A Study on New Sequence of Functions Involving $H$-Function

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Abstract A remarkably large number of operational techniques have drawn the attention of several researchers in the study of sequence of functions and polynomials. Very recently, Agarwal and Chand gave certain new sequence of functions involving the special functions in their series of papers. In this sequel, here, we aim to introduce a new sequence of functions involving the Generalized Mellin-Barnes Type of Contour Integrals by using operational techniques. Some generating relations and finite summation formulae of the sequence presented here are also considered. These generating relations and finite summation formulae are unified in nature and act as key formulae from which, we can obtain as their special cases.

Keywords: Special function, generating relations, $H$-function, sequence of functions


1. Introduction

The idea of representing the processes of calculus, derivation, and integration, as operators is called operational technique; it is also known as operational Calculus. Many operational technique involving various special functions, have found significant importance and applications in various sub field of applicable mathematical analysis. Several applications of operational techniques can be found in the problems in analysis, in particular differential equations are transformed into algebraic problems, usually the problem of solving a polynomial equation. Since last four decades, a number of workers like Chak [6], Gould and Hopper [10], Chatterjea [9], Singh [23], Srivastava and Singh [24], Mittal [15,16,17], Joshi and Parjapat [11], Patil and Thakare [18] and Srivastava and Singh [26] have studied in depth, the properties, applications and different extensions of the various operational techniques.

The key element of the operational technique is to consider differentiation as an operator acting on functions. Linear differential equations can then be recast in the form of an operator valued function $F(D)$ of the operator $D$ acting on the unknown function equals the known function. Solutions are then obtained by making the inverse operator of $F$ act on the known function.

Indeed, a remarkably large number of sequences of functions involving a variety of special functions have been developed by many authors (see, for example, [19]; for a very recent work, see also [2,3,4,22]). Here, we aim at presenting a new sequence of functions involving the $H$ – function, by using operational techniques. Some generating relations and finite summation formula are also obtained.

For our purpose, we begin by recalling some known functions and earlier works.

In 1971, the Rodrigues formula for generalized Lagurre polynomials is given by Mittal [15] as follows:

$$T_{kn}(x) = \frac{1}{n!} \cdot x^{-\alpha} \exp\left(p_k(x)\right) \cdot \theta^\alpha \left[\exp(-p_k(x))\right],$$

where $p_k(x)$ is a polynomial in $x$ of degree $k$.

Mittal [16] also proved following relation for (1) as follows:

$$T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!} \cdot x^{-\alpha-s} \exp\left(p_k(x)\right) \cdot \theta^\alpha \left[\exp(-p_k(x))\right],$$

where $s$ is constant and $T_{s} = x(s + xD)$.

In 1979, Srivastava and Singh [24] studied a sequence of functions $V_{n}^{(\alpha)}(x;a,k,s)$ defined by:

$$V_{n}^{(\alpha)}(x;a,k,s) = \frac{x^{-\alpha}}{n!} \cdot \exp\left(p_k(x)\right) \cdot \theta^\alpha \left[\exp(-p_k(x))\right].$$

By employing the operator $\theta = x^s(s + xD)$, where $s$ is constant and $p_k(x)$ is a polynomial in $x$ of degree $k$. 


A new sequence of function \( V_{n}^{(M,N;P,Q;\alpha)}(x;a,k,s) \) involving the well-known \( \overline{H} \) – function, introduced in this paper is defined as follows:
\[
V_{n}^{(M,N;P,Q;\alpha)}(x;a,k,s) = \frac{1}{n!} x^{-a} \overline{H}_{P,Q}^{M,N} \left[ (p_{k}(x))^T \right] x^{\alpha} \overline{H}_{P,Q}^{M,N} \left[ -p_{k}(x) \right],
\]
where \( T_{x}^{a,s} = x^{a}(s+xD), D = \frac{d}{dx} \), \( a \) and \( s \) are constants, \( \beta \geq 0 \), \( k \) is finite and non-negative integer, \( p_{k}(x) \) is a polynomial in \( x \) of degree \( k \) and \( \overline{H}_{P,Q}^{M,N} [x] \) is a well known \( \overline{H} \) -function is defined and represented in the following manner (see, [12] and see also, [1,5]):
\[
\overline{H}_{P,Q}^{M,N} [x] = \overline{H}_{P,Q}^{M,N} \left[ \sum_{j=1}^{M} (a_{j},\alpha_{j},A_{j})_{1,N} \nabla^{N} (a_{j},\alpha_{j})_{N+1,P} \right]
\]
where
\[
\overline{\phi}(\xi) = \frac{\prod_{j=m+1}^{M} \Gamma(b_{j} - \beta_{j} - \xi)}{\prod_{j=m+1}^{Q} \Gamma(1-b_{j} + \beta_{j} - \xi)} \prod_{j=m+1}^{P} \Gamma(a_{j} - \alpha_{j} - \xi). \]

It may be noted that the \( \overline{\phi}(\xi) \) contains fractional powers of some of the gamma function and \( M,N,P,Q \) are integers such that \( 1 \leq M \leq Q, \) \( 1 \leq N \leq P \) \( (\alpha_{j})_{1,P} \), \( (\beta_{j})_{1,Q} \) are positive real numbers and \( (A_{j})_{1,N} \), \( (B_{j})_{1,M+1,Q} \) may take non-integer values, which we assume to be positive for standardization purpose. \( (\alpha_{j})_{1,P} \) and \( (\beta_{j})_{1,Q} \) are complex numbers.

The nature of contour \( L \), sufficient conditions of convergence of defining integral (5) and other details about the \( \overline{H} \) -function can be seen in the papers [1,5,12,13].

The behavior of the \( \overline{H} \) -function for small values of \(|z|\) follows easily from a result given by Rathie [19]:
Third generating relation:
\[
\sum_{n=0}^{\infty} \binom{m+n}{m} V_n^{(M,N,P,Q;\alpha)}(x;a,k,s) t^n \\
= (1 - at)^{\frac{\alpha + s}{a}} \overline{H}_{P,Q}^{M,N}(p_k(x)) \\
V_n^{(M,N,P,Q;\alpha)}\left(x(1 - at)^{-1/a};a,k,s\right).
\]
(18)

Proof of the first generating relation:
From (4), Let us consider
\[
\sum_{n=0}^{\infty} V_n^{(M,N,P,Q;\alpha)}(x;a,k,s) t^n \\
= x^{-\frac{\alpha + s}{a}} \overline{H}_{P,Q}^{M,N}(p_k(x)) \exp\left(\frac{\alpha + s}{a} \overline{H}_{P,Q}^{M,N}(-p_k(x)) \right).
\]
Using operational technique (11), above equation (19) reduces to
\[
\sum_{n=0}^{\infty} V_n^{(M,N,P,Q;\alpha)}(x;a,k,s) t^n \\
= x^{-\frac{\alpha + s}{a}} \overline{H}_{P,Q}^{M,N}(p_k(x)) \exp\left(\frac{\alpha + s}{a} \overline{H}_{P,Q}^{M,N}(-p_k(x)) \right)
\]
(20)

after replacing t by tx^{-a} , we get the desired result (16).

Proof of the second generating relation:
Again from (4), we have
\[
\sum_{n=0}^{\infty} x^{-an} V_n^{(M,N,P,Q;\alpha-an)}(x;a,k,s) t^n \\
= x^{-\frac{\alpha + s}{a}} \overline{H}_{P,Q}^{M,N}(p_k(x)) \exp\left(\frac{\alpha + s}{a} \overline{H}_{P,Q}^{M,N}(-p_k(x)) \right)
\]
Using the (12), we get
\[
\sum_{n=0}^{\infty} x^{-an} V_n^{(M,N,P,Q;\alpha-an)}(x;a,k,s) t^n \\
= x^{-\frac{\alpha + s}{a}} \overline{H}_{P,Q}^{M,N}(p_k(x)) \exp\left(\frac{\alpha + s}{a} \overline{H}_{P,Q}^{M,N}(-p_k(x)) \right)
\]
(21)

\[
= (1 + at)^{\frac{\alpha + s}{a}} \overline{H}_{P,Q}^{M,N}(p_k(x)) \\
\times \overline{H}_{P,Q}^{M,N}\left[-p_k\left(x(1 + at)^{1/a}\right)\right].
\]
(22)

This complete the proof of second generating relation.

Proof of the third generating relation:
We can write (4) as follows
\[
\sum_{m=0}^{\infty} \binom{m+n}{m} V_n^{(M,N,P,Q;\alpha)}(x;a,k,s) t^m \\
= n!x^a \overline{H}_{P,Q}^{M,N}(p_k(x)) V_n^{(M,N,P,Q;\alpha)}(x;a,k,s),
\]
(23)

or
\[
\exp\left(t^{\alpha,s}\right)\overline{H}_{P,Q}^{M,N}(p_k(x)) V_n^{(M,N,P,Q;\alpha)}(x;a,k,s)
\]

3. Finite Summation Formulas
In this section, we establish certain finite summation formulas, some of which are presumably (new) ones.

**First finite summation formula:**

\[ V_{n}^{(M,N,P,Q;\alpha)}(x;a,k,s) = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{\alpha}{a} \right)^{m} \left( x^a \right)^m \sum_{n=0}^{m} \binom{m}{n} V_{n-m}^{(M,N,P,Q;0)}(x;a,k,s). \]  

(28)

**Second finite summation formula:**

\[ V_{n}^{(M,N,P,Q;\alpha)}(x;a,k,s) = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{\alpha-\beta}{a} \right)^{m} \left( x^a \right)^m \sum_{n=0}^{m} \binom{m}{n} V_{n-m}^{(M,N,P,Q;\beta)}(x;a,k,s). \]  

(29)

**Proof of First finite summation formula:**

From equation (4), we have

\[ V_{n}^{(M,N,P,Q;\alpha)}(x;a,k,s) = \frac{1}{n!} x^{-\alpha} \overline{H}_{P,Q}^{M,N}(p_k(x)) \left( T_s^{(a,s)} \right)^n \times \left[ x^{-a-1} \overline{H}_{P,Q}^{M,N}(-p_k(x)) \right]. \]  

(30)

Using the operational technique (13) on (30), we get

\[ V_{n}^{(M,N,P,Q;\alpha)}(x;a,k,s) = \frac{1}{n!} x^{-\alpha} \overline{H}_{P,Q}^{M,N}(p_k(x)) x \sum_{m=0}^{n} \binom{n}{m} \left( T_s^{(a,s)} \right)^m (x^{-1}). \]  

(31)

Finally from (28) and (34), we get the desired result (28).

**Proof of second finite summation formula:**

Equation (4) can be written as

\[ \sum_{n=0}^{\infty} V_{n}^{(M,N,P,Q;\alpha)}(x;a,k,s) t^n = x^{-\alpha} \overline{H}_{P,Q}^{M,N}(p_k(x)) \exp \left( T_s^{(a,s)} \right) \left[ x^{-\alpha} \overline{H}_{P,Q}^{M,N}(-p_k(x)) \right]. \]  

(35)

Applying the (11) on (35), we get

\[ \sum_{n=0}^{\infty} V_{n}^{(M,N,P,Q;\alpha)}(x;a,k,s) t^n = x^{-\alpha} \overline{H}_{P,Q}^{M,N}(p_k(x)) \exp \left( T_s^{(a,s)} \right) \left[ x^{-\alpha} \overline{H}_{P,Q}^{M,N}(-p_k(x)) \right]. \]  

(36)

Applying (15) on (36) then equation (36) reduces in to the form

\[ \sum_{n=0}^{\infty} V_{n}^{(M,N,P,Q;\alpha)}(x;a,k,s) t^n = x^{-\alpha} \overline{H}_{P,Q}^{M,N}(p_k(x)) \exp \left( T_s^{(a,s)} \right) \left[ x^{-\alpha} \overline{H}_{P,Q}^{M,N}(-p_k(x)) \right]. \]  

(37)
Now equating the coefficient of $t^n$ both the sides, we get
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (t^n)
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p(x))}{G_{P,Q}^{M,N} (p(x))} \] (38)

Finally, by using the equation (4) on (37), we get the result (29).

4. Special Cases

a. If we put $A_j = B_j = 1$, -function reduces to Fox’s H -function [25], p. 10, Eqn. (2.1.1), then the equation (16), (17) and (18) takes the following form:

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (p_k(x))
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p_k(x))}{G_{P,Q}^{M,N} (p_k(x))} \] (39)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (p_k(x))
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p_k(x))}{G_{P,Q}^{M,N} (p_k(x))} \] (40)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (p_k(x))
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p_k(x))}{G_{P,Q}^{M,N} (p_k(x))} \] (41)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (p_k(x))
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p_k(x))}{G_{P,Q}^{M,N} (p_k(x))} \] (42)

b. If we put $A_j = B_j = 1, a_j = b_j = 1$, then the $H$ - function reduces to general type of G-function [14] i.e.

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (p_k(x))
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p_k(x))}{G_{P,Q}^{M,N} (p_k(x))} \] (43)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (p_k(x))
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p_k(x))}{G_{P,Q}^{M,N} (p_k(x))} \] (44)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (p_k(x))
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p_k(x))}{G_{P,Q}^{M,N} (p_k(x))} \] (45)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (p_k(x))
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p_k(x))}{G_{P,Q}^{M,N} (p_k(x))} \] (46)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{a+s}{a} \right)^n T_{P,Q}^{M,N} (p_k(x))
= (1+at)^{-1} \int \frac{G_{M,N}^{P,Q} (p_k(x))}{G_{P,Q}^{M,N} (p_k(x))} \] (47)

5. Conclusion

In this paper, we have presented a new sequence of functions involving the $H$ -function by using operational
techniques. With the help of our main sequence formula, some generating relations and finite summation formulae of the sequence are also presented here. Our sequence formula is important due to presence of $H$-function. On account of the most general nature of the $H$-function, a large number of sequences and polynomials involving simpler functions can be easily obtained as their special cases but due to lack of space we cannot mention here.

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References


