| p_k - q \right\|
\]

(3.7)

for every \( T \in \tau \). It follows that \( d(q, T q) = 0 \) for every \( T \in \tau \) and thus \( q \in F(\tau) \). It implies that (3.4) \( \{ x_{n_k} \} \) converges strongly to \( q \). Since \( \lim_{n \to \infty} \| x_n - q \| \) exists, it follows that \( \{ x_n \} \) converges strongly to \( q \). This completes the proof.

We know that if \( T \) is a quasi nonexpansive multivalued mapping, then \( F(\tau) \) is closed. So we have the following result:

**Corollary 3.4.** Let \( D \) be a closed and convex subset of a uniformly convex Banach space \( E \). Let \( \{ T_n \} \) and \( \tau \) be two families of nonexpansive multivalued mappings from \( D \) into \( P(D) \) with \( F(\tau) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset \). Let \( \{ x_n \} \) be a sequence in \( (0, 1) \) such that 0 \( \lim \inf_{n \to \infty} a_n \leq \lim \sup_{n \to \infty} a_n < 1 \). Let \( \{ x_n \} \) be generated by (1.2). Assume that for each \( n \in \mathbb{N} \),

\[
H \left( P_{T_n} x, P_{T_n} p \right) \leq \| x - p \| \quad \forall x \in D, p \in F(\tau);
\]

(B1) the best approximation operator \( P_{T_n} \) is nonexpansive for every \( T \in \tau \);

(B2) \( P_{T_n} \) is a sequence in \( (0, 1) \) such that 0 \( \lim \inf_{n \to \infty} a_n \leq \lim \sup_{n \to \infty} a_n < 1 \). Let \( \{ x_n \} \) be generated by (1.2). Assume that for each \( n \in \mathbb{N} \),

\[
H \left( P_{T_n} x, P_{T_n} p \right) \leq \| x - p \| \quad \forall x \in D, p \in F(\tau)\) and the best approximation operator \( P_{T_n} \) is nonexpansive for every \( T \in \tau \).

If \( \{ T_n \} \) satisfies the SC-condition and Condition (A), then \( \langle x_n \rangle \) converges strongly to an element in \( F(\tau) \).

References:


