On A Class of New Type Generalized Difference Sequences Related to the P-Normed $l^p$ Space Defined By Orlicz Functions

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Abstract The idea of difference sequence spaces were defined by Kizmaz [6] and generalized by Et and Colak [5]. Later Esi et al. [4] introduced the notion of the new difference operator $\Delta_n^m$ for fixed $n,m \in \mathbb{N}$. In this article we introduce new type generalized difference sequence space $(M, \Delta_n^m, \varphi, p)$ using by the Orlicz function. We give various properties and inclusion relations on this new type difference sequence space.

Keywords: Orlicz function, difference sequence space, solid space, symmetric space

1. Introduction

Throughout the article $w$, $l_\infty$ and $l^p$ denote the spaces all, bounded and $p$ absolutely summable sequences, respectively. The zero sequence is denoted by $\Theta = (0,0,0,...)$. The sequence space $m(\varphi)$ was introduced by Sargent [11], who studied some of its properties and obtained its relationship with the space $l^p$. Later on it was investigated from sequence space point of view by Rath [9], Rath and Tripathy [10], Tripathy and Sen [15], Tripathy and Mahanta [14], Esi [2] and others.

An Orlicz function is a function $M: [0,\infty) \rightarrow [0,\infty)$, which is continuous, non-decreasing and convex with $M(0)=0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

An Orlicz function is said to satisfy $\Delta_2$-condition for all values of $u$, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)$, $u \geq 0$.

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda \leq 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct the sequence space
\[ l_M = \left\{ (x_k) : \sum_k M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}. \]

The space $l_M$ with the norm $\|x\|_M = \inf \left\{ \rho > 0 : \sum_k M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$ becomes a Banach space which is called an Orlicz sequence space. The space $l_M$ is closely related to the space $l^p$, which is an Orlicz sequence space with $M(x) = x^p$, $1 \leq p < \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Tripathy and Mahanta [14], Esi [1], Esi and Et [3], Parashar and Choudhary [8], and many others.

Kizmaz [6] defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ as follows:
\[ Z(\Delta) = \{ (x_k) : (Ax_k) \in Z \}, \]
for $Z = \ell_\infty$, $c$ and $c_0$, where $Ax_k = (\Delta x_k) = (x_k - x_{k+1})$ for all $k \in \mathbb{N}$.
The above spaces are Banach spaces, normed by
\[ \|x\|_\Delta = \|x\| + \sup_k \|Ax_k\| \]

Later, the difference sequence spaces were generalized by Et and Colak [5] as follows: Let $n \in \mathbb{N}$ be fixed integer, then $X(\Delta^n) = \{ (x_k) : (\Delta^n x_k) \in X \}$ for $X = l_\infty, c$ and $c_0$, where $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$ and so $\Delta^n x_k = \sum_{i=0}^n (-1)^i \binom{n}{i} x_{k+i}$.

They showed that the above spaces are Banach spaces, normed by
\[ \|x\|_\Delta = \|x\| + \sup_k \|\Delta^n x_k\| \]
After then, the notion new type of difference sequence spaces were further generalized Esi and et.al. [4] as follows:

Let \( m,n \in N \) be fixed integers, then
\[
X(\Lambda_m^n) = \{(x_k) : (\Lambda_m^n x_k) \in X\}
\]
for \( X = l_{\infty}, c \text{ and } c_0 \), where \( \Lambda_m^n x_k = \Lambda_m^n x_{k} - \Lambda_m^n x_{k+m} \) and \( \Delta_0^0 x_k = x_k \) for all \( k \in N \). The new type generalized difference has the following binomial representation:
\[
\Delta_m^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k+mi}
\]

They showed that the above spaces are Banach spaces, normed by
\[
\|(x_k)\|_{\Delta_m^n} = \sum_{i=1}^{r} \max_{k \geq i} |\Delta_m^n x_k|
\]
where, \( r = mn \) for \( m,n \geq 1 \); \( r = n \) for \( m = 0 \) and \( r = m \) for \( n = 0 \).

2. Definitions and Background

Throughout the article \( \phi_p \) denotes the set of all subsets of \( N \), the set of natural numbers, those do not contain more than \( s \) elements. Further \( (\phi_p) \) will denote a non-decreasing sequence of positive real numbers such that \( n \phi_{n+1} \leq (n+1) \phi_n \) for all \( n \in N \). The class of all the sequences \( (\phi_p) \) satisfying this property is denoted by \( \Phi \).

The space \( m(\phi) \) introduced and studied by Sargent [11] is defined as follows:
\[
m(\phi) = \left\{(x_k) : \max_{k \geq 1} \frac{1}{\phi_k} \sum_{l=1}^{k} |x_l| \leq \infty \right\}
\]
Recently Tripathy and Mahanta [13] defined and studied the following sequence space: Let \( M \) be an Orlicz function, then
\[
m(M,\Delta,\phi) = \left\{(x_k) : \sup_{k \geq 1} \frac{1}{\phi_k} \sum_{l=1}^{k} M \left( \frac{|x_l|}{\rho_l} \right) < \infty \text{ for some } \rho > 0 \right\}.
\]

The purpose of this paper is to introduce and study a class of new type generalized difference sequences related to the space \( l^p(\Delta) \) using by Orlicz function.

In this article we introduce the following sequence space: Let \( M \) be an Orlicz function and \( p= (p_k) \) be bounded sequence of strictly positive real numbers and \( m,n \geq 0 \) be fixed integers, then
\[
m(M,\Delta_m^n,\phi, p) = \left\{(x_k) : \sup_{k \geq 1} \frac{1}{\phi_k} \sum_{l=1}^{k} M \left( \frac{|x_l|}{\rho_l} \right)^{p_k} < \infty \text{ for some } \rho > 0 \right\}.
\]

Taking \( p_k = 1 \) for all \( k \) and \( m=n=1 \) i.e., considering only first difference we have the following difference sequence space which were defined and studied by Tripathy and Mahanta [13]
\[
m(\Delta_m,\phi) = \left\{(x_k) : \sup \frac{1}{\phi_k} \sum_{k \geq 1} M \left( \frac{|x_k - x_{k+m}|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}.
\]

Taking \( p_k = 1 \) for all \( k \), \( M(x) = x \) and \( m = n = 1 \) i.e., considering only first difference we have the following difference sequence space which were defined and studied by Tripathy [12]
\[
m(\Delta,\phi) = \left\{(x_k) : \sup \frac{1}{\phi_k} \sum_{k \geq 1} |x_k| < \infty \right\}.
\]

The space \( l^p(\Delta) \) for \( 0 < p < 1 \) is defined by Rath [9] as follows:
\[
l^p(\Delta) = \left\{(x_k) : \sup_{k \geq 1} \|x_k\|^p < \infty \right\}
\]

Let \( x = (x_k) \) be a sequence, then \( S(X) \) denotes the set of all permutations of the elements of \( (x_k) \) i.e. \( S(X) = \{ (x_{\pi(k)}) : \pi (k) \text{ is a permutation on } N \} \). A sequence space \( E \) is said to be symmetric if \( S(X) \subset E \) for all \( x \in E \).

A sequence space \( E \) is said to be monotone, if it contains the canonical pre-images of its step spaces.

The following inequality will be used throughout the paper
\[
|y_k + y_k|^{p_k} \leq C (|y_k|^{p_k} + |y_k|^{p_k})
\]
where \( x_k \) and \( y_k \) are complex numbers,
\[
C = \max \{1, 2H^{-1}\} \text{ and } H = \sup_k p_k < \infty.
\]

3. Main Results

In this section we prove some results involving the sequence space \( m(M,\Delta_m^n,\phi, p) \).

**Theorem 1.** Let \( p= (p_k) \) be bounded sequence of strictly positive real numbers. Then the space \( m(M,\Delta_m^n,\phi, p) \) is a linear space over the complex field \( C \).

**Proof:** Let \( (x_k), (y_k) \in m(M,\Delta_m^n,\phi, p) \). Then there exists positive numbers \( \rho_1 \) and \( \rho_2 \) such that
\[
\sup_{k \geq 1} \frac{1}{\phi_k} \sum_{k \geq 1} M \left( \frac{|x_k - x_{k+m}|}{\rho_1} \right)^{p_k} < \infty
\]
and
\[
\sup_{k \geq 1} \frac{1}{\phi_k} \sum_{k \geq 1} M \left( \frac{|y_k - y_{k+m}|}{\rho_2} \right)^{p_k} < \infty
\]
and
\[
\sup_{k \geq 1} \frac{1}{\phi_k} \sum_{k \geq 1} M \left( \frac{|x_k - x_{k+m}|}{\rho} \right)^{p_k} < \infty
\]
and
\[
\sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho_3} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k + \beta y_k}{\rho_3} \right)^{\gamma \rho k} \right] < \infty
\]

Let \( \rho_3 = \max \left( 2|\alpha|, 2|\beta| \rho_2 \right) \). Since \( M \) is non-decreasing and convex
\[
\sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k + \beta y_k}{\rho_3} \right)^{\gamma \rho k} \right] \leq \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k + \beta y_k}{\rho_3} \right)^{\gamma \rho k} \right] \leq C \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k}{\rho_1} \right)^{\gamma \rho k} \right] + C \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m y_k}{\rho_2} \right)^{\gamma \rho k} \right] \]
\[
\Rightarrow
\sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho_3} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k + \beta y_k}{\rho_3} \right)^{\gamma \rho k} \right] \leq \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho_3} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k}{\rho_1} \right)^{\gamma \rho k} \right] + \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho_3} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m y_k}{\rho_2} \right)^{\gamma \rho k} \right] \]
\[
< \infty
\]

Hence \( \alpha \left( x_k \right) + \beta \left( y_k \right) \in m \left( M, \Lambda^n_m, \varphi, p \right) \).

**Theorem 2.** Let \( p^m = \left( p_k \right) \) be bounded sequence of strictly positive real numbers and \( H = \max \left( 1, \sup_k \rho_k \right) \). Then \( m \left( M, \Lambda^n_m, \varphi, p \right) \) is a linear topological space paranormed by
\[
g \left( x \right) = \left( \sum_{i=1}^n \left| \Lambda^n_m x_i \right|^p \right)^{\gamma / \rho_k} + \]
\[
\inf_{\rho / H} \left\{ \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k}{\rho} \right)^{\gamma \rho k} \right] \right\} \leq 1, n, m = 1, 2, 3, \ldots
\]
where \( r = mn \) for \( m \geq 1, n \geq 1 \); \( r = n \) for \( m = 0 \) and \( r = m \) for \( n = 0 \).

**Proof:** Clearly \( g \left( x \right) = g \left( -x \right) \). Next \( \left( x_k \right) = \varnothing \) implies \( \Lambda^n_m x_k = 0 \) and such as \( M \left( 0 \right) = 0 \), therefore \( g \left( \varnothing \right) = 0 \). It can be easily shown that \( g \left( x \right) = 0 \Rightarrow \left( x_k \right) = \varnothing \).

Next, let \( \rho_1 > 0 \) and \( \rho_2 > 0 \) be such that
\[
\sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho_1} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k}{\rho_1} \right)^{\gamma \rho k} \right] \leq 1
\]
and
\[
\sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho_2} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m y_k}{\rho_2} \right)^{\gamma \rho k} \right] < \infty
\]
Let \( \rho = \rho_1 + \rho_2 \). Then we have
\[
\sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k + \beta y_k}{\rho} \right)^{\gamma \rho k} \right] \leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho_1} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k}{\rho_1} \right)^{\gamma \rho k} \right] + \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho_2} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m y_k}{\rho_2} \right)^{\gamma \rho k} \right] \leq 1
\]
Since the \( \rho \)'s are non-negative, we have
\[
g \left( x + y \right) = \left( \sum_{i=1}^n \left| \Lambda^n_m x_i + y_i \right|^p \right)^{\gamma / \rho_k} + \]
\[
\inf_{\rho / H} \left\{ \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m x_k + \beta y_k}{\rho} \right)^{\gamma \rho k} \right] \right\} \leq 1, n, m = 1, 2, 3, \ldots
\]
where \( r = mn \) for \( m \geq 1, n \geq 1 \); \( r = n \) for \( m = 0 \) and \( r = m \) for \( n = 0 \).

Next, for \( \lambda \in C \), without loss of generality, let \( \lambda \neq 0 \), then
\[
g \left( \lambda x \right) = \left( \sum_{i=1}^n \left| \Lambda^n_m \lambda x_i \right|^p \right)^{\gamma / \rho_k} + \]
\[
\inf_{\rho / H} \left\{ \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m \lambda x_k}{\rho} \right)^{\gamma \rho k} \right] \right\} \leq 1, n, m = 1, 2, 3, \ldots
\]
where \( r = mn \) for \( m \geq 1, n \geq 1 \); \( r = n \) for \( m = 0 \) and \( r = m \) for \( n = 0 \).

Next, for \( \lambda \in C \), without loss of generality, let \( \lambda \neq 0 \), then
\[
g \left( \lambda x \right) = \left( \sum_{i=1}^n \left| \Lambda^n_m \lambda x_i \right|^p \right)^{\gamma / \rho_k} + \]
\[
\inf_{\rho / H} \left\{ \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\rho} \sum_{k \geq 0} \left[ M \left( \frac{\Lambda^n_m \lambda x_k}{\rho} \right)^{\gamma \rho k} \right] \right\} \leq 1, n, m = 1, 2, 3, \ldots
\]
where \( r = \frac{\rho}{|k|} \)

\[
g(kx) = \max \left\{ |k| \left( \sum_{i=1}^{n} \left| \frac{x_i}{s} \right|^{|k|} \right)^{|k|/|k|} \right\}
\]

\[
\max \left\{ \left| k \right| \left( \sum_{i=1}^{n} \left| \frac{x_i}{s} \right|^{\rho/k} \right)^{\rho/k} \right\} \leq \left( \sum_{i=1}^{n} \left| \frac{x_i}{s} \right|^{\rho/k} \right)^{\rho/k} \leq \frac{1}{\left( \sum_{i=1}^{n} \left| \frac{x_i}{s} \right|^{\rho/k} \right)^{\rho/k}}
\]

So, the continuity of the scalar multiplication follows from the above inequality.

**Theorem 3.** \( m(M, \Delta_m^{n, \varphi, p}) \subseteq m(M, \Delta_m^{n, \Psi, p}) \) if and only if \( \sup_{s \geq 1} \frac{\varphi_s}{\Psi_s} < \infty \).

**Proof:** Let \( \sup_{s \geq 1} \frac{\varphi_s}{\Psi_s} < \infty \) and \( (x_k) \in m(M, \Delta_m^{n, \varphi, p}) \).

Then

\[
\sup_{s \geq 1} \frac{1}{\Psi_s} \sum_{k \in \sigma} \left[ M \left( \frac{\Delta_m^{n, x_k}}{\rho} \right) \right]^{\rho/k} < \infty,
\]

for some \( \rho > 0 \).

So,

\[
\sup_{s \geq 1} \frac{1}{\Psi_s} \sum_{k \in \sigma} \left[ M \left( \frac{\Delta_m^{n, x_k}}{\rho} \right) \right]^{\rho/k} \leq \left( \sum_{k \in \sigma} \frac{\varphi_k}{\Psi_k} \right) \sup_{s \geq 1} \frac{1}{\Psi_s} \sum_{k \in \sigma} \left[ M \left( \frac{\Delta_m^{n, x_k}}{\rho} \right) \right]^{\rho/k} < \infty.
\]

Therefore \( (x_k) \in m(M, \Delta_m^{n, \varphi, p}) \).

Conversely, let \( m(M, \Delta_m^{n, \varphi, p}) \subseteq m(M, \Delta_m^{n, \Psi, p}) \).

Suppose that \( \sup_{s \geq 1} \frac{\varphi_s}{\Psi_s} = \infty \).

Then there exists a sequence of natural numbers \( (s_i) \) such that

\[
\lim_{i \to \infty} \frac{\varphi_{s_i}}{\Psi_{s_i}} = \infty.
\]

Let \( (x_k) \in m(M, \Delta_m^{n, \varphi, p}) \). Then there exists \( \rho > 0 \) such that

\[
\sup_{s \geq 1} \frac{1}{\Psi_s} \sum_{k \in \sigma} \left[ M \left( \frac{\Delta_m^{n, x_k}}{\rho} \right) \right]^{\rho/k} < \infty.
\]

Now we have

\[
\sup_{s \geq 1} \frac{1}{\Psi_s} \sum_{k \in \sigma} \left[ M \left( \frac{\Delta_m^{n, x_k}}{\rho} \right) \right]^{\rho/k} \geq \left( \sup_{s \geq 1} \frac{\varphi_{s_i}}{\Psi_{s_i}} \right) \sup_{s \geq 1} \frac{1}{\Psi_s} \sum_{k \in \sigma} \left[ M \left( \frac{\Delta_m^{n, x_k}}{\rho} \right) \right]^{\rho/k} = \infty.
\]

Therefore \( (x_k) \notin m(M, \Delta_m^{n, \Psi, p}) \). As such we arrive at a contradiction. Hence \( \sup_{s \geq 1} \frac{\varphi_s}{\Psi_s} < \infty \).

The following result is a consequence of Theorem 3.

**Corollary 4:** Let \( M \) be an Orlicz function. Then \( m(M, \Delta_m^{n, \varphi, p}) = m(M, \Delta_m^{n, \Psi, p}) \) if and only if \( \sup_{s \geq 1} \frac{\varphi_s}{\Psi_s} < \infty \) and \( \sup_{s \geq 1} \frac{\varphi_s}{\Psi_s} < \infty \) for all \( s = 1, 2, 3, \ldots \).

**Theorem 5:** Let \( p = (p_k) \) be bounded sequence of strictly positive real numbers and let \( M \) and \( M_1 \) be Orlicz functions satisfying \( \Delta_2 \)-condition. Then

\[
m(M, \Delta_m^{n, \varphi, p}) \subseteq m(M_1, \Delta_m^{n, \varphi, p})
\]

**Proof:** Let \( (x_k) \in m(M_1, \Delta_m^{n, \varphi, p}) \). Then we have

\[
\sup_{s \geq 1} \frac{1}{\Psi_s} \sum_{k \in \sigma} \left[ M \left( \frac{\Delta_m^{n, x_k}}{\rho} \right) \right]^{\rho/k} < \infty,
\]

for some \( \rho > 0 \).

Let \( 0 < \varepsilon < 1 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that

\[
M(\delta) < \varepsilon \quad \text{for} \quad 0 \leq t \leq \delta.
\]

Let \( y_k = M_1 \left( \frac{\Delta_m^{n, x_k}}{\rho} \right) \) for all \( m \) and \( n \) and for any \( \sigma \in P_s \), let

\[
\sum_{k \in \sigma} \left[ M \left( y_k \right) \right]^{\rho/k} = \sum_{k \in \sigma} \left[ M \left( y_k \right) \right]^{\rho/k} + \sum_{k \in \sigma} \left[ M \left( y_k \right) \right]^{\rho/k}
\]

where the first summation is over \( y_k \leq \delta \) and the second is over \( y_k > \delta \). For the first summation above, we can write

\[
\sum_{k \in \sigma} \left[ M \left( y_k \right) \right]^{\rho/k} \leq \sum_{k \in \sigma} \left[ M \left( \delta \right) \right]^{\rho/k} \sum_{k \in \sigma} \left[ \left( y_k \right) \right]^{\rho/k} (1)
\]

(by using Remark)

For the second summation, we will make following procedure. For \( y_k > \delta \), we have

\[
y_k < 1 + \frac{y_k}{\delta}
\]

Since \( M \) is non-decreasing and convex, it follows that

\[
M \left( y_k \right) < M \left( 1 + \frac{y_k}{\delta} \right) \leq \frac{1}{2} M(2) + \frac{1}{2} M \left( \frac{1}{2} \frac{y_k}{\delta} \right)
\]

Since \( M \) satisfies \( \Delta_2 \)-condition, we can write

\[
M \left( y_k \right) \leq K \frac{1}{2} M(2) + \frac{1}{2} M \left( \frac{y_k}{\delta} \right) = K M(2) \left( \frac{y_k}{\delta} \right)
\]

Hence

\[
\sum_{k \in \sigma} \left[ M \left( y_k \right) \right]^{\rho/k} \leq \sum_{k \in \sigma} \left[ M \left( \delta \right) \right]^{\rho/k} \sum_{k \in \sigma} \left[ \left( y_k \right) \right]^{\rho/k} (2)
\]
By (1) and (2), we have \((x_k) \in m(MoM_1, \Delta_m^n, \varphi, p)\).

Taking \(M_1(x) = x\) in Theorem 5, we have the following result.

**Corollary 6:** Let \(p^m = (p_k)\) be bounded sequence of strictly positive real numbers and let \(M\) be an Orlicz function satisfying \(\Delta_\infty\)-condition. Then
\[
m\left(\Delta_m^n, \varphi, p\right) \subseteq m\left(M, \Delta_m^n, \varphi, p\right)
\]

From Theorem 3 and Corollary 6, we have

**Corollary 7:** Let \(p^m = (p_k)\) be bounded sequence of strictly positive real numbers and let \(M\) be an Orlicz function satisfying \(\Delta_\infty\)-condition. Then
\[
m\left(\Delta_m^n, \varphi, p\right) \subseteq m\left(M, \Delta_m^n, \Psi, p\right)
\]

if and only if \(\sup_{x \in \Delta_\infty} \frac{\varphi_k}{\Psi_s} < \infty\).

**Corollary 8:** The space \(m\left(M, \Delta_m^n, \varphi, p\right)\) is not solid and symmetric in general.

**Proof:** To show this space is not solid and symmetric in general, consider the following examples, respectively.

**Example 1.** Let \(m = n = 1, \varphi_k = 1, p_k = 1\) and \(x_k = 1\) for all \(k \in N\). Consider \(\lambda = \left(\lambda_k\right) = \left((-1)^k\right)\) for all \(k \in N\) and \(M(x) = x\). Then \((x_k) \in m\left(M, \Delta_m^n, \varphi, p\right)\) but \((\lambda_k x_k) \notin m\left(M, \Delta_m^n, \varphi, p\right)\). Hence the space is not solid in general.

**Example 2.** Let \(m = n = 1, \varphi_k = k^{-1}, p_k = 1\) and \(x_k = 1\) for all \(k \in N\) and \(M(x) = x\). Then the sequence \((x_k)\) define \(x_k = k\) for all \(k \in N\) is in \(m\left(M, \Delta_m^n, \varphi, p\right)\). Consider the sequence \((y_k)\), the rearrangement of \(x = (x_k)\) define as follows
\[
y_k = (x_1, x_2, x_4, x_3, x_5, x_6, x_7, x_9, x_8, x_{10}, x_{11}, \ldots)
\]

Then \((y_k) \notin m\left(M, \Delta_m^n, \varphi, p\right)\). Hence the space is not symmetric in general.

Finally, in this section, we consider that \(p = (p_k)\) and \(q = (q_k)\) are any bounded sequences of strictly positive real numbers. We are able to prove below results only under additional conditions.

**Corollary 9:** a) If \(0 < \inf_k p_k \leq p_k \leq 1\) for all \(k\), then
\[
m\left(M, \Delta_m^n, \varphi, p\right) \subseteq m\left(M, \Delta_m^n, \varphi\right)
\]

b) If \(1 \leq p_k \leq \sup_k p_k = H < \infty\) for all \(k\), then
\[
m\left(M, \Delta_m^n, \varphi\right) \subseteq m\left(M, \Delta_m^n, \varphi, p\right)
\]

c) Let \(0 < p_k \leq q_k\) for all \(k\) and \(q_k/p_k\) be bounded, then
\[
m\left(M, \Delta_m^n, \varphi, q\right) \subseteq m\left(M, \Delta_m^n, \varphi, p\right)
\]

**Proof:** Using the same technique as in Theorem 4 in [1], it is easy to prove the Corollary 9.

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**References**


[12] Tripathy, B.C., On a class of difference sequences related to the p-normed space \(F\), Demonstratio Mathematica, 36(4) (2003), 867-872.

